Digital Communication

(Code : 304181)

Semester V – Electronics and Telecommunication Engineering
(Savitribai Phule Pune University)

Strictly as per the New Credit System Syllabus (2015 Course)
Savitribai Phule Pune University w.e.f. academic year 2017-2018

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Preface

Dear students,

I am extremely happy to present the book of **"Digital Communication"** for you. I have divided the subject into small chapters so that the topics can be arranged and understood properly. The topics within the chapters have been arranged in a proper sequence to ensure smooth flow of the subject.

I am thankful to Shri. Pradeep Lunawat and Shri. Sachin Shah for the encouragement and support that they have extended to me. I am also thankful to the staff members of Tech-Max Publications and others for their efforts to make this book as good as it is. We have jointly made every possible efforts to eliminate all the errors in this book. However if you find any, please let us know, because that will help me to improve further.

I am also thankful to my family members and friends for their patience and encouragement.

\- **J. S. Katre**

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Syllabus...

Savitribai Phule Pune University, Pune

Third Year of Electronics & Telecommunication Engineering (2015 Course)

(304181) Digital Communication

Teaching Scheme		**Examination Scheme**	
Lecture	**: 04 hrs/week**	**In-Sem**	**: 30 Marks**
		End-Sem	**: 70 Marks**

Course Objectives :

- To understand the building blocks of digital communication system.
- To prepare mathematical background for communication signal analysis.
- To understand and analyze the signal flow in a digital communication system.
- To analyze error performance of a digital communication system in presence of noise and other interferences.
- To understand concept of spread spectrum communication system.

Course Outcomes :

On completion of the course, student will be able to :

1) Understand working of waveform coding techniques and analyse their performance.
2) Analyze the performance of a baseband and pass band digital communication system in terms of error rate and spectral efficiency.
3) Perform the time and frequency domain analysis of the signals in a digital communication system.
4) Design of digital communication system.
5) Understand working of spread spectrum communication system and analyze its performance.

Course Contents

Unit I

Digital Transmission of Analog Signal :

Introduction to Digital Communication System: Block Diagram and transformations, Basic Digital Communication Nomenclature. Digital Versus Analog Performance Criteria, Sampling Process, PCM Generation and Reconstruction, Quantization Noise, Non-uniform Quantization and Companding, PCM with noise: Decoding noise, Error threshold, Delta Modulation, Adaptive Delta Modulation, Delta Sigma Modulation, Differential Pulse Code Modulation, LPC speech synthesis. **(Refer chapters 1 and 2)**

Unit II

Baseband Digital Transmission :

Digital Multiplexing: Multiplexers and hierarchies, Data Multiplexers. Data formats and their spectra, synchronization: Bit Synchronization, Scramblers, Frame Synchronization, Inter-symbol interference, Equalization.

(Refer chapter 3)

Unit III

Random Signal and Noise :

Introduction, Mathematical definition of a random process, Stationary processes, Mean, Correlation and Covariance function, Ergodic processes, Transmission of a random process through a LTI filter, Power spectral density, Gaussian process, noise, Narrow band noise, Representation of narrowband noise in terms of in phase and quadrature components. **(Refer chapter 4)**

Unit IV

Baseband Receiver :

Signal space representation : Geometric representation of signal, Conversion of continuous AWGN channel to vector channel, Likelihood functions, Coherent Detection of binary signals in presence of noise, Optimum Filter, Matched Filter, Probability of Error of Matched Filter, Correlation receiver. **(Refer chapter 5)**

Unit V

Passband Digital Transmission :

Passband transmission model, Signal space diagram, Generation and detection, Error Probability derivation and Power spectra of coherent BPSK, BFSK and QPSK. Geometric representation, Generation and detection of - M-ary PSK, M-ary QAM and their error probability, Non-coherent BFSK, DPSK. **(Refer Chapter 6)**

Unit VI

Spread Spectrum Modulation :

Introduction, Pseudo noise sequences, A notion of spread spectrum, Direct sequence spread spectrum with coherent BPSK, Signal space dimensionality and processing gain, Probability of error, Concept of jamming, Frequency hop spread spectrum. **(Refer chapter 7)**

□□□

Unit I

Chapter 1 : Introduction to Digital Communication

1-1 to 1-22

Syllabus : Introduction to digital communication system : Block diagram and transformations, Basic digital communication, Nomenclature, Digital versus analog performance criteria.

Unit I

Chapter 2 : Digital Transmission of Analog Signals

2-1 to 2-76

Syllabus : Sampling process, PCM generation and reconstruction, Quantization noise, Non-uniform quantization and companding, PCM with noise : Decoding noise, Error threshold, Delta modulation, Adaptive delta modulation, Delta sigma modulation, Differential pulse code modulation, LPC speech synthesis.

Unit II

Chapter 3 : Baseband Digital Transmission 3-1 to 3-72

Syllabus : Digital multiplexing : Multiplexers and hierarchies, Data multiplexers, Data formats and their spectra, Synchronization : Bit synchronization, Scramblers, Frame synchronization, Inter-symbol Interference, Equalization.

Unit III

Chapter 4 : Random Signal & Noise 4-1 to 4-41

Syllabus : Introduction, Mathematical definition of a random process, Stationary processes, Mean, Correlation and Co-variance function, Ergodic processes, Transmission of a random process through a LTI filter, Power spectral density, Gaussian process, Noise, Narrowband noise, Representation of narrowband noise in terms of in phase and Quadrature components.

Unit IV

Chapter 5 : Baseband Receivers 5-1 to 5-42

Syllabus : Signal space representation, Geometric representation of signal, Conversion of continuous AWGN channel to vector channel, Likelihood functions, Coherent detection of binary signals in presence of noise, Optimum filter, Matched filter, Probability of error of matched filter, Correlation receiver.

Unit V

Chapter 6 : Passband Digital Transmission 6-1 to 6-89

Syllabus : Passband transmission model, Signal space diagram, Generation and detection, Error probability derivation and power spectra of coherent BPSK, BFSK and QPSK, Geometric representation, Generation and detection of M-ary PSK, M-ary QAM and their error probability, Non-coherent BFSK, DPSK.

Unit VI

Chapter 7 : Spread Spectrum Modulation 7-1 to 7-58

Syllabus : Introduction, Pseudo noise sequences, A notion of spread spectrum, Direct sequence spread spectrum with coherent BPSK, Signal space dimensionality and processing gain, Probability of error, Concept of jamming, Frequency hop spread spectrum.

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CHAPTER 1

Unit I

Introduction to Digital Communication

Syllabus :

Introduction to digital communication system : Block diagram and transformations, Basic digital communication, Nomenclature, Digital versus analog performance criteria.

1.1 Introduction to Digital Communication System (DCS) :

- In the previous semester you have learnt some important concepts of analog communication.
- Basically a communication system can be analog or digital type.
- In an analog communication system the information signal at the input is continuously varying in amplitude and time and it is used to proportionally change a characteristic such as amplitude, frequency or phase of a sinusoidal carrier. This produces a modulated signal.
- In the digital communication system, the information signal is a discrete in nature.

1.2 Sources and Signals :

- A source of information generates an information signal called message.
- The examples of message signals are as follows :
  1. Voice signal
  2. TV picture
  3. Teletype data
  4. Temperature or pressure.
  5. Many other sources
- The message signals mentioned above are all non-electrical signals.

  Hence a **transducer** is used to convert them into their electrical equivalent. Such electrical equivalent of a message is called as **baseband signal.**

1.2.1 Analog to Digital Conversion :

- The message signal can be analog or digital type. An analog signal can always be converted into a digital signal.
- The analog to digital conversion (A/D) can be achieved by using the system shown in Fig. 1.2.1.
- This system consists of three blocks namely sampler, quantizer and encoder.

Sampler :

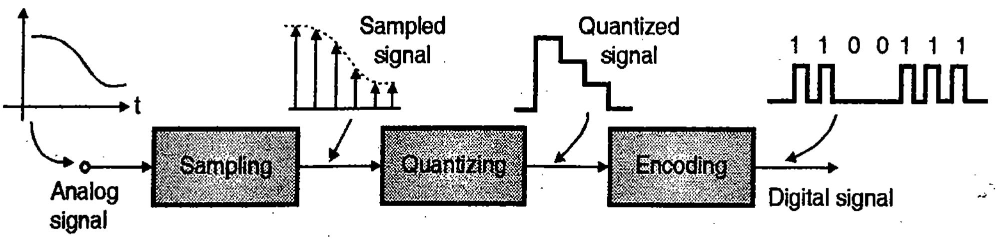
- The analog signal is applied at the input of the sampler.
- The sampler is a switch which samples the input signal at regular intervals of time and produces the discrete version of the input signal.

Quantizer :

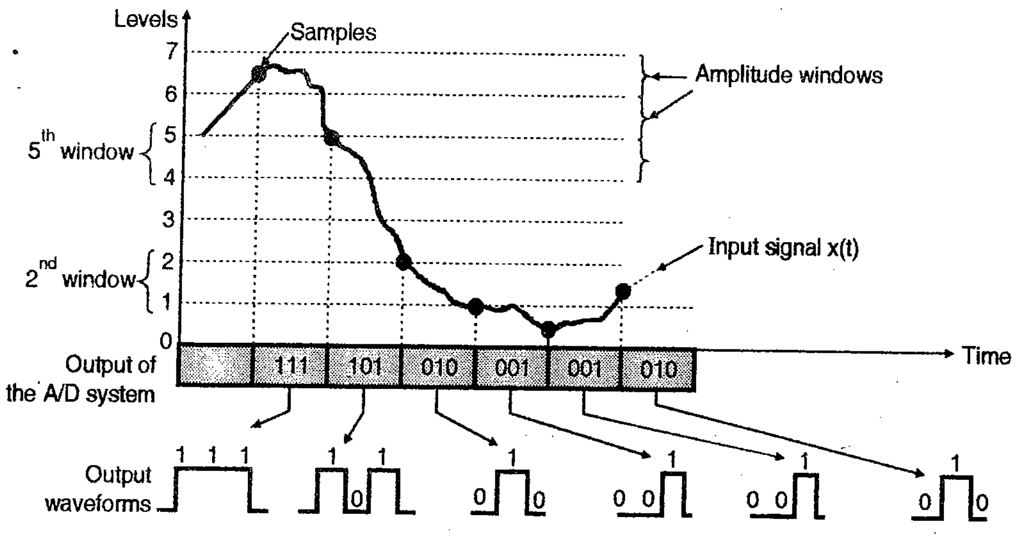
- Quantization is a process of approximation or rounding off.
- Quantization process approximates each sample to its nearest standard voltage level called quantization level. We get the approximated version of the sampled signal at the output of the quantizer
- The number of quantization levels is finite and generally it is a power of 2 i.e. 2, 4, 8, 16, 32 ......

Encoder :

- An encoder converts each quantized sample into a separate code word of length say N bits.
- Thus at the output of the encoder we get digital code words.



(E-1) **Fig. 1.2.1 : Analog-to-digital conversion**



(E-2) Fig. 1.2.2 : Input and output waveforms of a PCM system

1.2.2 Graphical Representation of A/D Conversion Process :

- Fig. 1.2.2 illustrates the A to D conversion process graphically.
- It is important to understand that the output is in the form of binary codes. Each transmitted binary code represents a particular amplitude of the input signal.
- Hence the "information" is represented in the form of a "code" which is being transmitted.
- The range of input signal magnitudes is divided into 8-equal levels (Y axis in Fig. 1.2.2). Each level is denoted by a three bit digital word between 000 and 111.
- Input signal x (t) is sampled. If the sample is in the 5th - window of amplitude then a digital word 101 is transmitted. If the sample is in the 2nd - window then the transmitted word is 010 and so on.
- In this illustration we have converted the sampled amplitudes into 3 bit codes, but in practice the number of bits per word can be as high as 8, 9 or 10.
- The codewords shown in Fig. 1.2.2 are three bit numbers. It is possible to introduce one more bit to indicate the "sign."

Error :

- Due to the approximation taking place in the quantization process, the A to D conversion introduces some error in the digital signal.
- Such errors cannot be reversed and it is not possible to produce an exact replica of the original analog signal at the receiver.
- However it is possible to minimize these errors by selecting a proper sampling rate and number of quantization levels.

1.3 Why Digital ?

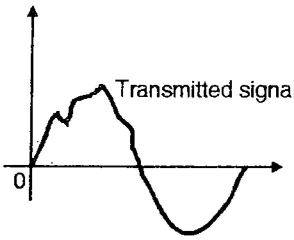
- In recent days the commercial as well as military communication systems are becoming digital.
- The reasons behind using the digital communication are as follows :

1. Digital signals can be easily regenerated.
2. Digital signals are less affected by noise.
3. It is possible to use regenerative repeaters. This will increase the range of communication.
4. Digital circuits are affected less by distortion and interference.
5. The error rates (errors introduced) with digital techniques are extremely low.
6. Digital circuits and systems are more reliable and more flexible.
7. It is easy for digital signals to undergo signal processing.

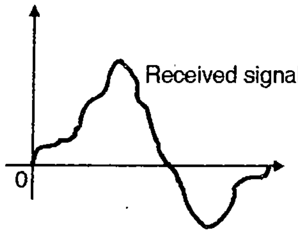
- We will discuss these advantages one by one.

1. Digital signals are less affected by noise :

- In the analog communication, are desiged is to transmit a waveform as shown in Fig. 1.3.1(a) because all information is contained in the shape of the waveform.
- But during the transmission the shape of transmitted signal gets distorted as shown in Fig. 1.3.1(b) due to shortcomings of the communication channel.

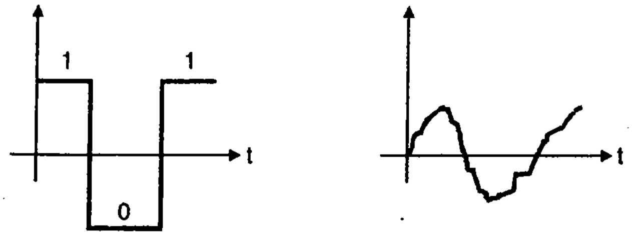


(a) Transmitted signal in analog communication



(b) Received signal in the analog communication

(E-4) **Fig. 1.3.1**



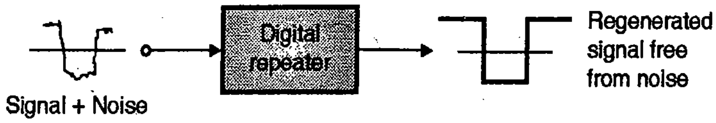
(c) Transmitted signal in digital transmission **(d) Received signal in digital transmission**

(E-5) **Fig. 1.3.1**

- Since all the information is contained in the shape of the waveform, it is necessary to preserve the shape of waveform.
- But the noise will distort the shape of the waveform as shown in Fig. 1.3.1(b). Therefore the received information will also be distorted.
- Now consider the transmission of digital signal as shown in Fig. 1.3.1(c). To transmit such a binary signal the objective is to transmit either 0 or 1.
- Fig. 1.3.1(d) shows the received digital signal with noise. The receiver has to decide whether a 0 is received or a 1 is received. Even for the noise contaminated received signal it is easier for the receiver to make a correct decision most of the times.
- Hence the digital transmission is more immune to noise as compared to the analog communication.

2. Easy regeneration :

- The digital signals that are contaminated with noise can be easily reproduced without noise using regenerative repeaters.
- The repeaters are basically regenerators of the signals. They receive the (signal + noise) at their input, separate out the signal from noise and regenerate the signal which is free from noise.
- Due to the use of repeaters the noise performance of digital communication systems is much better than that of the analog systems. Note that repeaters cannot be used for the analog communication system because it is impossible to separate noise once it gets mixed with the analog signal.
- The operation of a digital repeater becomes clear from Fig. 1.3.2.
- Thus it is possible to undo the effect of noise on the signal. This improves the noise immunity and system performance in presence of noise.



(E-7) **Fig. 1.3.2 : A digital repeater**

3. Digital communication is suitable for long distances :

- The digital communication becomes cost effective over analog communication if it is used for longer distances.
- Consider a long distance communication link shown in Fig. 1.3.3.
- As the signal from the source travels a distance from source, its amplitude will reduce due to attenuation. In addition to that, the signal will become increasingly distorted. The noise from various sources gets added to the signal.
- In digital communication system repeaters are introduced as shown in Fig. 1.3.3 between the destination and source to regenerate noise free signal.

4. Less distortion and interference :

- Digital circuits are less subject to distortion and interference as compared to the analog circuits. This is because binary digital signals have two states : 0 or 1.
- A disturbance (noise) must be large enough to change the state of the signal from 0 to 1 or vice versa.
- It is also possible to use the regenerative repeaters as discussed earlier.
- Due to all this the noise and other disturbances do not accumulate during the digital transmission.

5. Low error rates :

- Due to use of regenerative repeaters used between the transmitter and receivers, in digital communication system and also due to the use of various error detection and correction procedures that we can use with the digital communication systems, the error rates are extremely low.
- That means even in presence of noise, the digital data can be received without introducing any error.

6. More reliability and flexibility :

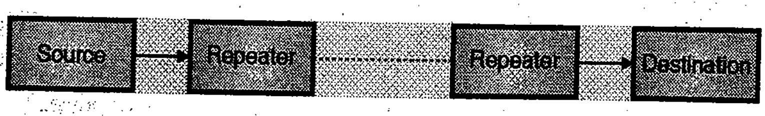
Digital circuits are more reliable and their cost of production is low. Also due to the use of digital hardware (microprocessor, LSI and digital switches) the circuit implementation has a better flexibility.

7. Use of TDM :

The time division multiplexing (TDM) technique can be used to transmit many digital signals over a single communication channel.

8. Signal processing :

Various signal processing functions can be carried out easily over the digital signals. Such processing will protect the signal against interference and jamming. They also provide encryption and privacy.



(E-6) **Fig. 1.3.3 : A long distance communication link**

1.3.1 Advantages of Digital Communication :

Some of the advantages of digital communication are as follows :

1. Due to the digital nature of the transmitted signal, the interference of additive noise does not introduce many errors. So digital communication has a better noise immunity.
2. Due to the channel coding techniques used in digital communication, it is possible to detect and correct the errors introduced during the data transmission.
3. Repeaters can be used between transmitter and receiver to regenerate the digital signal. This improves the noise immunity further and also extends the range of communication.
4. Due to the digital nature of the signal, it is possible to use the advanced data processing techniques such as digital signal processing, image processing, data compression etc.
5. TDM (Time Division Multiplexing) technique can be used to transmit many voice channels over a single common transmission channel. Thus digital telephony is possible to achieve.
6. Digital communication is suitable in military applications where only a few permitted receivers can receive the transmitted signal.
7. Digital communication is becoming simpler and cheaper as compared to the analog communication due to the invention of high speed computers and integrated circuits (ICs).

1.3.2 Disadvantages :

Some of the important disadvantages of digital communication are :

1. The bit rates of digital systems are high. Therefore they require a larger channel bandwidth as compared to analog systems.
2. Digital modulation needs synchronization in case of synchronous modulation.

1.4 Typical Block Diagram and Transformations in DCS :

SPPU : Dec. 11, May 13, Dec. 13, Dec. 15

University Questions

Q. 1 Explain with a neat sketch, the block diagram of digital communication system and discuss the various formatting techniques involved in it.

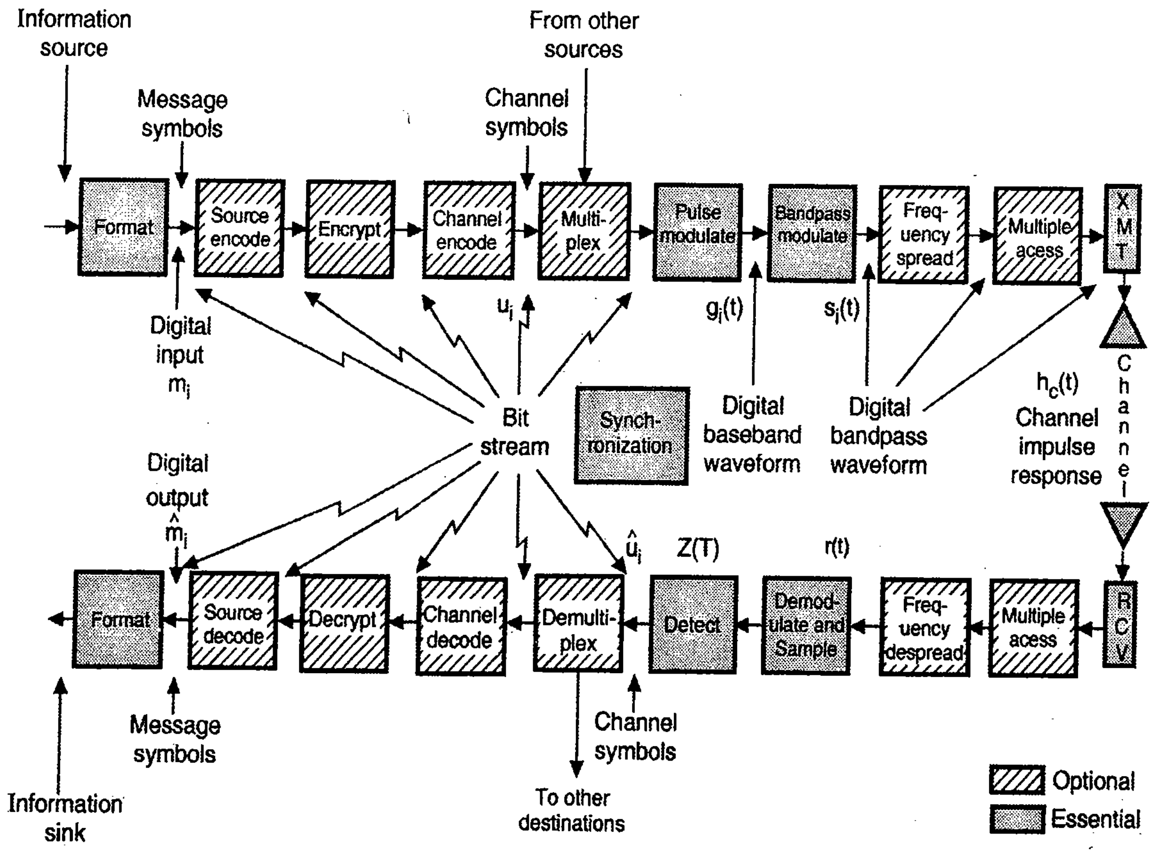
(Dec. 11, May 13, 8 Marks)

Q. 2 With the help of detail diagram explain function of each block of digital communication system.

(Dec. 13, 8 Marks)

Q. 3 Explain with the help of block diagram formatting and transmission of baseband signal.

(Dec. 15, 8 Marks)



(E-1350) **Fig. 1.4.1 : Block diagram of a typical digital communication system (DCS)**

- Fig. 1.4.1 shows the functional block diagram of the digital communication system (DCS). It basically consists of a transmitter (upper blocks), a receiver (lower blocks) and a communication channel.
- The signal processing steps carried out at the receiver are exactly opposite to those taking place at the transmitter.
- The modulate and demodulate/detect blocks together are known as **MODEM.**
- In Fig. 1.4.1 some blocks are optional while the other are essential.

1.4.1 Transmitter :

SPPU : Dec. 15

University Questions

Q. 1 Explain with the help of block diagram formatting and transmission of baseband signal.

(Dec. 15, 8 Marks)

- The input information source is applied to the **format** block which samples and converts the information into a digital signal.
- This digital signal is then applied to the **source encoder block.** In source coding the encoder converts the digital signal generated at the source output into another signal in digital form. Source encoding is used to reduce or eliminate redundancy for ensuring an efficient representation of the source output. Different source coding techniques are PCM, DM, ADM etc.
- The conversion of signal from one form to the other is called as mapping. Such a mapping is usually of one as to one type.
- Due to elimination of redundancy the source coding represents the source output very efficiently.
- In Fig. 1.4.1, the only essential blocks are as follows : Formatting, Modulation, De-Modulation / Detection and Synchronization.
- **Encryption** is the process of converting the digital signal at the source encoder output into another secret coded signal. This is an optional block. Encryption is required to ensure the communication privacy.
- The encrypted signal is applied to the **channel coding block.**
- Channel encoding is done to minimize the effect of channel noise.
- This will reduce the number of errors in the received data and will make the system more reliable. Channel coding technique introduces some redundancy.
- The channel encoder maps the incoming digital signal into a channel input. Channel coding for the given data rate can reduce the probability of error.
- The output of channel coder is applied to a **multiplexer** which combines other signals originating from some other sources.
- Upto this point the signal is in the form of the bit stream. At **modulator** this bit stream is converted into waveforms, that are compatible with the transmission channel (this is also called as line coding).
- The **pulse modulation** process will convert the bit stream at its input into suitable line codes. The **Bandpass Modulation** is used for providing an efficient transmission of the signal over the channel. The modulator can use any of the CW digital modulation techniques such as ASK (Amplitude Shift Keyings), FSK (Frequency Shift Keying) or PSK (Phase Shift Keying).
- The demodulator is used for demodulation.
- The **frequency spreading** (spread spectrum technique) will produce a signal that is immune to interference or any kind.
- The modulated waveform is passed through the optional multiple access block (FDMA / TDMA or CDMA) and applied to the transmission channel.

1.4.2 Receiver :

- At the receiver the transmitted signal alongwith the noise added to it while travelling over the channel is received.
- The blocks used at the receiver perform exactly the opposite operation as that are performed at the transmitter.

- The received signal is first passed through the **multiple access** decoder to separate out the signals.
- Then the signal is passed through the **frequency despreading** block which recovers the original signal from the spread spectrum signal. Frequency dispreading process is opposite to the signal spreading process.
- It is then passed through the **demodulator/detector** which is opposite to the modulation process. The **demultiplexer** will separate out the multiplexed signals and pass on the desired (selected) signal to the channel decoder.

Channel decoder :

- The channel decoder is present at the receiver and it maps the channel output into a digital signal in such a way that effect of channel noise is reduced to a minimum.
- Thus channel encoder and decoder together provide a reliable communication over a noisy channel. This is achieved by introducing redundancy (parity bits) in a predecided form, at the transmitter.
- The output of the channel encoder is a series of codewords which include the message and some parity bits. These additional parity bits introduce redundancy.
- The channel decoder converts these codewords back into digital messages.
- The channel coder output is applied to **decryption** block. Decryption is a process which is exactly opposite to the encryption process. The decrypted signal is applied to the source decoder.

Source decoder :

- Source decoder is at the receiver and it behaves exactly in an inverse way to the source encoder.
- It delivers the destination (user) the original digital source output.

Main advantage of using the source coding is that it reduces the bandwidth requirement.

- The source decoder output is finally passed through the format block to recover the original information signal back.

Thus in source coding the redundancy is removed whereas in channel coding the redundancy is introduced in a controlled manner.

- It is possible to opt for only source encoding or for only channel encoding. It is not essential to perform both but in many systems both these are performed together.
- It is possible to change the sequence in which channel encoding and source encoding are being performed.
- Channel and source encoding improve the system performance but they increase the circuit complexity as well.

Synchronization :

- Synchronization is essential in DCS. Its key element is clock signal. This clock signal is involved in the control of all signal processing functions within the DCS.
- In Fig. 1.4.1 synchronization block is drawn without any connecting lines, to show that it has got a role to play in each and every block of DCS.

1.4.3 Signal Processing Functions or Transformations :

- Following are some of the important signal processing functions or transformations that are related to the digital communication :
  1. Formatting and source coding
  2. Baseband signaling
  3. Bandpass signaling
  4. Equalization
  5. Channel coding
  6. Multiplexing and multiple access
  7. Spreading
  8. Encryption
  9. Synchronization
- All these transformations have been discussed in detail in the subsequent chapters of this book.

1.5 Basic Digital Communication Nomenclature : SPPU : Dec. 15

University Questions

Q. 1 Define the terms related to digital communication :

1. Messages 2. Characters 3. Symbols

(Dec. 15, 6 Marks)

Some of the basic digital signal nomenclature which is used frequently in digital communication literature is as follows :

1. Information source :

- It is the device which produces information which is to be communicated using DCS. Information source can be analog or discrete.
- The analog information can be transformed into digital by means of A to D conversion discussed earlier. It uses the sampling and quantization blocks. Sampling is a formatting process while quantization is a source coding technique.

2. Textual message :

- Textual message is a sequence of characters as shown in Fig. 1.5.1(a). It consists of alphabets, numbers and special characters or symbols.

Send Rs. 5000/- to me tomorrow

Ok. I will.

Fig. 1.5.1(a) : Textual message

- For digital transmission, each character in the textual message is converted into a sequence of digits.

3. Character :

- A character is an alphabet or a number or a special symbol. The examples of a character are shown in Fig. 1.5.1(b).

9, 8, A, m, #, @

Fig. 1.5.1(b) : Characters

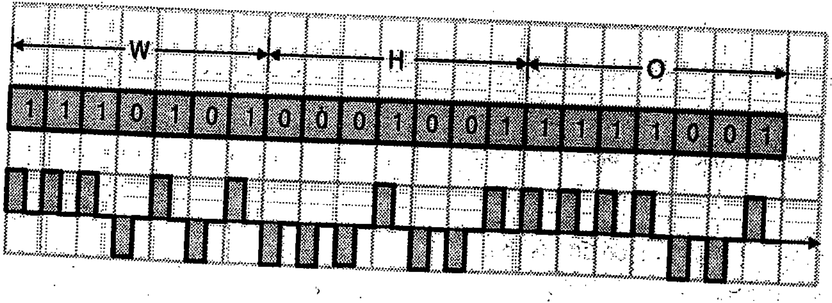
- Each character is converted into a sequence of binary digits. This is called **character encoding.** Some of the standard codes used for character encoding are ASCII (American Standard Code for Information Exchange) or EBCDIC (Extended Binary Coded Decimal Interchange Code) or Hollerith, Baudot codes, etc.

4. Binary digit (bit) :

- It is the fundamental information unit of all DCS. The bit is also used as unit of information. A binary digital can have two possible values i.e. zero or one.

5. Bit stream :

- It is defined as the sequence of binary digits i.e. zeros and ones. A bit stream is generally called as the **baseband signal in DCS.**
- Fig. 1.5.1(c) shows a bit stream for the message WHO which is represented with a 7-bit ASCII code.



(E-1348) **Fig. 1.5.1(c) : Bit stream using 7-bit ASCII**

6. Symbol :

- A symbol is also called as a **digital message** and it is a group of k bits considered one unit.
- Such a unit is referred to as **symbol m_i** (i = 1, 2, ..., M). Thus there will be M symbols with k bits per symbol. The relation between M and k is as follows :

$$\therefore \quad M = 2^k$$

- Fig. 1.5.1(d) shows some examples of symbols.

1 or 0 ....Binary symbol (k = 1, M = 2)

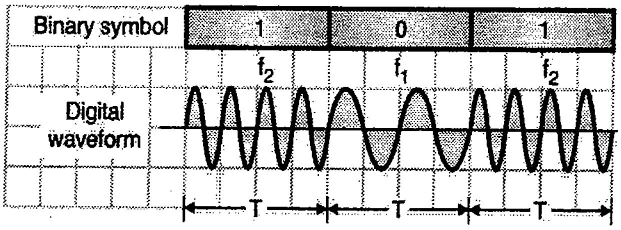
10, 00, 01 or 11 ....Quaternary symbol (k = 2, M = 4)

000, 001, ..., 111 ....8-ary symbol (k = 3, M = 8)

Fig. 1.5.1(d) : Examples of symbols

7. Digital waveform :

- Digital waveform is a voltage or current waveform which represents a digital symbol. Fig. 1.5.1(e) shows an example of such waveform.
- Even though the waveform shown in Fig. 1.5.1(e) is a sinusoidal waveform. It is called as a digital waveform because it is encoded with digital information.



(E-1349) **Fig. 1.5.1(e) : Digital waveform**

- The digital waveform of Fig. 1.5.1(e) is a Binary Frequency Shift Keying (BFSK) waveform, drawn to represent binary symbols with the help of frequency value.
- Symbol "0" is represented by frequency f_1 and symbol "1" is represented by frequency f_2. In Fig. 1.5.1(e), T represents the duration of each symbol.

8. Data rate :

- Data rate is defined as the number of bits transmitted by a DCS per second. It is expressed in bits per second and denoted by R.

$$R = \frac{k}{T}$$

$$= \frac{1}{T} \log_2 M \text{ bits/second}$$

1.6 Digital Versus Analog Performance Criteria :

- The performance of a digital communication system is evaluated in a completely different manner as compared to that of an analog communication system.
- An analog communication system is evaluated on the basis of the following parameters :

1. Signal to noise ratio.
2. Percentage distortion.
3. Expected mean square error between transmitted and received waveforms.

- A DCS transmits signals which represent digits. The DCS performance is evaluated on the basis of probability of error (P_E) value.
- P_E is defined as the probability of incorrectly detecting a digit.

1.6.1 Comparison of Analog and Digital Modulation :

Sr. No.	Analog modulation	Digital modulation
1.	Transmitted modulated signal is analog in nature.	Transmitted signal is digital i.e. train of digital pulses.
2.	Amplitude, frequency or phase variations in the transmitted signal represent the information or message.	Amplitude, width or position of the transmitted pulses is constant. The message is transmitted in the form of code words.
3.	Noise immunity is poor for AM, but improved for FM and PM.	Noise immunity is excellent.
4.	It is not possible to separate out noise and signal. Therefore repeaters cannot be used.	It is possible to separate signal from noise. So repeaters can be used.
5.	Coding is not possible.	Coding techniques can be used to detect and correct the errors.
6.	Bandwidth required is lower than that for the digital modulation methods.	Due to higher bit rates, higher channel bandwidths is needed.
7.	FDM is used for multiplexing.	TDM is used for multiplexing.
8.	Not suitable for transmission of secret information in military application.	Due to coding techniques, it is suitable for military applications.
9.	Analog modulation systems are AM, FM, PM, PAM, PWM etc.	Digital modulation systems are PCM, DM, ADM, DPCM etc.

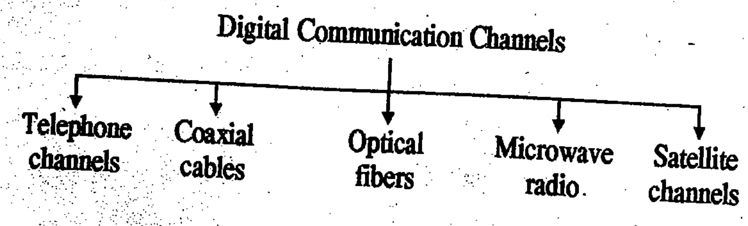
1.6.2 Channels for Digital Communications :

- The type of modulation and coding used in a digital communication system is largely dependent on the channel characteristics and application areas, in which the DCS is being used.
- Some of the important characteristics of a channel are :
  1. Power required to achieve the desired S/N ratio.
  2. Bandwidth of the channel.
  3. Amplitude and phase response of channel.
  4. Type of channel (Linear or Nonlinear)
  5. Effects of external interference on the channel.

Classification of channels :

- The digital communication channels are classified into five categories as follows :

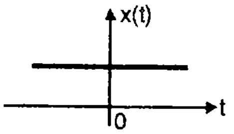
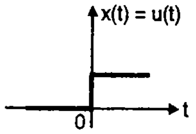
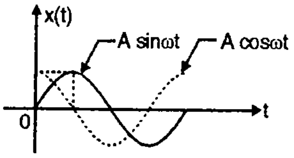
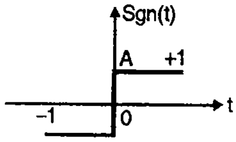
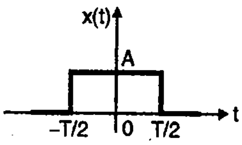
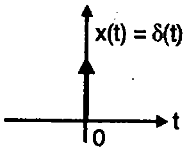
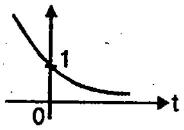
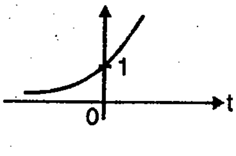
(E-576)

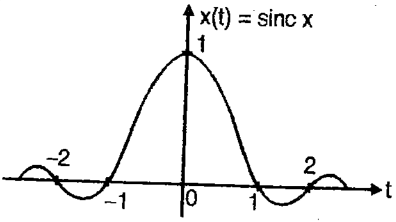


1.7 Signals :

- In the communication systems this word "Signal" is going to be used every now and then. Therefore we must clearly understand its exact meaning.
- The "Signal" is defined as the function of one or more independent variables which contains some information.
- The very familiar examples of electrical signals are voltage and current which are functions of time.
- The signal is a general term and it is not necessary that always it will be used for electrical circuits.
- Some of the important signals are shown in the following table alongwith their mathematical expressions.

Important Signals :

Sr. No.	Name of signal and mathematical representation	Waveforms
1.	**DC signal** $x(t) = A, -\infty < t < \infty$	 (E-583)
2.	**Unit step signal** $x(t) = u(t) \begin{rcases} = 1 \\ = 0 \end{rcases} \begin{matrix} t \geq 0 \\ t < 0 \end{matrix}$	
3.	**Sinusoidal signals** $x(t) = A \sin \omega t$ $x(t) = A \cos \omega t$	
4.	**Signum function** $\text{sgn}(t) = 1 \quad \text{for } t > 0$ $= -1 \quad \text{for } t < 0$ $= \lvert 1 \rvert \quad \text{for } t = 0$	
5.	**Rectangular pulse** $A \text{ rect}[t/T] \quad = A \quad \ldots -T/2 \leq t \leq T/2$ $= 0 \quad \ldots \text{ elsewhere}$	
6.	**Delta function** $\delta(t) = 0 \quad \text{for } t \neq 0$ Area under delta function : $\int_{-\infty}^{\infty} \delta(t)\, dt = 0$ Shifting property : $\int_{-\infty}^{\infty} x(t) \cdot \delta(t - t_d)\, dt = x(t_d)$ Replication property : $x(t) * \delta(t) = x(t)$	
7.	**Decaying exponential function** $x(t) = e^{-\alpha t}$	
8.	**Rising exponential function** $x(t) = e^{\alpha t}$	

Sr. No.	Name of signal and mathematical representation	Waveforms
9.	**sinc function :** $\text{sinc } x = \frac{\sin(\pi x)}{\pi x}$ sinc (0) = 1 and sinc (x) = 0 at $x = \pm 1, \pm 2 \ldots$	

1.7.1 Energy and Power Signals :

Power signal :

A signal is called as a power signal if its "average normalized power" is non-zero and finite. It has been observed that almost all the periodic signals are power signals.

Energy signals :

A signal having a finite non-zero total normalized energy is called as an energy signal. It has been observed that almost all the non-periodic signals defined over a finite period, are energy signals. As these signals are defined over a finite period, they are called as time limited signals.

1.7.2 Average Normalized Power :

Average normalized power,

$$P = \langle x^2(t) \rangle$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt \qquad \ldots(1.7.1)$$

The above definition can be generalized for a complex signal x(t) as,

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt \qquad \ldots(1.7.2)$$

For a periodic signal with a period T_o, the Equations (1.7.1) and (1.7.2) gtes modified to,

$$P = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} x^2(t)\, dt \qquad \ldots(1.7.3)$$

For a complex periodic signal x (t) the average normalized power is given by,

$$P = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} |x^2(t)|\, dt \qquad \ldots(1.7.4)$$

1.7.3 Energy :

The total normalized energy for a "real" signal x (t) is given by,

$$E = \int_{-\infty}^{\infty} x^2(t)\, dt \qquad \ldots(1.7.5)$$

However if the signal is complex then the expression for total normalized energy is given by,

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt \qquad \ldots(1.7.6)$$

These equations indicate that the energy is the area under the $x^2(t)$ curve over $(-\infty \leq t \leq \infty)$ hence it is always positive.

1.8 A Review of Fourier Series and Fourier Transform :

- In the field of communication engineering we need to analyze a given signal. To do so we have to express the signal in its frequency domain.
- The translation of a signal from time domain to frequency domain is obtained by using the tools such as Fourier series and Fourier transform.

1.8.1 Fourier Series :

- Sine waves and cosine waves are the basic building functions for any periodic signal.
- That means any periodic signal basically consists of sine waves having different amplitudes, of different frequencies and having different relative phase shifts.
- Fourier series represents a periodic waveform in the form of sum of infinite number of sine and cosine terms. It is a representation of the signal in a time domain series form.
- Fourier series is a "tool" used to analyze any periodic signal. After the "analysis" we obtain the following information about the signal :
  1. What all frequency components are present in the signal ?
  2. Their amplitudes and
  3. The relative phase difference between these frequency components.

All the "frequency components" are nothing else but sine waves at those frequencies.

1.8.2 Exponential Fourier Series [OR Complex Exponential Fourier Series] :

- Substituting the sine and cosine functions in terms of exponential function in the expression for the quadrature Fourier series, we can obtain another type of Fourier series called the exponential Fourier series.
- A periodic signal x(t) is expressed in the exponential Fourier series form as follows :

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j\,2\pi n\,t/T_0} \qquad ...(1.8.1)$$

$$\text{Where, } C_n = \frac{1}{T_0}\int_{t}^{t+T_0} x(t)\cdot e^{-j\,2\pi n\,t/T_0}\,dt \qquad ....(1.8.2)$$

Amplitude and Phase Spectrums :

- The amplitude spectrum of the signal x(t) is denoted by,

$$|C_n| = [(\text{Real part of } C_n)^2 + (\text{Imaginary part of } C_n)^2]^{1/2} \qquad ...(1.8.3)$$

- The phase spectrum of x(t) is denoted by,

$$\phi_n = \arg(C_n)$$

$$= \tan^{-1}\left[\frac{\text{Imaginary part of } C_n}{\text{Real part of } C_n}\right] \qquad ...(1.8.4)$$

- The amplitude spectrum is a symmetric or even function. That means $|C_n| = |C_{-n}|$. But the phase spectrum is an asymmetric or odd function. That means $\arg(C_n) = -\arg(C_{-n})$.

1.8.3 Fourier Transform :

- A Fourier transform is the limiting case of Fourier series. It is used for the analysis of non-periodic signals.
- The Fourier transform of a signal x(t) is defined as follows :

Fourier transform :

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt \qquad ...(1.8.5)$$

$$\text{OR} \quad X(f) = \int_{-\infty}^{\infty} x(t)\,e^{-j2\pi f t}\,dt$$

These equations are known as "analysis" equations.

1.8.4 Inverse Fourier Transform :

- The signal x (t) can be obtained back from its Fourier transform X(f) by using the inverse Fourier transform. The Inverse Fourier Transform (IFT) is defined as follows :

Inverse Fourier transform :

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,e^{j\omega t}\,d\omega \qquad ...(1.8.6)$$

$$\text{OR} \quad x(t) = \int_{-\infty}^{\infty} X(f)\,e^{j2\pi f t}\,df$$

Amplitude and phase spectrums :

1. The amplitude and phase spectrums are continuous and not discrete in nature. Out of them, the amplitude spectrum of a real valued function x(t) exhibits an even symmetry.

$$\therefore \quad X(f) = X(-f) \qquad ...(1.8.7)$$

2. And the phase spectrum has an odd symmetry. That means,

$$\theta(f) = -\theta(-f) \qquad ...(1.8.8)$$

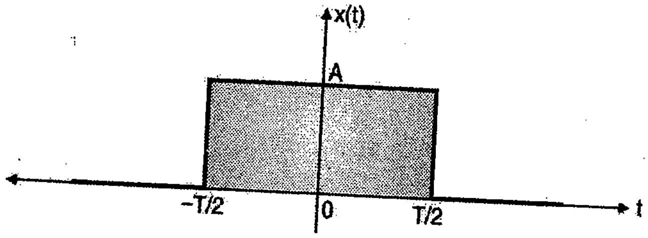
1.8.5 Properties of Fourier Transform :

Table 1.8.1 : Fourier transform properties

Sr. No.	Property	Mathematical expression
1.	Linearity or superposition	$[a_1\, x_1(t) + a_2\, x_2(t)] \leftrightarrow [a_1\, X_1(f) + a_2\, X_2(f)]$ a_1 and a_2 are constants.
2.	Time scaling	$x(\alpha t) \overset{F}{\leftrightarrow} \frac{1}{\lvert\alpha\rvert} X(f/\alpha)$. α is constant.
3.	Duality or symmetry	If $x(t) \overset{F}{\leftrightarrow} X(f)$ then $X(t) \overset{F}{\leftrightarrow} x(-f)$
4.	Time shifting	$x(t-t_d) \overset{F}{\leftrightarrow} e^{-j2\pi f t_d} X(f)$
5.	Area under x(t)	$\int_{-\infty}^{\infty} x(t)\,dt = X(0)$
6.	Area under X(f)	$\int_{-\infty}^{\infty} X(f)\,df = x(0)$
7.	Frequency shifting	$e^{j2\pi f_c t}\,x(t) \overset{F}{\leftrightarrow} X(f-f_c)$

Sr. No.	Property	Mathematical expression
8.	Differentiation in time domain	$\frac{d}{dt}x(t) \overset{F}{\leftrightarrow} j\,2\pi f\,X(f)$
9.	Integration in time domain	$\int_{-\infty}^{t} x(\lambda)\,d\lambda \overset{F}{\leftrightarrow} \frac{1}{j2\pi f}X(f)$
10.	Conjugate functions	If $x(t) \overset{F}{\leftrightarrow} X(f)$ then $x^*(t) \overset{F}{\leftrightarrow} X(-f)$
11.	Multiplication in time domain	$x_1(t)\,x_2(t) \overset{F}{\leftrightarrow} \int_{-\infty}^{\infty} X_1(\lambda)\,X_2(f-\lambda)\,d\lambda$
12.	Convolution in time domain	$x_1(t) * x_2(t) \overset{F}{\leftrightarrow} X_1(f)\,X_2(f)$

Ex. 1.8.1 : Obtain the Fourier transform of a rectangular pulse of duration T and amplitude A as shown in Fig. P. 1.8.1(a).



(E-11) **Fig. P. 1.8.1(a) : Rectangular pulse**

Soln. :

- The rectangular pulse shown in Fig. P. 1.8.1(a) can be expressed mathematically as,

$$\text{rect}(t/T) = A \quad \text{for} -T/2 \le t \le T/2$$
$$= 0 \quad \text{elsewhere}$$

This is also known as the gate function.

- Therefore the Fourier transform will be,

$$F[x(t)] = X(f)$$

$$= \int_{-\infty}^{\infty} x(t)\cdot e^{-j2\pi ft}\,dt$$

...by definition of F.T.

$$= \int_{-T/2}^{T/2} A\,e^{-j2\pi ft}\,dt$$

$$= \frac{A}{-j2\pi f}\left[e^{-j2\pi ft}\right]_{-T/2}^{T/2}$$

$$= \frac{A}{-j2\pi f}\left[e^{-j\pi fT} - e^{j\pi fT}\right]$$

$$= \frac{A}{j2\pi f}\left[e^{j\pi fT} - e^{-j\pi fT}\right]$$

$$= \frac{A}{\pi f}\left[\frac{e^{j\pi fT} - e^{-j\pi fT}}{2j}\right] \quad ...(1)$$

As per the Euler's theorem,

$$\sin\theta = \frac{(e^{j\theta} - e^{-j\theta})}{2j}$$

Applying this to Equation (1),

we get, $$F[x(t)] = \frac{A}{\pi f}[\sin(\pi f T)] \quad ...(2)$$

Multiply and divide the RHS of Equation (1) by T to get,

$$F[x(t)] = AT\frac{\sin(\pi f T)}{\pi f T} \quad ...(3)$$

In the above equation,

$$\frac{\sin(\pi fT)}{\pi fT} = \text{sinc}(fT) \quad ...(4)$$

$$\therefore \quad F[x(t)] = AT\,\text{sinc}(fT)$$

$$\therefore \quad A\,\text{rect}(t/T) \overset{F}{\leftrightarrow} AT\,\text{sinc}(fT)$$

Thus the rectangular pulse transforms into a sinc function.

Amplitude spectrum :

The amplitude spectrum of the rectangular function is as shown in Fig. P. 1.8.1(b).

$$\text{As,} \quad \text{sinc}(0) = 1$$

$$\therefore \quad AT\cdot\text{sinc}(0) = AT$$

The sinc function will have zero value for the following values of "f T":

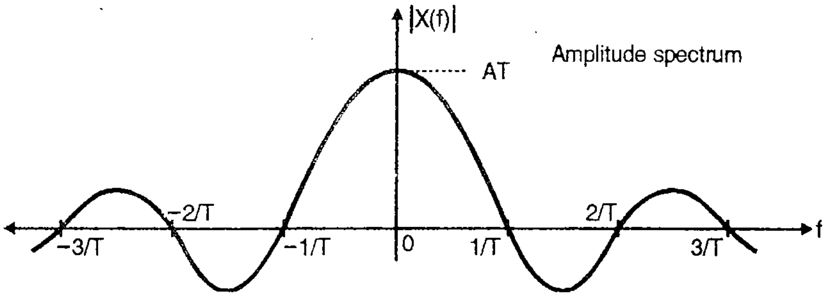
$$\text{sinc}(fT) = 0 \quad \text{for } fT = \pm 1, \pm 2, \pm 3, ...$$

$$\text{i.e.} \quad \text{for } f = \pm\frac{1}{T}, \pm\frac{2}{T}, \pm\frac{3}{T} ...$$

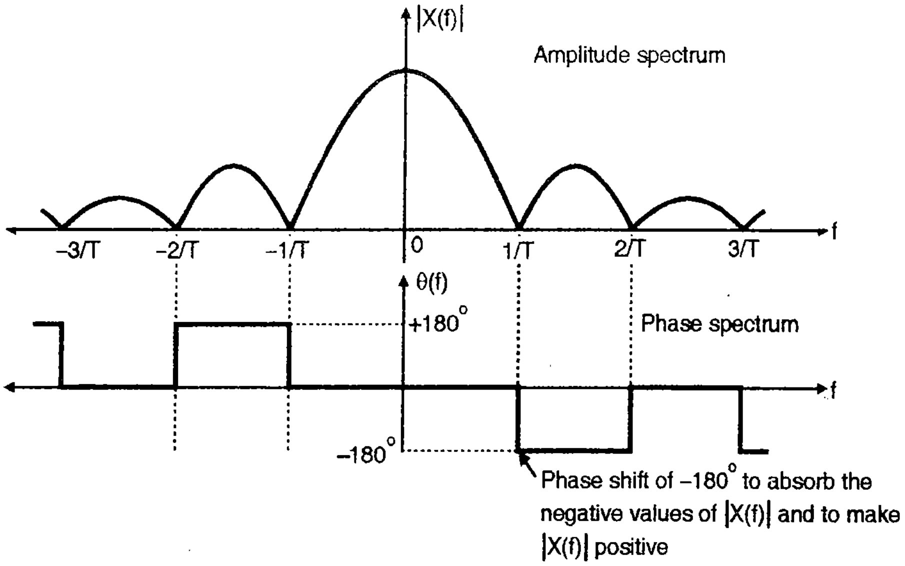
The phase spectrum has not been shown as it has zero value for all the values of f.

To absorb negative values of |X(f)| in the phase shift :

The negative amplitude of the amplitude spectrum |X(f)| has been absorbed by introducing a ± 180° phase shift as shown in Fig. P. 1.8.1(c).

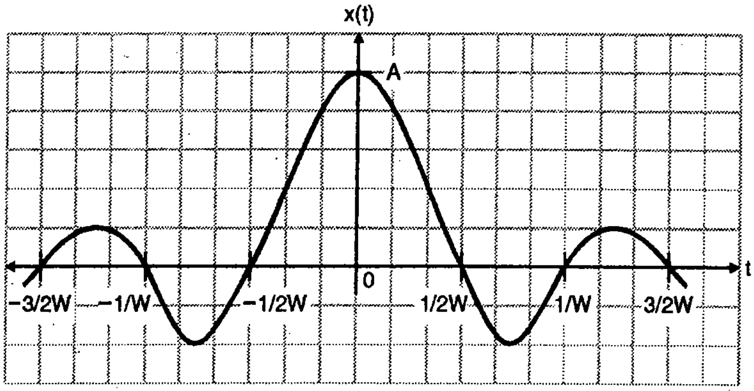


(E-12) **Fig. P. 1.8.1(b) : Amplitude spectrum of a rectangular pulse**



(E-13) **Fig. P. 1.8.1(c) : Amplitude and phase spectrums for a rectangular pulse. Negative values of |X(f)| have been absorbed in the additional phase shift of ± 180° in the phase spectrum**

Ex. 1.8.2 : For the sinc function shown in Fig. P. 1.8.2(a), obtain the Fourier transform and plot the spectrum.



(E-14) **Fig. P. 1.8.2(a) : A sinc pulse**

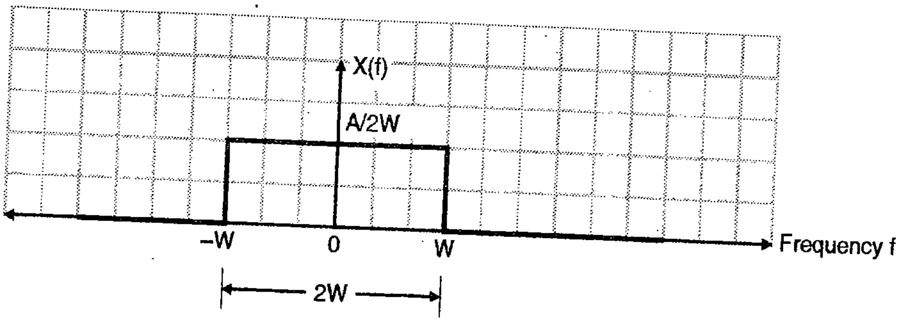
Soln. :

1. The sinc signal shown in Fig. P. 1.8.2(a) can be expressed mathematically as,

$$x(t) = A \operatorname{sinc}(2\,Wt) \qquad \ldots(1)$$

2. To evaluate the Fourier transform of this function, we are going to apply the duality and time scaling properties of the Fourier transform. Refer to example 1.8.1 where we have obtained the Fourier transform of a rectangular pulse of amplitude A and duration T, as,

$$A \operatorname{rect}[t/T] \overset{F}{\leftrightarrow} AT \operatorname{sinc}(fT) \qquad \ldots(2)$$



(E-15) **Fig. P. 1.8.2(b) : Spectrum of a sinc pulse**

3. Using the duality property we can write that,

$$AT\ \text{sinc}\ (tT) \overset{F}{\leftrightarrow} A\ \text{rect}\ [f / T] \quad ...(3)$$

Compare the LHS of Equation (3) with the RHS of Equation (1) which states the expression for x(t), we can write that,

$$2A\ W\ \text{sinc}\ (2\ Wt) \overset{F}{\leftrightarrow} A\ \text{rect}\ [f / 2\ W]$$

$$\therefore \quad A\ \text{sinc}\ (2\ Wt) \overset{F}{\leftrightarrow} \frac{A}{2W}\ \text{rect}\ [f/2W] \quad ...\textbf{Ans.}$$

Thus a sinc pulse in the time domain is transformed into a rectangular pulse in the frequency domain. The spectrum of sinc pulse is shown in Fig. P. 1.8.2(b).

1.8.6 Fourier Transform for the Periodic Signals :

- Sometimes it is essential to obtain the FT of periodic signals. For example, sampling theorem as we have already discussed.
- FT of a periodic signal $x_p(t)$ is given by,

$$F\ [x_p\ (t)] = \sum_{n=-\infty}^{\infty} C_n\ \delta\ (f - n\ f_o) \quad ...(1.8.9)$$

Where C_n is the Fourier coefficient given by,

$$C_n = \frac{1}{T_o} \int_{t}^{t+T_o} x_p(t) \cdot e^{-j2\pi nf_o t}\ dt \quad ...(1.8.10)$$

where T_o = 1-cycle period of the signal $x_p(t)$.

- The FT of $x_p(t)$ defined in Equation (1.8.9), enables us to obtain the spectrum of $x_p(t)$ in the frequency domain.

1.8.7 Fourier Transforms of Standard Signals :

- The F.T. of some standard signals is given in the following table.

Table 1.8.2 : F.T. of standard signals

Sr. No.	Signal	Mathematical representation	Fourier transform
1.	Rectangular pulse of amplitude A and duration T.	$x(t) = A\ \text{rect}\left(\frac{t}{T}\right)$	$X(f) = AT\ \text{sinc}\ (fT)$
2.	Sinc pulse	$x(t) = A\ \text{sinc}\ (2\ Wt)$	$X(f) = \frac{A}{2\ W}\ \text{rect}\left(\frac{f}{2\ W}\right)$
3.	Decaying exponential signal for t > 0.	$x(t) = e^{-\alpha t}\ u(t) \cdot a > 0$	$\frac{1}{\alpha + j\ 2\pi f}$
4.	Rising exponential pulse for t < 0.	$x(t) = e^{\alpha t}\ u(-t)$	
5.	Double exponential pulse	$x(t) = e^{-\alpha \lvert t \rvert}, a > 0$	$X(f) = \frac{2\alpha}{\alpha^2 + (2\ \pi f)^2}$
6.	Unit impulse	$\delta(t)$	$X(f) = 1$
7.	DC signal	$x(t) = 1$	$X(f) = \delta(f)$

Sr. No.	Signal	Mathematical representation	Fourier transform
8.	Cosine signal	$x(t) = \cos(2\pi f_o t)$	$\frac{1}{2}[\delta(f - f_o) + \delta(f + f_o)]$
9.	Sine signal	$x(t) = \sin(2\pi f_o t)$	$\frac{1}{2j}[\delta(f - f_o) + \delta(f + f_o)]$
10.	Signum function	$x(t) = \text{sgn}(t)$	$X(f) = \frac{1}{j\pi f}$
11.	Unit step	$x(t) = u(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$

1.9 Power and Energy Theorems :

In this section, we are going to learn two important theorems regarding the energy and power of signals. The two theorems are :

1. Parseval's power theorem and
2. Rayleigh's energy theorem.

1.9.1 Parseval's Power Theorem :

This theorem states the relation between the average power P of a periodic signal and its "Fourier series" coefficients. The Parseval's power theorem states that the total average power of a periodic signal

x(t) is equal to the sum of the average powers of the individual Fourier coefficients i.e. C_n.

$$\therefore \quad \text{Average power of } x(t) = (\text{Power of } C_1) + (\text{Power of } C_2) + \ldots \qquad \ldots(1.9.1)$$

$$\text{or total average power : } P = \sum_{n=-\infty}^{\infty} |C_n|^2 \qquad \ldots(1.9.2)$$

1.9.2 Rayleigh's Energy Theorem :

The Rayleigh's energy theorem is analogous to the Parseval's power theorem. It states that the total energy of the signal x(t) is equal to the sum of energies of its individual spectral components in the frequency domain. The total normalized energy of a signal x(t) is given by,

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

According to the Rayleigh's energy theorem,

Total normalized energy :

$$E = \int_{-\infty}^{\infty} |X(f)|^2 \, df \qquad \ldots(1.9.3)$$

Thus the total normalized energy is equal to the area under the signal corresponding to the square of the amplitude spectrum | X (f) | of the signal.

1.10 Spectral Density Functions :

The spectral density of a signal, is used for defining the distribution of energy or power per unit bandwidth as a function of frequency.

The spectral density of energy signals is called as "Energy Spectral Density (ESD)" while that of the power signals is called as "Power Spectral Density (PSD)".

1.10.1 Energy Spectral Density (ESD) :

According to Rayleigh's energy theorem, the total energy of a signal is given by,

$$E = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

The Rayleigh energy theorem not just provides a useful method of evaluating the total energy but it also tells us that $|X(f)|^2$ can provide us the "distribution of energy" of signal x(t) in the frequency domain.

Definition of ESD :

Therefore the squared amplitude spectrum $|X(f)|^2$ is called as the "Energy Spectral Density (ESD)" or "Energy Density Spectrum".

$$\therefore \quad \text{ESD} = \psi(f) = |X(f)|^2 \qquad \ldots(1.10.1)$$

The most important property of ESD is that the area under the ESD curve represents the energy of a signal.

$$\text{i.e.} \int_{-\infty}^{\infty} \psi(f) \, df = E$$

1.10.2 Power Spectral Density (PSD) :

- We now define the power spectral density of a periodic signal x(t) as follows :

Power spectral density :

$$S(f) = \frac{1}{T_o^2} \sum_{n=-\infty}^{\infty} |X(nf_o)|^2 \delta(f - nf_o) \qquad \ldots(1.10.2)$$

- Equation (1.10.2) shows that power spectral density is a discrete function of frequency. This is indicated by the term $\delta\ (f - nf_0)$ in Equation (1.10.2). The power spectral density function is present only at the harmonic frequencies of f_0.
- The most important property of PSD is that the area under PSD is equal to the average power of the signal x(t).

$$\therefore \quad P = \int_{-\infty}^{\infty} S(f)\, df \qquad \ldots(1.10.3)$$

1.11 Correlation of Energy Signals :

The correlation function is used as a measure of similarity between two signals. Higher the value of correlation function is, more is the degree of similarity. The correlation function is of two types :

1. Auto-correlation function
2. Cross-correlation function

1.11.1 Auto-Correlation Function for the Energy Signals :

The auto-correlation function for an energy signal x(t) provides the measure of similarity between the signal x(t) and its time delayed version $x(t - \tau)$. The auto-correlation function for the real valued energy signal is given by,

$$R(\tau) = \int_{-\infty}^{\infty} x(t) \cdot x(t-\tau)\, dt \qquad \ldots(1.11.1)$$

$$\text{OR} \quad R(\tau) = \int_{-\infty}^{\infty} x(t+\tau) \cdot x(t)\, dt \qquad \ldots(1.11.2)$$

If the signal x(t) is complex valued then the auto-correlation function for it is defined as follows :

$$R(\tau) = \int_{-\infty}^{\infty} x(t) \cdot x^*(t-\tau)\, dt \qquad \ldots(1.11.3)$$

Relation between ESD and autocorrelation :

The autocorrelation and energy spectral density form a Fourier transform pair. That means,

$$R(\tau) \overset{F}{\leftrightarrow} \psi(f)$$

1.11.2 Cross-Correlation of Energy Signals :

The "cross-correlation function" can be used to obtain the measure of similarity between a signal and the time delayed version of a second signal. The cross correlation between two real valued energy signals $x_1(t)$ and $x_2(t)$ is given by,

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t) \cdot x_2(t-\tau)\, dt \qquad \ldots(1.11.4)$$

We can define the second cross-correlation function of the energy signals $x_1(t)$ and $x_2(t)$ as follows :

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t) \cdot x_1(t-\tau)\, dt \qquad \ldots(1.11.5)$$

If the signals $x_1(t)$ and $x_2(t)$ are complex valued signals of finite energy then the cross-correlation is defined as :

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t) \cdot x_2^*(t-\tau)\, dt \qquad \ldots(1.11.6)$$

1.11.3 Auto-Correlation Function of Power Signals :

The auto-correlation function of power signals is a measure of similarity between a periodic signal x(t) and its delayed version $x(t - \tau)$. If the signal x(t) is periodic with a period T_0, then the auto-correlation function over one complete period T_0 is defined as follows :

Auto-correlation function of a $R(\tau)$ periodic signals

$$= \frac{1}{T_0} \int_{-T_0/2}^{T/2} x(t) \cdot x(t-\tau)\, dt$$

...For x(t) to be real valued. ...(1.11.7)

Note that this expression is same as that for the auto-correlation function of energy signals, except for the inclusion of $1/T_0$ and change in the limits of integration.

For any period T it is defined as,

$$R(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t) \cdot x(t-\tau)\, dt$$

For x(t) to be real valued.

For any period T with x(t) complex,

$$R(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t) \cdot x^*(t-\tau)\, dt$$

For any period T with x(t) complex and τ in negative direction,

$$R(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t+\tau) \cdot x^*(t)\, dt \quad \text{...(1.11.8)}$$

Both $x(t)$ and $x^*(t-\tau)$ are advanced by τ to get $x(t+\tau)$ and $x^*(t)$ respectively.

Relation between correlation and PSD :

The autocorrelation of a power signal x(t) and its power spectral density form a Fourier transform pair. That means,

$$R(\tau) \overset{F}{\leftrightarrow} S(f)$$

1.11.4 Cross-Correlation of Power Signals :

Cross-correlation is defined for two different power signals $x_1(t)$ and $x_2(t)$. If $x_1(t)$ and $x_2(t)$ represent two different power signals then the cross-correlation between them is defined as follows :

$$R_{12}(\tau) = \lim_{T \to \infty} \int_{-T/2}^{+T/2} x_1(t)\, x_2^*(t-\tau)\, dt \quad \text{...(1.11.9)}$$

Similarly we can define a second cross-correlation function as,

$$R_{21}(\tau) = \lim_{T \to \infty} \int_{-T/2}^{+T/2} x_2(t)\, x_1^*(t-\tau)\, dt \quad \text{...(1.11.10)}$$

If the two periodic signals $x_1(t)$ and $x_2(t)$ have the same time period T_0, then the cross-correlation is defined as,

$$R_{12}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_1(t) \cdot x_2^*(t-\tau)\, dt \quad \text{...(1.11.11)}$$

and the second cross-correlation for the two periodic signals is defined as,

$$R_{21}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_2(t)\, x_1^*(t-\tau)\, dt \quad \text{...(1.11.12)}$$

Note that the equations for the auto-correlation function and cross-correlation function are identical except for the fact that it has been defined for two different power signals $x_1(t)$ and $x_2(t)$ whereas the auto-correlation is defined for the same signal x(t).

1.12 Random Variables :

Definition of a random variable (R.V.) :

- The term "Random Variable" is used to signify a rule by which a real number is assigned to each possible outcome of an experiment.
- Random variables can be of two types namely, the discrete random variables and continuous random variables.

Discrete random variables :

- As we have already defined, a random variable associated with an experiment is a rule or relationship X(s) that assigns a real number X to every sample point "s".
- If the sample space "S" contains a countable (finite) number of sample points then X(s) will be a discrete random variable. A discrete random variable will thus have a countable number of distinct values.

Continuous random variable :

- A continuous random variable is not restricted to a finite number of distinct values. Instead it can have any value within a certain range.
- Thus the continuous random variable has an "uncountable" number of possible values.
- Such a random variable is defined for systems which generate an infinite number of outputs (outcomes) within a finite period of time.

1.12.1 Cumulative Distribution Function (CDF) :

- The cumulative distribution function (CDF) of a random variable is defined as the probability that the random variable X takes values less than or equal to x.

i.e. CDF : $F_X(x) = P(X \leq x)$ ...(1.12.1)

1.12.2 Probability Density Function (PDF) :

SPPU : May 08

University Questions

Q. 1 Define the term related to random process : Probability Density Function. **(May 08, 2 Marks)**

The Cumulative Distribution Function (CDF) can give useful information about discrete as well as continuous random variables. However, the Probability Density Function (PDF) is a more convenient way of describing a continuous random variable. The probability density function $f_X(x)$ is defined as the derivative of the cumulative distribution function. Thus,

$$\text{PDF :} \quad f_X(x) = \frac{d}{dx} F_X(x) \qquad \text{...(1.12.2)}$$

Some of the important properties of CDF are as follows :

Table 1.12.1 : Definition and properties of CDF

Sr. No.	Property	Remarks
1.	CDF : $F_X(x) = P(X \leq x)$	Definition of CDF.
2.	$0 \leq F_X(x) \leq 1$	CDF is always bounded between 0 and 1.
3.	$F_X(\infty) = 1$	As $X \leq \infty$ is a "certain event".
4.	$F_X(-\infty) = 0$	As $X \leq -\infty$ is a "null event".
5.	$F_X(x_1) \leq F_X(x_2)$ for $x_1 < x_2$	CDF is a monotone non-decreasing function.
6.	$F_X(x) = 0$ for $x \leq x_1$ $= \sum_{i=1}^{k} P(X = x_i)$ for $x_1 \leq x \leq x_k$ $= 1$ for $x > x_k$	CDF for discrete random variable.

Table 1.12.2 : Definition and properties of PDF

Sr. No.	Description	Remarks
1.	PDF : $f_X(x) = \frac{d}{dx} F_X(x)$	Definition of PDF
2.	$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx$	CDF is obtained by integrating PDF.
3.	$f_X(x) \geq 0$ ...for all x.	PDF is a non-negative function.
4.	$\int_{-\infty}^{\infty} f_X(x) = 1$	Area under PDF curve is 1.
5.	$P(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f_X(x)\, dx$	Probability of obtaining X between x_1 and x_2 is equal to the area under the PDF curve between the values x_1 and x_2.

1.13 Statistical Averages :

- For many of the applications, the PDF provides more information about a random variable than needed.
- Therefore it is always simpler and more convenient to describe a random variable by few characteristic numbers. These numbers are the various statistical averages or mean values.

Mean value / Average value / Expected value :

$$\therefore \text{ Mean value of } X = m_x = \overline{X} = E[X] = \sum_{i=1}^{k} x_i P(x_i) \qquad ...(1.13.1)$$

The mean value is denoted by m_x. It is also called as the expected value of X i.e. E[X]. One more way to denote the mean value is $\overline{X}$.

Mean value of a continuous random variable :

The mean value for a continuous random variable is defined as,

$$\therefore \quad m_x = \int_{-\infty}^{\infty} x f_X(x)\, dx \qquad ...(1.13.2)$$

This is the expression for the mean or average value of a continuous random variable.

Second moment of X (Mean square value) :

The second moment is also called as the mean squared value of the random variable X and is given by,

$$E[X^2] = \overline{X^2} = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx \qquad ...(1.13.3)$$

Variance of a random variable (σ_x^2) :

$$\therefore \quad \sigma_x^2 = E[X^2] - m_x^2 \qquad ...(1.13.4)$$

Thus

variance = Mean square value – Square of the mean.

Standard deviation (σ_x) :

Standard deviation (σ_x) of a random variable is defined as the square root of the variance σ_x^2.

$$\therefore \quad \text{Standard deviation} = \sqrt{\text{Variance}}$$

$$\therefore \quad \sigma_x = \sqrt{E[X^2] - m_x^2} \qquad ...(1.13.5)$$

The standard deviation σ_x, of a random variable X is the measure of the width of its PDF. The larger the value of σ_x the wider is the PDF.

1.14 Probability Models :

Need to use standard probability models :

- In the communication field we come across various types of random variables.
- They will have their own probability density functions which are different from each other.
- Practically it is very difficult to study all these probability distribution functions.
- Therefore the PDFs of all the random variables are approximated to some standard probability density functions.
- Such commonly used standard PDFs are as follows :

1.14.1 Gaussian Distribution (or Normal Distribution) :

- The Gaussian distribution is used for continuous random variables. It is perhaps the most important PDF in the area of communication.
- The majority of noise processes observed in practice are Gaussian and many naturally occurring experiments are characterized by continuous random variables with Gaussian PDF.
- The PDF of a continuous random variable having Gaussian distribution is given by :

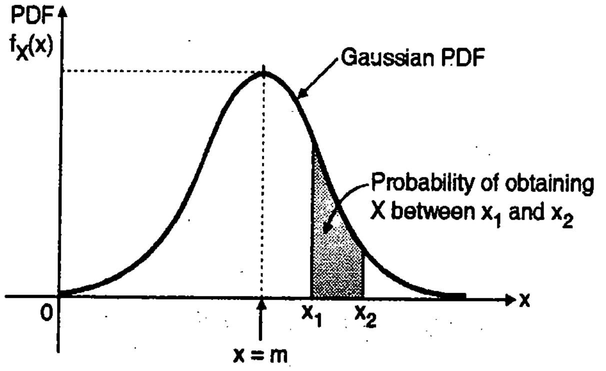
Gaussian PDF :

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2} \quad -\infty < x < \infty \qquad ...(1.14.1)$$

Where, m = Mean of the random variable

σ^2 = Variance of the random variable.

Gaussian PDF is also called as "Normal PDF". The shape of Gaussian PDF is bell type shown in Fig. 1.14.1.



(E-74) **Fig. 1.14.1 : The Gaussian PDF**

The important points about the Gaussian PDF are as follows :

1. The Gaussian PDF is a bell shaped function with a peak at x = m i.e. corresponding to the mean value of the random variable X.
2. The Gaussian PDF has an even symmetry about the peak.

$$\therefore \quad f_X(x = m - \sigma) = f_X(x = m + \sigma) \qquad ...(1.14.2)$$

3. Probability of obtaining "X" above and below the mean value is equal i.e. 1/2.

$$\therefore \quad P(X \leq m) = P(X > m) = 1/2 \quad ...(1.14.3)$$

4. Area under the Gaussian PDF is 1.

$$\therefore \quad \int_{-\infty}^{\infty} f_X(x)\,dx = 1 \quad ...(1.14.4)$$

5. Probability of observing "X" between x_1 and x_2 can be obtained by integrating the Gaussian PDF between the limits x_1 to x_2. This is shown by the shaded area in Fig. 1.14.1.

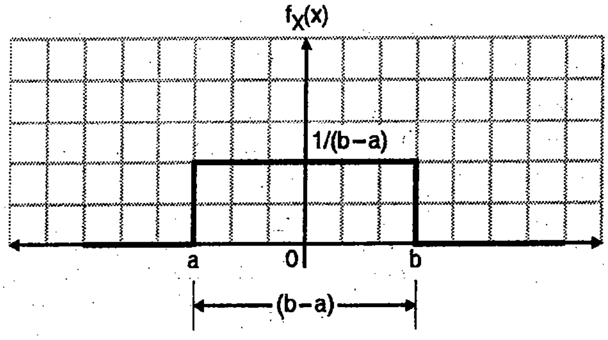
$$\therefore \quad P(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f_X(x)\,dx \quad ...(1.14.5)$$

$$= \int_{x_1}^{x_2} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(x-m)^2/2\sigma^2}\,dx \quad ...(1.14.6)$$

$$\therefore \quad F_X(x) = \frac{1}{2}\,\text{erfc}\left[\frac{m-x}{\sigma\sqrt{2}}\right] \quad ...(1.14.7)$$

1.14.2 Uniform Distribution :

- If a continuous random variable X is equally likely to be observed in a finite range and is likely to have a zero value outside this finite range then the random variable is said to have a uniform distribution.
- The PDF of uniform distribution is as shown in Fig. 1.14.2.



(E-75) **Fig. 1.14.2 : Uniform distribution**

The PDF of a random variable having a uniform PDF is given by,

$$\text{Uniform PDF,} \quad f_X(x) = 1/(b-a) \quad a \leq x \leq b$$
$$= 0 \text{ elsewhere} \quad ...(1.14.8)$$

$$\therefore \quad m_x = \frac{(a+b)}{2} \quad ...(1.14.9)$$

Variance of uniform distribution :

Variance of a random variable is defined as,

$$\sigma_x^2 = E[X^2] - m_x^2 \quad ...(1.14.10)$$

$$\therefore \quad \sigma_x^2 = \frac{(b-a)^2}{12} \quad ...(1.14.11)$$

1.14.3 Error Function :

The error function is defined as :

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\,du \quad ...(1.14.12)$$

The important properties of error function are as follows :

1. The value of error function is always between 0 and 1 inclusive of the two extreme values.

$$\therefore \quad 0 \leq \text{erf}(x) \leq 1 \quad ...(1.14.13)$$

2. The value of error function is 0 at x = 0 and 1 at x = ∞.

$$\left.\begin{array}{ll} \text{i.e.} \quad \text{erf}(x) = 0 & \text{at } x = 0 \\ \text{and} \quad \text{erf}(x) = 1 & \text{at } x = \infty \end{array}\right\} \quad ...(1.14.14)$$

3. Symmetry property : This property states that the error function is an anti symmetrical function i.e.,

$$\text{erf}(x) = -\text{erf}(-x) \quad ...(1.14.15)$$

1.14.4 Complementary Error Function :

The complementary error function is defined as,

$$\text{erfc}(x) = 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-u^2}\,du \quad ...(1.14.16)$$

The complementary error function is a monotonically decreasing function. That means with increase in the value of x, the value of erfc (x) decreases.

Review Questions

Q. 1 State various sources of information signals.

Q. 2 What is a transducer ? Why is it used ?

Q. 3 With the help of block diagram explain the process of A to D conversion.

Q. 4 Write a note on : Graphical representation of A/D conversion.

Q. 5 State advantages and disadvantages of digital representation of a signal.

Q. 6 With the help of block diagram explain the operation of digital communication system.

Q. 7 Explain how digital communication is superior to the analog communication ?

Q. 8 State advantages and disadvantages of digital communication.

Q. 9 Compare analog and digital communication.

Q. 10 Write a short note on : Channels for digital communications.

Q. 11 Compare various channels.

Q. 12 Write a short note on : Optical fiber as communication medium.

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CHAPTER 2

Digital Transmission of Analog Signals

Unit I

Syllabus :

Sampling process, PCM generation and reconstruction, Quantization noise, Non-uniform quantization and companding, PCM with noise : Decoding noise, Error threshold, Delta modulation, Adaptive delta modulation, Delta sigma modulation, Differential pulse code modulation, LPC speech synthesis.

2.1 Introduction :

- Using the waveform coding technique we convert the analog PAM signal into a digital signal.
- This digital signal is in the form of a train or stream of binary digits 0 and 1.
- Thus with waveform coding techniques we enter into the world of digital communication.
- After sampling an analog signal, the next step in its digital transmission is the generation of the "coded version" (digital representation) of the signal.
- Pulse Code Modulation (PCM) provides one method to meet such a requirement.
- In PCM, the message signal is sampled and amplitude of each sample is approximated (rounded off) to the nearest one of a finite set of discrete levels.
- This will enable us to represent both time and amplitude in discrete form.
- Hence it is possible to transmit the message signal by means of a digital (coded) waveform.
- Conceptually PCM is simple to understand. It was the first method which was developed for the digital coding of the waveforms.
- PCM is the most applied of all the digital coding systems in use today.
- PCM is therefore widely accepted as the standard against which the other digital coder systems are calibrated.

2.1.1 Advantages of Digital Representation of a Signal :

The digital representation of a signal has following advantages :

1. Immunity to transmission noise and interference.
2. Regeneration of the coded signal along the transmission path is possible.
3. Communication can be kept "private" and "secured" through the use of encryption.
4. It is possible to use a uniform format for different kinds of baseband signals.
5. It is possible to store the signal and process it further.
6. Digital signals are better suited for processing and multiplexing.
7. Digital transmission systems are more immune to noise.
8. Measurement and evaluation of digital signals is simpler.
9. It is possible to evaluate error performance of digital systems.

2.1.2 Disadvantages :

1. The required bandwidth is increased due to digital technology.
2. System complexity is increased.
3. In order to convert the analog signal to digital prior to transmission and then from digital to analog at the receiver, we need to use the additional encoders and decoder circuits.
4. Synchronization is necessary for the digital systems (between transmission and receiver clocks).
5. Digital transmission systems are not compatible to the older analog transmission systems.

Difference between waveform coding and source coding :

- The waveform coders are in principle designed to be signal independent.
- The waveform coders are different from the source coders (e.g. linear predictive coders). The source coders depend on parameterization of the analog signal in accordance with an appropriate model for the generation of the signal.

2.2 Sampling Process :

2.2.1 Introduction to Sampling :

- In Chapter 1 we have discussed that a message signal can originate from a digital or analog source.
- In the pulse modulation and digital modulation systems, the signal to be transmitted must be in the discrete time form.
- If the message signal is coming from a digital source (e.g. a digital computer) then it is in the proper form for processing by a digital communication system.
- But this is not always the case. The message signal can be analog in nature (e.g. speech or video signal).
- In such a case it has to be first converted into a discrete time signal. We use the "sampling process" to do this.
- Thus using the sampling process we convert a continuous time signal into a discrete time signal.
- For the sampling process to be of practical utility it is necessary to choose the sampling rate properly.
- The sampling process should satisfy the following requirements :
  - Sampled signal should represent the original signal faithfully.
  - We should be able to reconstruct the original signal from its sampled version.
- Fig. 2.2.1 summarizes the sampling process.

Thus sampling is the process of converting a continuous analog signal to a discrete analog signal and the sampled signal is the discrete time representation of the original analog signal.

2.2.2 Band Limited and Time Limited Signal :

The signals can be classified into two categories namely the time limited signals and bandlimited signals.

2.2.3 Time Limited Signal :

- A time limited signal is the signal which exists only over a certain time slot or duration. Outside that slot the signal does not exist.
- An example of a time limited signal is the rectangular pulse shown in Fig. 2.2.2(a). It is mathematically represented as follows :

$$x(t) = A \quad ..... -T/2 \leq t \leq T/2$$

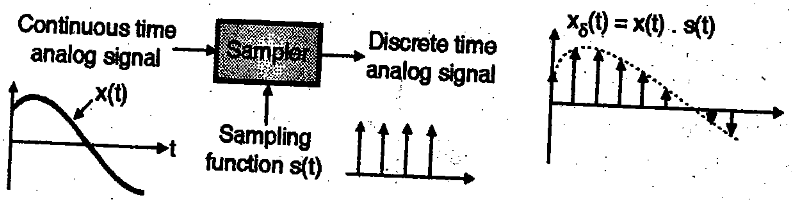
$$= 0 \quad ..... \text{elsewhere}$$

2.2.4 Bandlimited Signal :

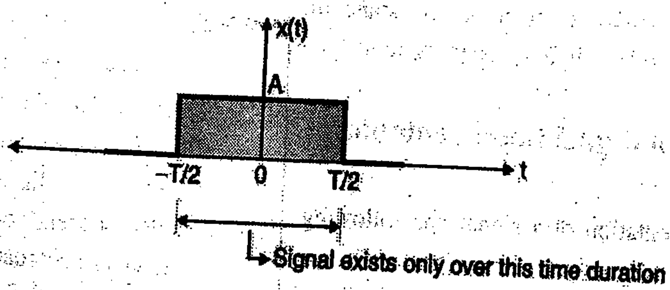
- A bandlimited signal is the signal having a frequency spectrum which exists only over a certain frequency range. The frequency spectrum of a bandlimited signal will be zero outside this frequency range.
- Fig. 2.2.2(b) shows the frequency spectrum of a bandlimited signal.
- The spectrum of Fig. 2.2.2(b) can be expressed mathematically as,

$$|x(f)| = A \quad ..... -W \leq f \leq W$$

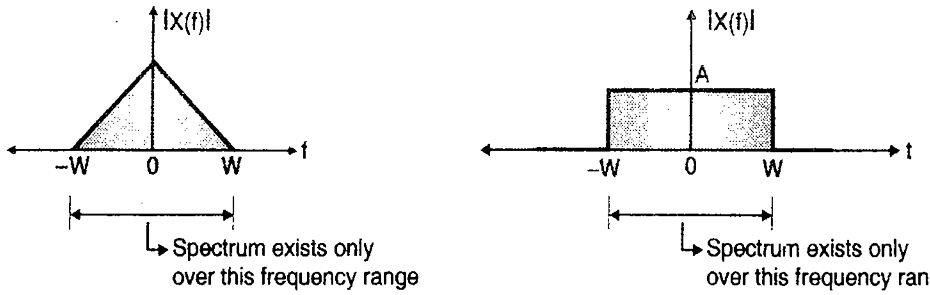
$$= 0 \quad ..... \text{elsewhere}$$



(E-591) Fig. 2.2.1 : Sampling process



(E-592) Fig. 2.2.2(a) : Time limited signal



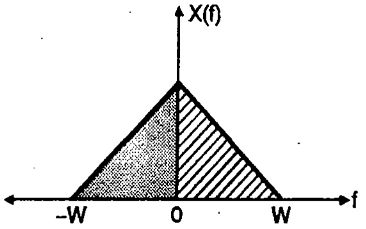
(E-593) **Fig. 2.2.2(b) : Frequency spectrums of bandlimited signals**

2.2.5 Narrow Band Signals :

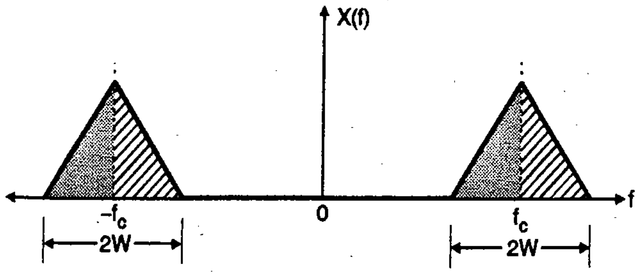
- Narrow band signals are the signals which occupy a small bandwidth. That means the range of frequencies in the frequency spectrum of the narrow band signals is small.
- We can obtain the narrow band signals by passing a wideband signal through a narrow bandpass filter.
- The voice signals occupy the slot of 0 to 3.2 kHz and they can be called as the narrowband signals.
- The video signal however, occupies 0 to 5 MHz slot. Therefore it is a wideband signal.
- Narrowband systems are the systems which are designed to process the narrowband signals.

2.2.6 Low Pass and Band Pass Signals :

- A low pass signal is defined as the signal which has a frequency spectrum extending right from 0 Hz to W Hz, if a single sided spectrum is plotted and it extends from – W to + W, if a double sided spectrum is plotted as shown in Fig. 2.2.3(a).



(a) Spectrum of a low pass signal



(b) Spectrum of a band pass signal

(D-406) **Fig. 2.2.3**

- A video signal which has a frequency spectrum from 0 to 5 MHz is an example of low pass signal.
- But the nature of the spectrum of a bandpass signal is completely different. It is as shown in Fig. 2.2.3(b). The bandwidth is 2W Hz but it is centered about a frequency ± f_c rather than zero. We assume that $f_c > W$, therefore $(f_c - W) > 0$.
- Thus bandpass signals are the signals having their spectrum extending from f_1 to f_2 with $f_2 > f_1$ and both f_1 and f_2 being of nonzero value. For example, the voice signal which has a spectrum extending from 20 Hz to 3.4 kHz is a bandpass signal.

2.3 Sampling Theorem for Low Pass Signals in Time Domain :

- In order to represent the original message signal "faithfully" (without loss of information), it is necessary to take as many samples of the original signal as possible.
- Higher the number of samples, closer is the representation.
- The number of samples depends on the "sampling rate" and the maximum frequency of the signal to be sampled.
- Sampling theorem was introduced to the communication theory in 1949 by Shannon. Therefore this theorem is also called as **"Shannon's sampling theorem"**.
- The statement of sampling theorem in time domain, for the bandlimited signals of finite energy is as follows :

Statement :

1. If a finite energy signal x (t) contains no frequencies higher than "W" Hz (i.e. it is a band limited signal) then it is completely determined by specifying its values at the instants of time which are spaced (1/2W) seconds apart.
2. If a finite energy signal x (t) contains no frequency components higher than "W" Hz then it may be completely recovered from its samples which are spaced (1/2W) seconds apart.

Combined statement of sampling theorem : A continuous time signal x (t) can be completely represented in its sampled form and recovered back from the sampled form if the sampling frequency $f_s \geq 2W$ where "W" is the maximum frequency of the continuous time signal x (t).

2.3.1 Proof of Sampling Theorem :

SPPU : May 07, Dec. 07, Dec. 08, May 09, May 15

University Questions

Q. 1 Explain sampling theorem for low pass signals in time domain. Also explain reconstruction of signal from samples. **(May 07, 10 Marks)**

Q. 2 With the help of mathematical expression and spectral diagrams explain the time domain and frequency domain approach of sampling theorem. **(Dec. 07, 10 Marks)**

Q. 3 A continuous time signal g(t) of finite energy and infinite duration which is strictly bandlimited to W Hz is ideally sampled $g_\delta(t)$ i.e.

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\,\delta(t-nT_s) \;;$$

$$\text{also,}\quad g_\delta(t) \Leftrightarrow f_s \sum_{m=-\infty}^{\infty} G(f-mf_s)$$

Determine the expression of $G_\delta(f)$ and G(f). Draw the spectrum of $G_\delta(f)$ and G(f) with and without Aliasing. Reconstruct the signal g(t) and derive interpolation formula, comment on it. **(Dec. 08, 6 Marks)**

Q. 4 Represent and discuss on the time domain and frequency domain approach of sampling theorem. **(May 09, 8 Marks)**

Q. 5 With suitable spectral diagram prove the sampling theorem and explain aliasing effect. **(May 15, 6 Marks)**

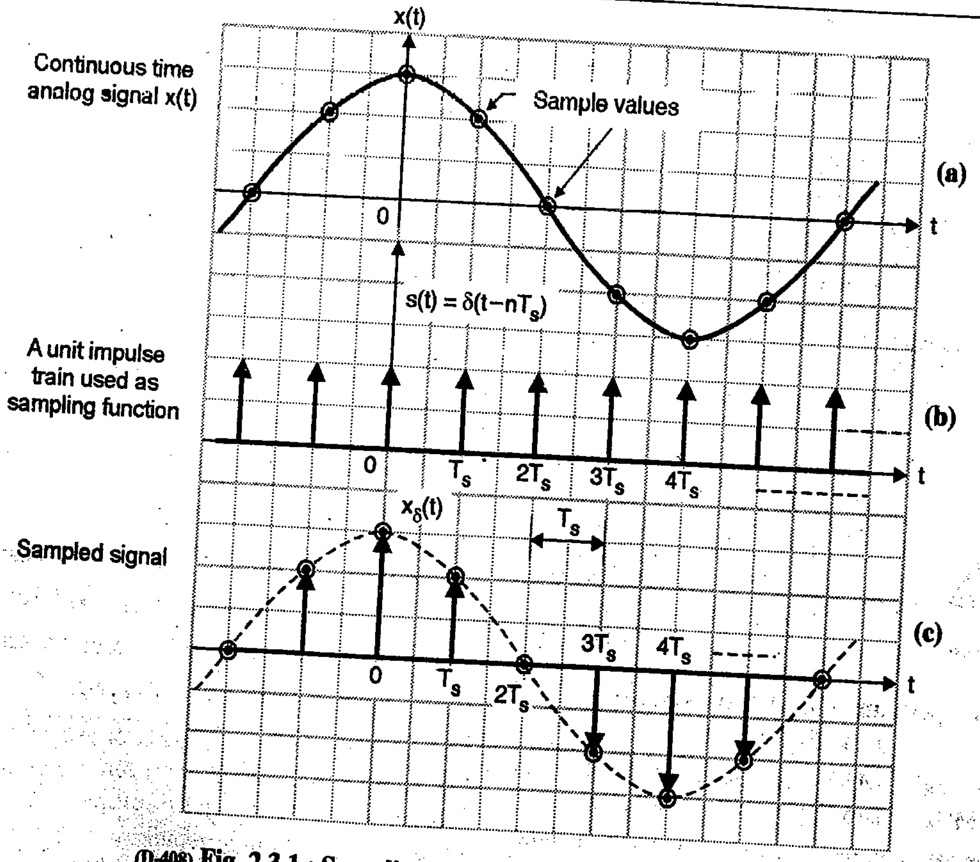
Let us now prove the sampling theorem in time domain. The assumptions made for this proof are as follows :

Assumptions :

- Let x (t) be a continuous time analog signal as shown in Fig. 2.3.1.
- Let x (t) be a signal with finite energy and infinite duration.
- Let x (t) be a strictly bandlimited signal. That means it does not contain any frequency components above "W" Hz.
- Let s (t) be the sampling function as shown in Fig. 2.3.1. It is a train of unit impulses, spaced by a period of T_s seconds. This sampling function samples the original signal at a rate of "f_s" samples per second. Therefore "T_s" represents the sampling period such that,

$$T_s = \frac{1}{f_s} = \text{Sampling period} \qquad \text{...(2.3.1)}$$

$$\text{and} \quad f_s = \frac{1}{T_s} = \text{Sampling rate.}$$



(D-408) **Fig. 2.3.1 : Sampling of a continuous time signal x (t)**

Procedure to be followed :

We are going to follow the steps given below to prove the sampling theorem :

Step 1 : Represent the sampling function s (t) mathematically.

Step 2 : Represent the sampled signal x_δ (t) mathematically.

Step 3 : Obtain the Fourier transform of the sampled signal.

Step 4 : Prove that the sampled signal x_δ (t) completely represents x (t).

Step 5 : Represent x (t) as summation of sinc functions (interpolation).

Step 6 : Graphical representation of the interpolation process.

Step 7 : Actual recovery of x (t) using an ideal low pass filter.

Part 1 : Sampling theorem :

Spectrum of the sampled signal :

Step 1 : Represent the sampling function s (t) mathematically :

- Fig. 2.3.1 shows the sampling function s (t) which is a train of unit impulses.
- The spacing between the adjacent unit impulses is T_s seconds, therefore the frequency of the sampling function is equal to the sampling frequency f_s.
- The sampled signal is denoted by x_δ (t) and it is as shown in Fig. 2.3.1.
- The sample function s (t) can be represented mathematically as follows :

$$s(t) = \ldots\ldots\ldots\delta(t+2T_s)+\delta(t+T_s)+\delta(t) + \delta(t-T_s)+\delta(t-2T_s)+\ldots\ldots$$

$$\therefore\ s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s) \qquad \ldots(2.3.2)$$

Step 2 : Represent the sampled signal x_δ (t) mathematically :

- Fig. 2.3.1 shows the sampled signal x_δ (t) graphically. It is present only at the sampling instants i.e. T_s, 2 T_s etc. and its instantaneous amplitude is equal to the amplitude of original signal x (t) at the sampling instants.
- This is shown by the encircled points in Fig. 2.3.1. Let us represent the instantaneous amplitude of x (t) at the various sampling points t = n T_s as x (n T_s). This is the amplitude of the encircled points of Fig. 2.3.1.
- Looking at the sampled signal x_δ (t) we can say that the sampled signal is obtained by multiplying x (t) and s (t).

$$\therefore\ x_\delta(t) = x(t)\times s(t) = x(nT_s)\times s(t) \qquad \ldots(2.3.3)$$

- Substituting the expression for s (t) from Equation (2.3.2) we get the mathematical expression for the sampled signal x_δ (t) as,

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\cdot\delta(t-nT_s) \qquad \ldots(2.3.4)$$

Step 3 : Obtain the Fourier transform of the sampled signal :

- The fourier transform of a train of impulses (dirac delta function) is given by,

$$X(f) = f_0 \sum_{n=-\infty}^{\infty} \delta(f-nf_0)$$

- Here we have the similar pulse train as sampling function s (t). Therefore the Fourier transform of the sampling function is given by,

$$S(f) = f_s \sum_{n=-\infty}^{\infty} \delta(f-nf_s) \qquad \ldots(2.3.5)$$

- Note that f_0 has been replaced by f_s in the above equation.
- The sampled signal in the time domain is represented as product of x (t) and s (t).

$$\text{i.e. } x_\delta(t) = x(t)\times s(t) \qquad \ldots(2.3.6)$$

- Taking the Fourier transform of both the sides we get,

$$\text{i.e. } X_\delta(f) = X(f) * S(f) \qquad \ldots(2.3.7)$$

- This is because the Fourier transform of the product of two signals in the time domain is the convolution of their Fourier transforms. Substituting the value of S(f) from Equation (2.3.5) we get,

$$X_\delta(f) = X(f) * \left[f_s \sum_{n=-\infty}^{\infty} \delta(f-nf_s) \right] \qquad \ldots(2.3.8)$$

where * denotes convolution. Interchanging the orders of convolution and summation results in :

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f) * \delta(f - nf_s) \quad \text{...(2.3.9)}$$

- From the properties of delta function, we find that the convolution of X (f) and δ (f − nf$_s$) is equal to X (f − nf$_s$). Hence the above equation can be simplified as follows :

F.T. of the sampled signal,

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \quad \text{...(2.3.10)}$$

where X (f) = Fourier transform of the original signal x (t).

Conclusion from Equation (2.3.10) :

1. The term X (f − nf$_s$) in Equation (2.3.10) represents the shifted version of the spectrum X (f) of the original signal x (t). Thus depending on the value of "n" (which extends from − ∞ to + ∞) we will get infinite number of original spectrums X (f) centered at frequencies 0, ± f$_s$, ± 2f$_s$, ± 3f$_s$, ± 4f$_s$ ...etc. In other words,

$$X(f - nf_s) = X(f) \text{ at } f = 0, \pm f_s, \pm 2f_s, \pm 3f_s \quad \text{...(2.3.11)}$$

2. This concept will be clear if we open Equation (2.3.10) and write the terms separately as shown below.

Now open the summation sign in Equation (2.3.10) to get,

(D-409)

$$X_\delta(f) = \ldots + \underbrace{f_s X(f+2f_s)}_{\text{X(f) shifted left by } 2f_s} + \underbrace{f_s X(f+f_s)}_{\text{X(f) shifted left by } f_s} + \underbrace{f_s X(f)}_{\text{Spectrum X(f)}} + \underbrace{f_s X(f-f_s)}_{\text{X(f) shifted right by } f_s} + \ldots$$

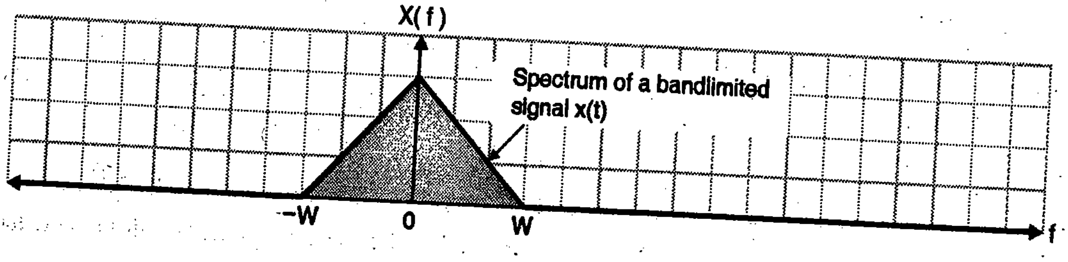
The spectrum X$_\delta$ (f) of the sampled signal is plotted as shown in Fig. 2.3.2.

Equation (2.3.10) can also be written as :

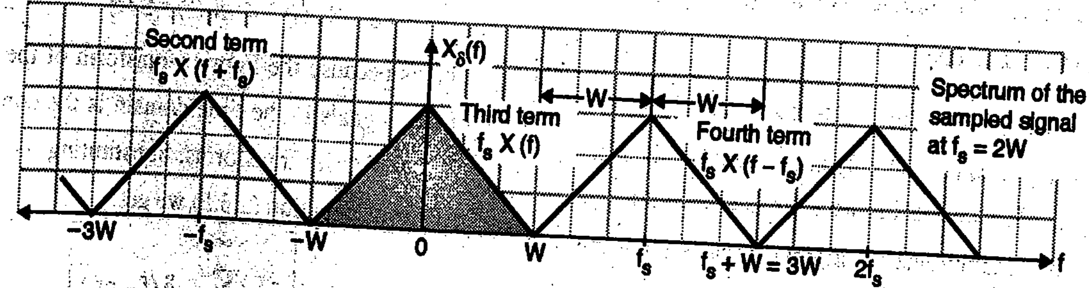
$$X_\delta(f) = f_s X(f) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} f_s X(f - nf_s) \quad \text{... (2.3.12)}$$

Comment :

From Equation (2.3.12) we conclude that the process of uniform sampling of a signal in the time domain results in a periodic spectrum in the frequency domain with a period equal to the sampling rate f$_s$.



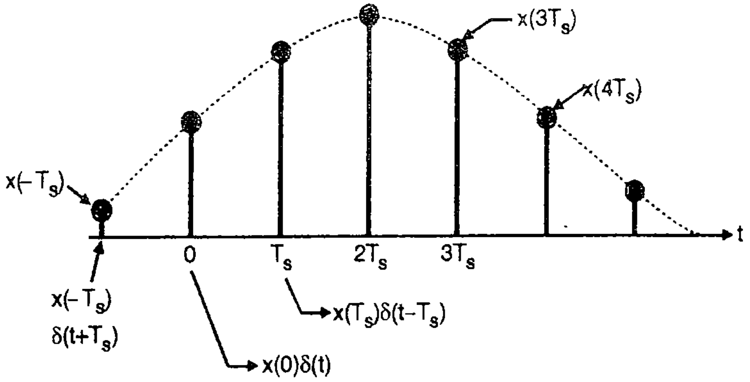
(a) Spectrum of the original signal x(t)



(b) Spectrum of the sampled signal x$_\delta$ (t) with f$_s$ = 2W, i.e. Sampling is done exactly at Nyquist rate

(D-410) Fig. 2.3.2

3. Prove that sampled signal $x_\delta(t)$ completely represents $x(t)$:



(D-411) **Fig. 2.3.3 : Sampled signal $x_\delta(t)$**

- $x_\delta(t)$ can be represented in the summation form as follows (Refer Fig. 2.3.3).

$$x_\delta(t) = \ldots x(-T_s)\,\delta(t+T_s) + x(0)\,\delta(t) + x(T_s)\,\delta(t-T_s) + \ldots \qquad \ldots(2.3.12(a))$$

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\cdot\delta(t-nT_s)$$

- We can obtain another useful expression for the fourier transform $X_\delta(f)$ by taking the fourier transform of both the sides of the equation stated above as,

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(nT_s)\,e^{-j2\pi nfT_s} \qquad \ldots(2.3.13)$$

- This equation is the fourier transform of a discrete time signal $x_\delta(t)$. Therefore it is called as the discrete fourier transform (DFT). Compare it with the definition of fourier transform of a continuous time signal. i.e.

$$X(f) = \int_{-\infty}^{\infty} x(t)\,e^{-j2\pi ft}\,dt$$

- As the signal is discrete, the integration sign has been replaced by the summation sign and "t" has been replaced by "nT_s".
- Now consider Equation (2.3.12)

$$X_\delta(f) = f_s X(f) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} f_s X(f-nf_s)$$

$$\therefore \quad X(f) = \frac{1}{f_s} X_\delta(f) - \sum X(f-nf_s)$$

- But in the range $-W \le f \le W$ the second term of the above expression will not be present

$$\therefore \quad X(f) = \frac{1}{f_s} X_\delta(f) \qquad \ldots(2.3.14)$$

- Substitute $f_s = 2W$ and $X_\delta(f)$ from Equation (2.3.13) to get,

$$X(f) = \frac{1}{2W} \sum_{n=-\infty}^{\infty} x(nT_s)\cdot e^{-j2\pi nfT_s} \qquad \ldots(2.3.15)$$

- This is the frequency spectrum of $x(t)$ in terms of $x(nT_s)$ i.e. the sampled signal.

  Substitute $T_s = 1/2W$ to get,

$$X(f) = \frac{1}{2W} \sum_{n=-\infty}^{\infty} x(n/2W)\cdot e^{-j2\pi nf/2W}$$

$$\ldots -W \le f \le W \qquad \ldots(2.3.16)$$

- This equation shows that the spectrum of $x(t)$ is same as the spectrum of $x_\delta(t)$ in the frequency range $-W$ to $+W$. Hence the sampled signal represents the original signal $x(t)$ successfully.

Thus if the sample values $x(n/2W)$ of the signal $x(t)$ are specified for all time, then the Fourier transform $X(f)$ of the original signal is uniquely determined by using the Equation (2.3.16). Because $x(t)$ is related to $X(f)$ by the inverse Fourier transform, it follows that the signal $x(t)$ is itself uniquely determined by the sample values $x(n/2W)$ for $-\infty \le n \le \infty$. Or in other words the sequence of samples $\{x(n/2W)\}$ contains all the information of $x(t)$.

- Thus we have proved first part of the sampling theorem.

Part 2 of the sampling theorem :

4. **Reconstruction of signal from samples :**

- This is the second part of the sampling theorem. From Equation (2.3.16) we can obtain x (t) by taking the inverse fourier transform (IFT).

$$x(t) = \text{IFT}\{X(f)\}$$

$$= \text{IFT}\left\{\frac{1}{2W}\sum_{n=-\infty}^{\infty} x(n/2W).\, e^{-j\pi fn/W}\right\}$$

- Using the definition of inverse Fourier transform,

$$x(t) = \int_{-W}^{W} \frac{1}{2W}\sum_{n=-\infty}^{\infty} x(n/2W)\cdot e^{-j\pi f n/W} e^{j2\pi ft}\, df$$

- Interchanging the order of summation and integration we get,

$$x(t) = \sum_{n=-\infty}^{\infty} x(n/2W)\frac{1}{2W}\int_{-W}^{W} e^{j2\pi f\left(t-\frac{n}{2W}\right)}\, df$$

$$x(t) = \sum_{n=-\infty}^{\infty} x(n/2W)\cdot\frac{1}{2W}\times\frac{1}{j2\pi\left[t-\frac{n}{2W}\right]}\cdot\left[e^{j2\pi f(t-n/2W)}\right]_{-W}^{W}$$

$$x(t) = \sum_{n=-\infty}^{\infty} x(n/2W)\frac{1}{j4\pi W\left[t-\frac{n}{2W}\right]}\cdot\left[e^{j2\pi W(t-n/2W)} - e^{-j2\pi W(t-n/2W)}\right]$$

$$= \sum_{n=-\infty}^{\infty} x(n/2W)\cdot\left[\frac{e^{j2\pi W(t-n/2W)} - e^{-j2\pi W(t-n/2W)}}{j4\pi W\left[t-\frac{n}{2W}\right]}\right]$$

- The term inside the square bracket is a "sine" function.

$$\therefore\ x(t) = \sum_{n=-\infty}^{\infty} x(n/2W)\frac{\sin(2\pi Wt-n\pi)}{(2\pi Wt-n\pi)} \quad ...(2.3.17)$$

- We can simplify the equation above by using the definition of the "sinc function". The sinc function is defined as :

$$\text{sinc}\, x = \frac{\sin(\pi x)}{\pi x} \quad ...(2.3.18)$$

- Therefore Equation (2.3.17) can be written as :

$$x(t) = \sum_{n=-\infty}^{\infty} x(n/2W)\,\text{sinc}\,(2\,Wt-n) \quad ...(2.3.19)$$

- Equation (2.3.19) provides an interpolation formula for reconstructing the original signal x (t) from the sequence of sample values {x (n /2W)}. The "sinc" function plays the role of an interpolation function. Each sample x (n / 2W) is multiplied by a delayed version of the interpolation function i.e. sinc function. Then all these resulting waveforms are added to obtain x (t).

5. **Graphical representation of the interpolation process :**

Let us re-arrange Equation (2.3.19) as follows :

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\text{sinc}\,2W\left(t-\frac{n}{2W}\right)$$

This is because $\frac{1}{2W} = T_s$

$$\therefore \quad x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\text{sinc}\,2W(t-nT_s) \qquad ...(2.3.20)$$

Let us expand this equation to write,

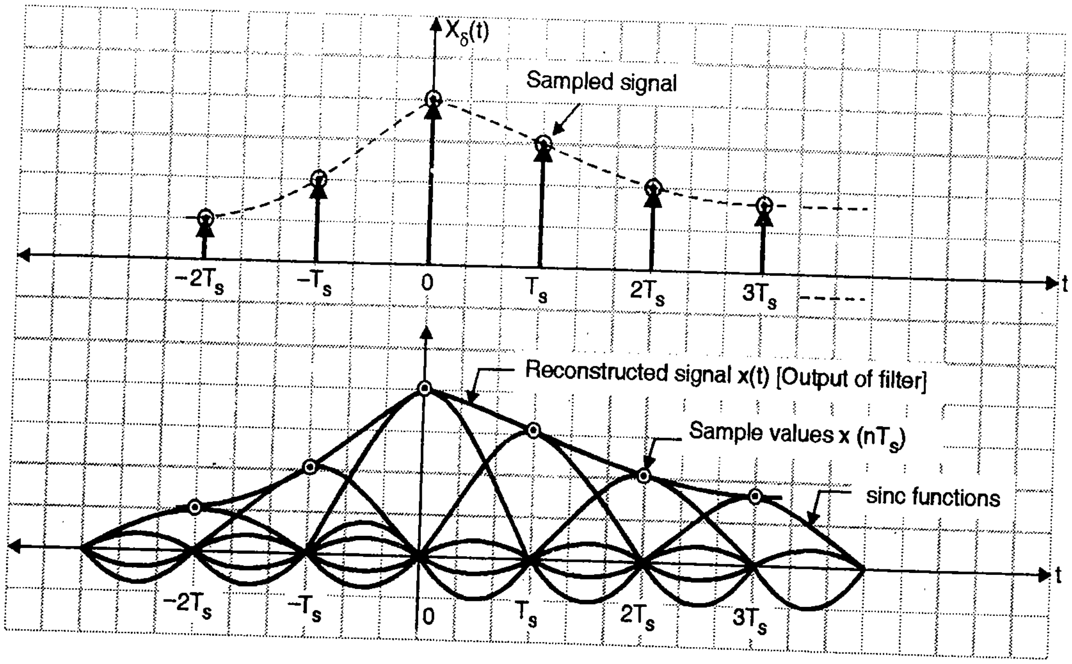
$$x(t) = x(0)\,\text{sinc}\,2Wt + x(\pm T_s)\,\text{sinc}\,2W(t \pm T_s) + x(\pm 2T_s)\,\text{sinc}\,2W(t \pm 2T_s) + \ldots \qquad .....(2.3.21)$$

(a) First term : x (0) sinc 2Wt :

- This will have a maximum amplitude at t = 0. The maximum amplitude is equal to the sample value x (0) at t = 0. This sinc function will pass through zeros at t = ± 1/2 W, ± 1/4 W ...etc. This is as shown in Fig. 2.3.4.

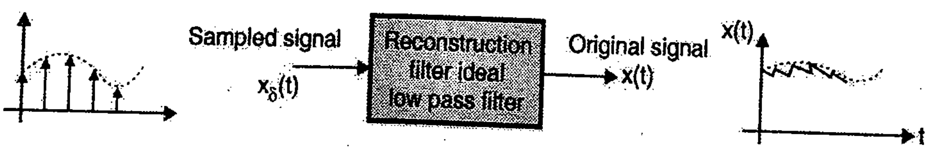
(b) Second term : $x(\pm T_s)$ sinc 2W $(t \pm T_s)$:

- This sinc function will have maximum amplitude at $t = \pm T_s$. The maximum amplitude is equal to the sample value $x(\pm T_s)$ at $t = \pm T_s$ respectively. Thus sinc 2W $(t \pm T_s)$ represents shifted sinc function i.e. "sinc 2Wt" by a period $\pm T_s$. This is as shown in Fig. 2.3.4.
- Similarly the third term, $x(\pm 2T_s)$ sinc 2W $(t \pm 2T_s)$ represents shifted sinc function "sinc 2Wt" by a period of $\pm 2T_s$ and so on. We can plot all these sinc functions along with the sampled signal $x_\delta(t)$ as shown in Fig. 2.3.4. Note that the peak amplitude of any sinc function is equal to the corresponding sample value $x(nT_s)$.

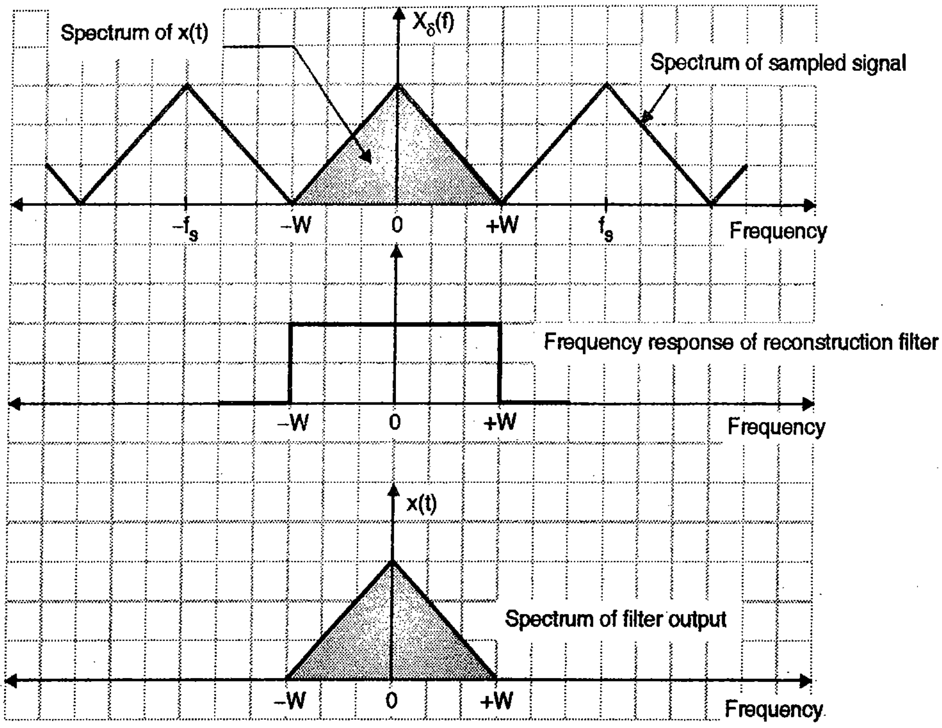


(D-412) **Fig. 2.3.4 : Reconstruction of the original signal x (t) from its samples using the interpolation**

6. Actual reconstruction of the original signal by using a low pass filter :



(D-413) **Fig. 2.3.5(a) : Reconstruction filter**



(D-414) **Fig. 2.3.5(b) : Operation of reconstruction**

- Thus the peaks of the sinc pulses represent the amplitudes of the samples.
- The signal x (t) expressed in Equation (2.3.19) is then passed through an ideal low pass filter to recover the original signal x (t). This low pass filter is therefore called as the reconstruction filter. This is shown in the graphical representation of Fig. 2.3.5(a).
- Assume that the cut-off frequency of the ideal low pass filter is adjusted precisely to W Hz. The frequency response of the reconstruction filter is shown in Fig. 2.3.5(b).
- When the sampled signal x_δ (t) is applied at the input, this filter will allow only the shaded portion in the spectrum of x_δ (t) to pass through to the output and will block all other frequency components.
- Thus the frequency components only corresponding to x (t) will be passed through to the output and the original signal x (t) is recovered.

2.3.2 Aliasing or Foldover Error :

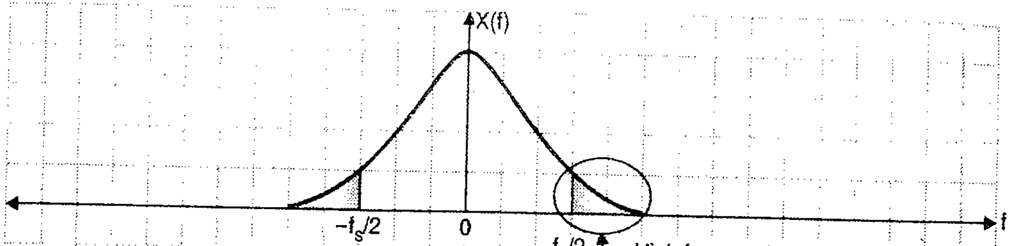
SPPU : May 06, May 15

University Questions

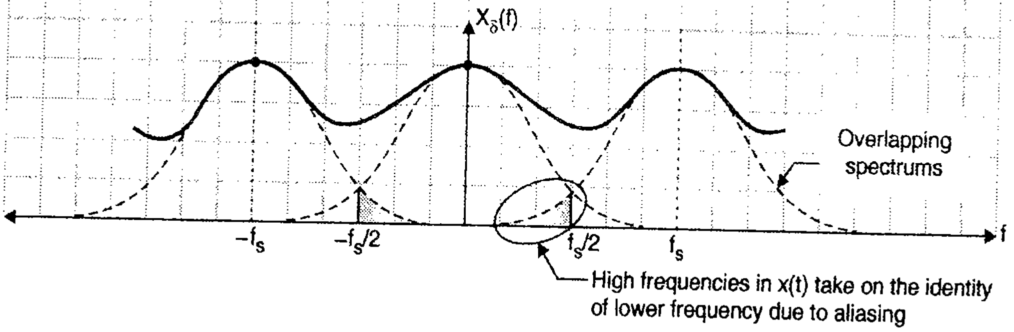
Q. 1 Assuming spectrum of baseband signal bandlimited to W Hz draw the following :
1. Spectrum of ideally sampled signal for $f_s > 2W$.
2. Spectrum of naturally sampled signal for $f_s = 2W$.
3. Spectrum of naturally sampled signal for $f_s < 2W$.
(May 06, 6 Marks)

Q. 2 With suitable spectral diagram prove the sampling theorem and explain aliasing effect.
(May 15, 6 Marks)

- If the signal x (t) is not strictly bandlimited and / or if the sampling frequency f_s is less than 2W, then an error called **aliasing** or **foldover error** is observed. The adjacent spectrums overlap if $f_s < 2W$. This is shown in Fig. 2.3.6(b).
- The signal x (t) is not strictly bandlimited. The spectrum of signal x (t) is shown in Fig. 2.3.6(b).
- The spectrum X_δ (f) of the discrete time signal x_δ (t) is shown in Fig. 2.3.6(b) which is nothing but the sum of X (f) and infinite number of frequency shifted replicas of it as explained earlier.
- Consider the two replicas of X (f) which are centered about the frequencies f_s and $-f_s$.
- If we use a reconstruction filter with its pass-band extending from $-f_s / 2$ to $+f_s / 2$ then its output will not be an undistorted version of the original signal x (t). Some distortion will be present in the filter output.
- The distortion occurs due to the overlapping of the adjacent spectrums as shown in Fig. 2.3.6(b). Due to this overlapping, it is seen that the portions of the frequency shifted replicas are "folded over" inside the desired spectrum.
- Due to this "fold over", high frequencies in X (f) are reflected into low frequencies in X_δ (f). This can be understood by comparing the shaded portions of the spectra shown in Figs. 2.3.6(a) and (b).
- **Aliasing** : This phenomenon of a high frequency in the spectrum of the original signal x (t), taking on the identity of lower frequency in the spectrum of the sampled signal x_δ (t) is called as aliasing or fold over error.



(a) Spectrum of a continuous time signal x(t)



(b) Spectrum of the sampled version of x(t) with $f_s < 2W$

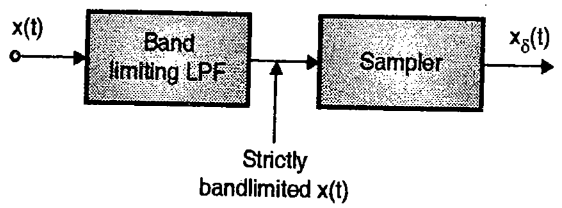
(D-415) **Fig. 2.3.6**

Effect of aliasing :

- Due to aliasing some of the information contained in the original signal x (t) is lost in the process of sampling.

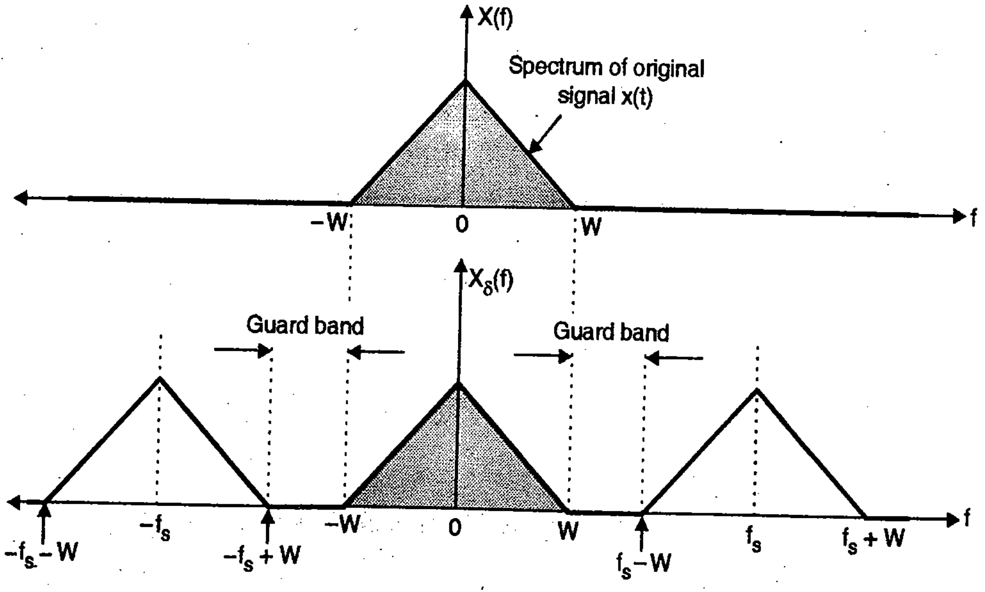
How to eliminate aliasing ?

Aliasing can be completely eliminated if we take the following action :



(D-416) **Fig. 2.3.7(a) : Use of a bandlimiting filter to eliminate aliasing**

- Use a bandlimiting low pass filter and pass the signal x (t) through it before sampling as shown in Fig. 2.3.7(a).
- This filter has a cutoff frequency at $f_c = W$, therefore it will strictly bandlimit the signal x (t) before sampling takes place. This filter is also called as antialiasing filter or prealias filter.
- Increase the sampling frequency f_s to a great extent i.e. $f_s >>> 2W$. Due to this, even though x (t) is not strictly bandlimited, the spectrums will not overlap. A guard band is created between the adjacent spectrums as shown in Fig. 2.3.7(b).
- Thus aliasing can be prevented by :
  1. Using an antialiasing or prealiasing filter and
  2. Using the sampling frequency $f_s > 2W$.



(D-417) **Fig 2.3.7(b) : Spectrum of a sampled signal for $f_s > 2W$**

2.3.3 Nyquist Rate and Nyquist Interval :

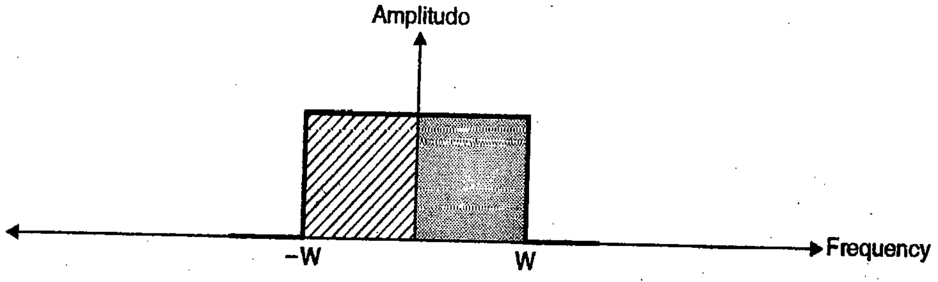
- The minimum sampling rate of "2W" samples per second for a signal x (t) having maximum frequency of "W" Hz is called as "Nyquist rate". The reciprocal of Nyquist rate i.e. 1/2W is called as the Nyquist interval.

Nyquist rate = 2W Hz

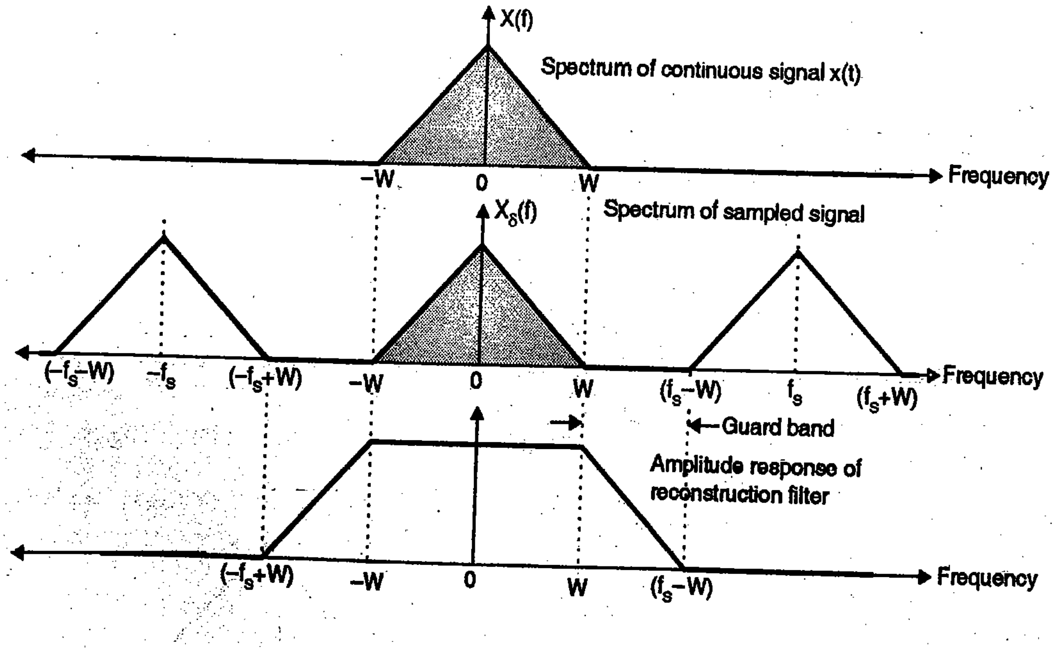
Nyquist interval = 1/2W seconds

2.3.4 Effect of Nonideal Filter :

- As mentioned earlier the reconstruction filter is a low pass filter. It is expected to pass all the frequencies in the range of (– W to + W) Hz.
- This is because the original signal x (t) is bandlimited to "W" Hz.
- Therefore the frequency response of a reconstruction filter should be as shown in Fig. 2.3.8. This is the frequency response of an ideal low pass filter.
- But it is not possible to practically realize an ideal low pass filter. Therefore a practical low pass filter with a frequency response as shown in Fig. 2.3.9 is used.
- It is possible to use the practical low pass filter without introducing any distortion due to the presence of the guard bands between the adjacent frequency spectrums as shown in Fig. 2.3.8. That is why it is necessary to have $f_s > 2W$.



(D-418) Fig. 2.3.8 : Frequency response of an ideal low pass filter used as a reconstruction filter



(D-419) Fig. 2.3.9 : Amplitude response of a practical reconstruction filter

Ex. 2.3.1 : Consider the signal { 3 cos (200 πt) + (5 sin 6000 πt) + 10 cos 1200 πt }. What is the Nyquist rate for this signal ?

May 2000, 3 Marks

Soln. :

The highest frequency component in the given signal is,

$$f_m = 6000 \text{ Hz ...corresponding to the last term.}$$

$$\therefore \quad \text{Nyquist rate} = 2 f_m = 12 \text{ kHz} \qquad \text{...Ans.}$$

Ex. 2.3.2 : If x (t) = 10 cos 2000 πt · cos 8000 πt, what is the minimum sampling rate ?

May 2000, 8 Marks

Soln. :

To calculate the minimum sampling rate :

$$x(t) = 10 \cos 2000 \pi t \cdot \cos 8000 \pi t$$

$$= 5 \cos [(2000 + 8000) \pi t\,]$$

$$+ 5 \cos [(8000 - 2000) \pi t]$$

$$\therefore \quad x(t) = 5 [\cos 10000 \pi t + \cos 6000 \pi t]$$

Thus the two frequency components present in the signal x (t) are $f_1 = 5000$ Hz and $f_2 = 3000$ Hz.

Therefore the minimum sampling rate is given by :

$$f_{s\,(min)} = 2 \times f_1 = 2 \times 5000 \text{ Hz}$$

$$= 10 \text{ kHz} \qquad \text{...Ans.}$$

Ex. 2.3.3 : Find the Nyquist rate and Nyquist interval for the following signal :

x (t) = 10 cos (4000 πt) cos (1000 πt)

Dec. 2000, 4 Marks

Soln. :

$$x(t) = 10 \cos (4000 \pi t) \cos (1000 \pi t)$$

$$= 5 [\cos 5000 \pi t + \cos (3000 \pi t)]$$

∴ x (t) consists of two frequency components, $f_1 = 2500$ Hz and $f_2 = 1500$ Hz.

$$\therefore \quad \text{Nyquist rate} = 2 \times f_1 = 5 \text{ kHz} \qquad \text{...Ans.}$$

$$\text{and} \quad \text{Nyquist interval} = 1/5 \text{ kHz} = 0.2 \text{ msec} \qquad \text{...Ans.}$$

Ex. 2.3.4 : Determine the Nyquist rate and Nyquist interval for the following signal :

x (t) = 5 cos (2000 t) + 7 sin (7000 t).

May 01, 4 Marks

Soln. :

The expression for x(t) shows that there are two frequency components in this signal.

$$f_1 = \frac{2000}{2\pi} = 318.3 \text{ Hz and } f_2 = 1114 \text{ Hz.}$$

$$\therefore \quad \text{Nyquist rate} = 2 f_2 = 2 \times 1114 \text{ Hz}$$

$$= 2228 \text{ Hz} \qquad \text{...Ans.}$$

$$\text{and} \quad \text{Nyquist interval} = 1 / (\text{Nyquist rate})$$

$$= 1/2228 \text{ Hz}$$

$$= 4.487 \times 10^{-4} \text{ sec.}$$

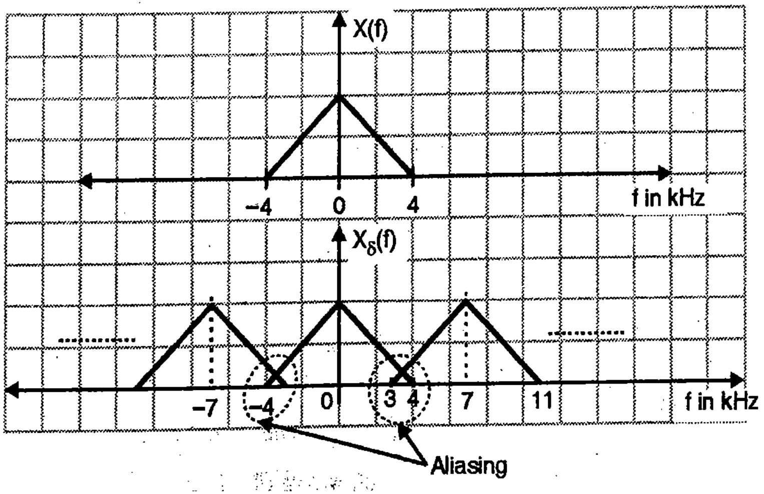
$$= 0.4487 \text{ m sec} \qquad \text{...Ans.}$$

Ex. 2.3.5 : A bandlimited signal with $f_{max} = 4$ kHz is sampled at $f_s = 7$ kHz. Calculate aliasing component and draw the spectrum of sampled wave.

Dec. 01, 6 Marks

Soln. :

The spectrum of sampled signal is as shown in Fig. P. 2.3.5.



(E-656) **Fig. P. 2.3.5 : Spectrum of sampled signal**

Ex. 2.3.6 : A waveform [10 + 10 sin (500 t + 30°)] is to be sampled periodically and reproduced from these sample values. Find the maximum allowable time interval between sample values. How many sample values are needed to be stored in order to reproduce one second of this waveform ?

May 02, 5 Marks, Dec. 07, 6 Marks

Soln. :

1. The input waveform is $x(t) = 10 + 10 \sin(500t + 30°)$. The first term represents a dc shift whereas the second term is a sinewave of frequency,

$$f_m = \frac{500}{2\pi} = 79.58 \text{ Hz.}$$

2. Therefore the minimum sampling rate is given by :

$$f_{s\,(min)} = 2 f_m = 2 \times 79.58$$
$$= 159.16 \text{ Hz.}$$

3. The maximum allowable time interval between the sample values is given by,

$$T_{s\,(max)} = \frac{1}{f_{s\,(min)}} = \frac{1}{159.16}$$

$$\therefore \quad T_{s\,(max)} = 6.28 \text{ msec}$$ **...Ans.**

4. The number of samples needed to be stored to produce 1 sec. is given by,

$$\text{Number of samples} = \frac{1 \text{ sec}}{6.28 \text{ msec}}$$
$$= 159.16 \text{ samples}$$ **...Ans.**

Ex. 2.3.7 : A cosine wave of 1 Volt and 5 kHz frequency is sampled at sampling frequency f_s = 8 kHz. Sampling is ideal. Output of the sampler are then passed through an ideal low pass filter of 0 Hz to 4 kHz bandwidth. The output of LPF is observed on CRO. Draw this output wave shape on graph paper to the scale comparing it with input wave shape. Interprete the result. Also draw the spectrum at the input, output of sampler and at the output of filter for the given case of cosine 5 kHz wave on the graph paper and to the scale.

Dec. 05, 10 Marks

Soln. :

Given : $x(t) = \cos(2\pi \times 5 \times 10^3 t)$

$\therefore f_m = 5$ kHz, $\quad f_s = 8$ kHz.

Sampling - ideal, Filter passband 0-4 kHz.

Part I : Spectrum of sampled signal :

$$x(t) = \cos(10\pi \times 10^3 t) = \cos(31415.92\, t)$$

Take the F.T. of both sides to get,

$$X(f) = F[\cos(2\pi \times 5 \times 10^3 t)]$$
$$= \frac{1}{2}[\delta(f - f_m) + \delta(f + f_m)]$$
$$= \frac{1}{2}[\delta(f - 5000) + \delta(f + 5000)]$$

The spectrum of x(t) is as shown in Fig. P. 2.3.7(a).

The spectrum of ideally sampled signal $X_\delta(f)$ is given by,

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - n f_s)$$

$$= 8000 \sum_{n=-\infty}^{\infty} X(f - 8000\, n)$$

$$X_\delta(f) = 8000[\ldots X(f + 8000) + X(f)$$
$$+ X(f - 8000) + X(f - 16000) \ldots]$$

X_δ is as shown in Fig. P. 2.3.7(a).

As we pass the sampled signal through a LPF having a passband from 0 to 4 kHz, the spectrum of LPF output is as shown in Fig. P. 2.3.7(a).

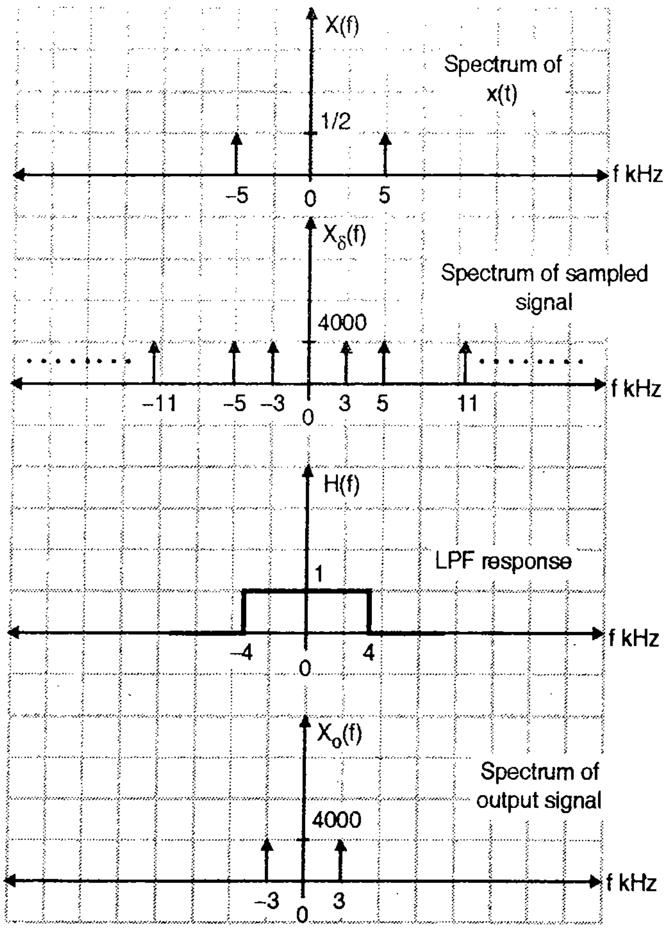
Part II : Waveform of the output signal :

The spectrum of output of LPF is as shown in Fig. P. 2.3.7(a). Mathematically it is expressed as,

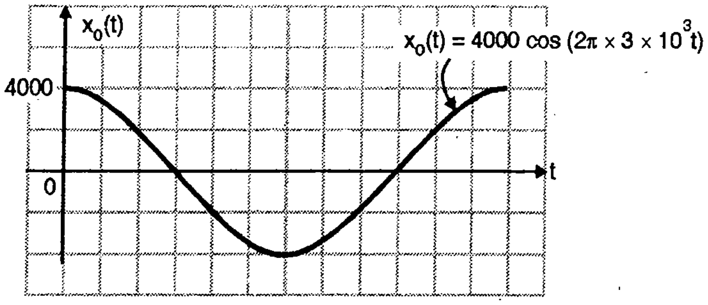
$$X_o(f) = 4000[\delta(f + 3000)$$
$$+ \delta(f - 3000)]$$

Take the IFT to get $x_o(t) = 4000 \cos(2\pi \times 3 \times 10^3 t)$.

The output signal $x_o(t)$ if observed on CRO, looks as shown in Fig. P. 2.3.7(b).



(E-657) **Fig. P. 2.3.7(a)**



(E-658) **Fig. P. 2.3.7(b) : Output signal**

Ex. 2.3.8 : A signal $x(t) = 10\cos(1000t + \pi/3) + 20\cos(2000t + \pi/6)$ is to be uniformly sampled.

1. What is the maximum allowable time interval between sample values that will ensure faithful signal reproduction ?
2. If one hours of signal to be reproduced, how many samples needs to be stored.

Dec. 06. 8 Marks

Soln. :

Step 1 : Calculate Nyquist rate $f_{s\,(min)}$: (E-659)

$$x(t) = 10\cos(\underbrace{1000t}_{\to f_1 = \frac{1000}{2\pi}} + \pi/3) + 20\cos(\underbrace{2000t}_{\to f_2 = \frac{2000}{2\pi}} + \pi/6)$$

$$\therefore \quad f_1 = \frac{1000}{2\pi} = 159.15 \text{ Hz and}$$

$$f_2 = \frac{2000}{2\pi} = 318.3 \text{ Hz.}$$

$$\therefore \quad \text{Nyquist rate } f_{s\,(min)} = 2 \times f_2$$

$$= 2 \times 318.3 \text{ Hz} = 636.6 \text{ Hz.}$$

Step 2 : **Minimum allowable interval between samples :**

This is nothing but the sampling interval.

$$\therefore \quad T_{s\,(max)} = \frac{1}{f_{s\,(min)}} = \frac{1}{636.6}$$

$$= 1.57 \text{ mS} \quad \text{...Ans.}$$

Step 3 : **Number of samples per hour :**

Number of samples per second = 636.6

∴ Number of samples per hour = 636.6 × 3600

$$= 2.2918 \times 10^6 \quad \text{...Ans.}$$

Ex. 2.3.9 : The signal $x(t) = \cos 200\,\pi t + 0.25 \cos 700\,\pi t$ is sampled at the rate of 400 samples per second. Sampled waveform is then passed through an ideal low pass filter with 200 Hz bandwidth. Write an expression for filter output. Sketch the frequency spectrum of sampled waveform.

May 07, May 15, 8 Marks

Soln. :

- It has been given that $x(t) = \cos 200\,\pi t + 0.25 \cos 700\,\pi t$ and $f_s = 400$ Hz. The spectrum of the ideally sampled signal is given by,

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - n f_s)$$

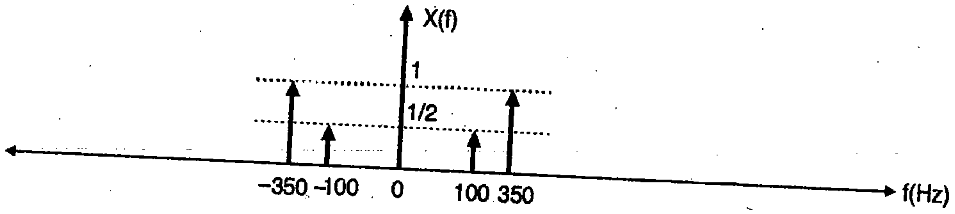
$$= 400 \sum_{n=-\infty}^{\infty} X(f - 400\,n)$$

$$= 400\,X(f) + 400\,X(f \pm 400) + 200\,X(f \pm 800) + \ldots \quad \text{...(1)}$$

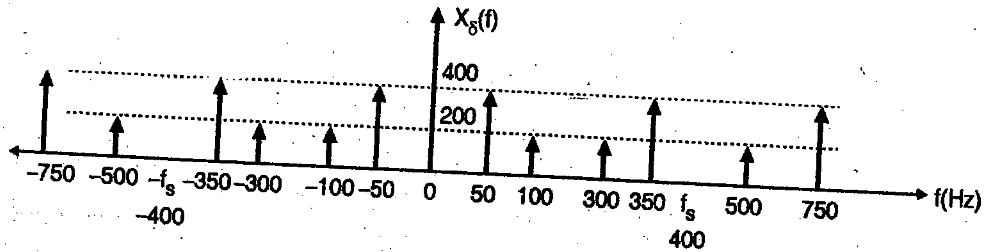
- The spectrum of x (t) is as shown in Fig. P. 2.3.9(a) which contains two frequency components $f_1 = 100$ Hz and $f_2 = 350$ Hz.
- The second term in Equation (1) shows X (f) centered about $\pm f_s = 400$ Hz.
- The spectrum shifted towards right (see Fig. P. 2.3.9(b)) consists of four frequency components.
- The Spectrum of low pass filter frequency response is shown in Fig. P. 2.3.9(c).
- It allows only the frequency components of 50 Hz and 100 Hz to pass through to the output.

$$\therefore \quad \text{Output} = 400 \cos (400\,\pi t) + 200\,(100\,\pi t)$$

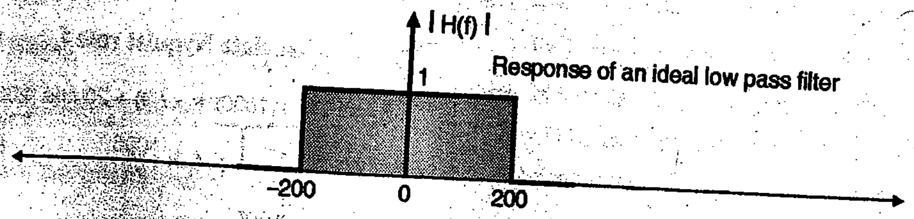
Note that cos (400 πt) is an undesired component.



(a) Spectrum of x(t)



(b) Spectrum of $x_\delta(t)$



(c) Response of an ideal low pass filter

(E-660) **Fig. P. 2.3.9**

Ex. 2.3.10 : Specify the Nyquist rate and Nyquist interval for each of the following signals :

1. $g(t) = \text{sinc}\,(200\,t)$
2. $g(t) = \text{sinc}^2\,(200\,t)$
3. $g(t) = \text{sinc}\,(200\,t) + \text{sinc}^2\,(200\,t)$

Dec. 08, 6 Marks

Soln. :

1. $g(t) = \text{sinc}\,(200\,t)$ **:**

$$\text{sinc}\,(200\,t) = \frac{\sin(200\,\pi\,t)}{200\,\pi\,t} = \frac{\sin(2\pi \times 100\,t)}{200\,\pi\,t}$$

$$\therefore\ f = 100\text{ Hz}$$

$$\therefore\ \text{Nyquist rate} = 2 \times 100 = 200\text{ Hz}$$ ...Ans.

$$\text{Nyquist interval} = \frac{1}{200} = 5\text{ mS}$$ ...Ans.

2. $g(t) = \text{sinc}^2\,(200\,t)$ **:**

$$\text{sinc}^2\,(200\,t) = \frac{\sin^2(200\,\pi\,t)}{200\,\pi\,t}$$

$$\text{But } \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\therefore\ \frac{\sin^2(200\,\pi\,t)}{200\,\pi\,t} = \frac{1-\cos(2 \times 200\,\pi\,t)}{2 \times 200\,\pi\,t} = \frac{1}{400\,\pi\,t} - \frac{\cos(2\pi \times 200\,t)}{400\,\pi t}$$

$$f = 200\text{ Hz}$$

$$\text{DC term } f = 0\text{ Hz}$$

$$\therefore\ \text{Nyquist rate} = 2 \times 200\text{ Hz} = 400\text{ Hz}$$ ...Ans.

$$\text{And Nyquist interval} = \frac{1}{400} = 2.5\text{ mS}$$ ...Ans.

3. $g(t) = \text{sinc}\,(200\,t) + \text{sinc}^2\,(200\,t)$ **:**

$$f_2 = 200\text{ Hz}$$

$$f_1 = 100\text{ Hz}$$

$$\therefore\ \text{Nyquist rate} = 2 \times f_2 = 2 \times 200 = 400\text{ Hz}$$ ...Ans.

$$\therefore\ \text{Nyquist interval} = \frac{1}{400} = 2.5\text{ mS}$$ ...Ans.

Ex. 2.3.11 : The signal $g(t) = 10\cos(40\,\pi t)\cos(400\,\pi t)$ is sampled at the rate of 500 samples/sec.

1. Determine the Nyquist rate.
2. Calculate the cut off frequency of ideal reconstruction filter.
3. Draw the spectrum of resulting sampled signal.
4. If g(t) is considered to be a band pass signal, determine the lowest permissible sampling rate.

Dec. 11, 8 Marks

Soln. :

Given : Signal

$$g(t) = 10\cos(40\,\pi t)\cos(400\,\pi t) = 5\,[\cos(440\,\pi t) + \cos(360\,\pi t)]$$

$$\therefore\ f_{max} = 220\text{ Hz}$$

$$f_{min} = 180\text{ Hz}$$

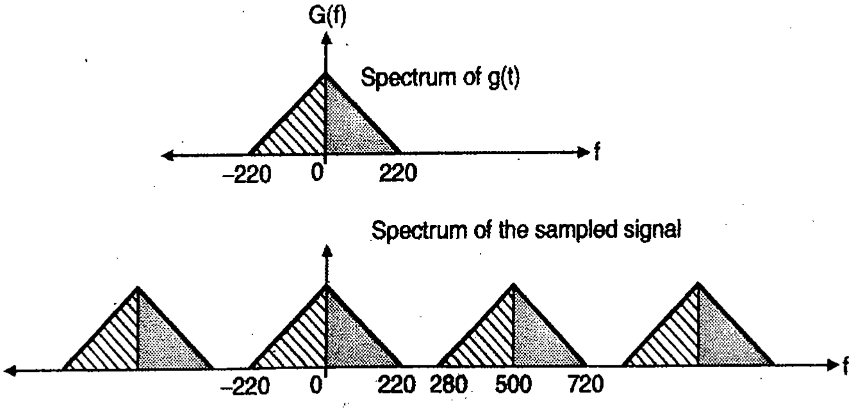
1. Nyquist rate :

$$f_{s\,(min)} = 2 \times f_{max} = 2 \times 220 = 440\text{ Hz}$$ ...Ans.

2. Cut off frequency of an ideal reconstruction filter :

$$f_c = 220\text{ Hz}$$ ...Ans.

3. Spectrum of the sampled signal :



(E-1342) **Fig. P. 2.3.11 : Spectrum of the sampled signal**

4. **Lowest sampling rate if the signal is a bandpass signal :**

The bandwidth B of this signal is given by,

$$B = f_{max} - f_{min} = 220 - 180 = 40 \text{ Hz}$$

$$\therefore \; f_{s\,(min)} = \frac{2 f_M}{k}$$

$$F_M = f_{max} = 220 \text{ Hz}$$

$$k = \frac{f_M}{B} = \frac{220}{40} = 5.5$$

∴ The value of k is 5

$$\therefore \; f_{s\,(min)} = \frac{2 \times 220}{5} = 88 \text{ Hz} \quad \text{...Ans.}$$

2.3.5 Sampling Theorem for Bandpass Signals :

The sampling theorem for the bandpass signals can be stated as follows :

A bandpass signal x (t), having a maximum bandwidth of 2W Hz can be completely represented in its sampled form and recovered back from the sampled form if it is sampled at a rate which is at least twice the maximum bandwidth. (i.e. $f_s \geq 4W$.)

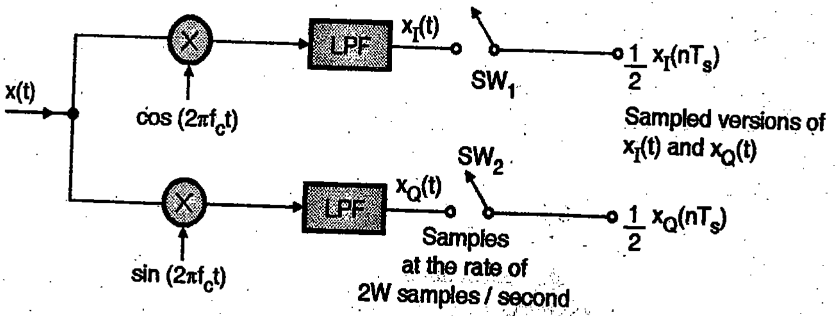
Quadrature sampling of bandpass signals :

- In this section, we consider a scheme called "quadrature sampling" for the uniform sampling of bandpass signals. This scheme is actually a natural extension of the sampling of low pass signals. The scheme is as follows :
- In this scheme, we do not sample the bandpass signal directly. Instead, before sampling we represent the bandpass signal x (t) in terms of its "in-phase" and "quadrature" components, x_I (t) and x_Q (t) respectively.
- The in-phase and quadrature components can be obtained by multiplying the bandpass signal x (t) by cos ($2\pi f_c$ t) and sin ($2\pi f_c$ t) respectively and then by suppressing the sum frequency components by means of low pass filters as shown in Fig. 2.3.10(a).
- If x_I (t) = In-phase component and x_Q (t) = Quadrature component.

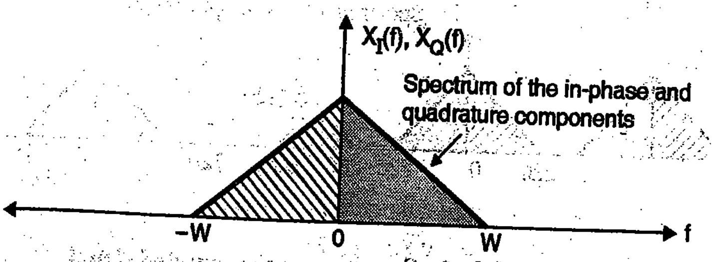
Then we can express the bandpass signal x (t) in terms of x_I (t) and x_Q (t) as follows :

$$x(t) = x_I(t) \cos(2\pi f_c t) - x_Q(t) \sin(2\pi f_c t) \quad \text{...(2.3.22)}$$

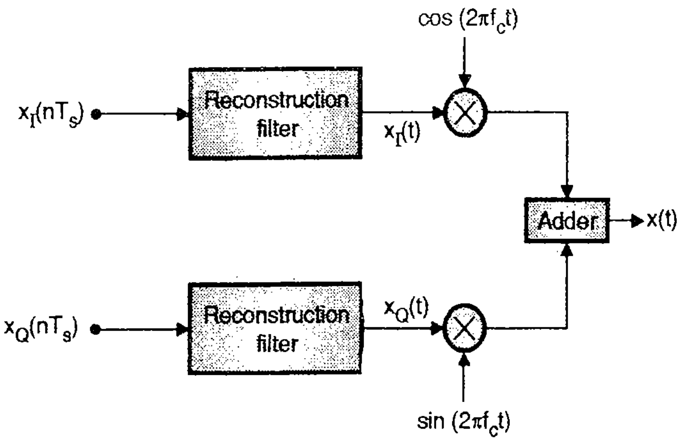
- Under the assumption of $f_c > W$, it is found that x_I (t) and x_Q (t) both are "low pass signals" extending from – W to + W as shown in Fig. 2.3.10(b).
- Then both the in-phase and quadrature components are separately sampled at a rate of 2W samples per second by the switches SW_1 and SW_2 as shown in Fig. 2.3.10(a) to obtain the sampled versions of x_I (t) and x_Q (t).
- In order to reconstruct the original bandpass signal from its quadrature sampled version, we first reconstruct the in-phase component x_I(t) and quadrature component x_Q (t) from their respective sampled versions x_I (nT_s) and x_Q (nT_s) by means of reconstruction filters. Then multiply x_I (t) and x_Q (t) by cos ($2\pi f_c$ t) and sin ($2\pi f_c$ t) respectively and add the result. The reconstruction process of x (t) is shown in Fig. 2.3.10(c).



(D-423) Fig. 2.3.10(a) : Generation of in-phase and quadrature samples from the bandpass signal x (t)



(D-424) Fig. 2.3.10(b) : Spectrum of the in-phase and quadrature components of x (t)



(D-425) **Fig. 2.3.10(c) : Reconstruction of the bandpass signal x(t)**

Ex. 2.3.12 : The given signal is m (t) = 10 cos 2000 πt cos 8000 πt :

(a) What is the minimum sampling rate based on the low pass uniform sampling theorem ?

(b) Repeat (a) based on the bandpass sampling theorem.

Soln. :

(a) $m(t) = 10 \cos 2000\pi t \cos 8000 \pi t$

$$\therefore \quad m(t) = 5 \cos 10000 \pi t + 5 \cos 6000 \pi t \quad \text{...(1)}$$

Thus the highest frequency present in m (t) is W = 5000 Hz. Therefore as per the low pass sampling theorem, the minimum sampling rate is given by,

$$f_{s\,min} = 2W = 2 \times 5000 = 10 \text{ kHz} \quad \text{...Ans.}$$

(b) Looking at Equation (1) it is clear that the given signal m (t) contains two frequency components which are,

$$f_1 = 3000 \text{ Hz and } f_2 = 5000 \text{ Hz}$$

The bandwidth of m (t) is,

$$B = f_2 - f_1 = 2000 \text{ Hz}$$

When f_1 and f_2 are not harmonically related to the sampling frequency f_s, the bandpass sampling theorem stated in section 2.3.5 is stated in a more generalized form as follows :

If a bandpass signal x (t) has a bandwidth "B" and the highest frequency "f_M" then x (t) can be recovered from its sampled version if $f_s = \frac{2f_M}{k}$ where k is the largest integer not exceeding $\frac{f_M}{B}$. All higher sampling rates are not necessarily usable unless they exceed $2f_M$.

Thus $f_M = f_2 = 5$ kHzand

$$B = 2 \text{ kHz}$$

$$\therefore \quad k = \frac{5}{2} = 2.5. \text{ Therefore the value of k is 2}$$

$$\therefore \quad f_s = \frac{2 \times 5 \text{ kHz}}{2} = 5 \text{ kHz} \quad \text{...Ans.}$$

Ex. 2.3.13 : A bandpass signal has a spectral range that extends from 20 kHz to 82 kHz. Find the sampling frequency f_s.

Soln. :

$$f_1 = 20 \text{ kHz and } f_2 = 82 \text{ kHz.}$$

$$\therefore \quad \text{Bandwidth } B = f_2 - f_1 = 82 - 20$$

$$= 62 \text{ kHz} \quad \text{...(1)}$$

$$\text{Let us assume that } f_s = 2B = 2 \times 62 = 124 \text{ kHz} \quad \text{...(2)}$$

From Equations (1) and (2) we observe that neither f_1 nor f_2 is harmonically related to f_s. Hence we have to use the general bandpass sampling theorem stated in the preceding example.

$$\therefore \quad k = \frac{f_M}{B} = \frac{82}{62}$$

$$= 1.32 \rightarrow 1$$

$$\therefore \quad f_s = \frac{2f_M}{k} = \frac{2 \times 82}{1}$$

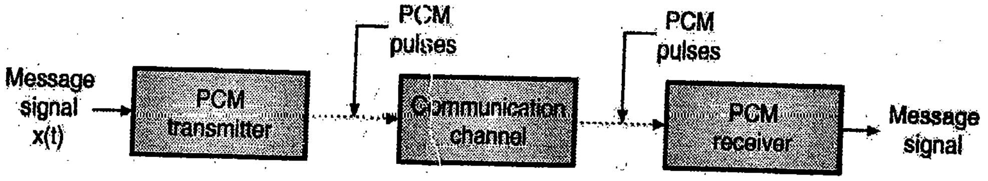
$$= 164 \text{ kHz} \quad \text{...Ans.}$$

2.4 Pulse Code Modulation (PCM) :

- PCM is a type of pulse modulation like PAM, PWM or PPM but there is an important difference between them. PAM, PWM or PPM are "analog" pulse modulation systems whereas PCM is a "digital" pulse modulation system.
- That means the PCM output is in the coded digital form. It is in the form of digital pulses of constant amplitude, width and position.
- The information is transmitted in the form of "code words". A PCM system consists of a PCM encoder (transmitter) and a PCM decoder (receiver).
- The essential operations in the PCM transmitter are sampling, quantizing and encoding.
- All these operations are usually performed in the same circuit called as analog-to-digital (A to D) converter.
- It should be understood that the PCM is not modulation in the conventional sense.
- Because in modulation, one of the characteristics of the carrier is varied in proportion with the amplitude of the modulating signal. Nothing of that sort happen in PCM.

2.4.1 Pulse Code Modulation (PCM) System :

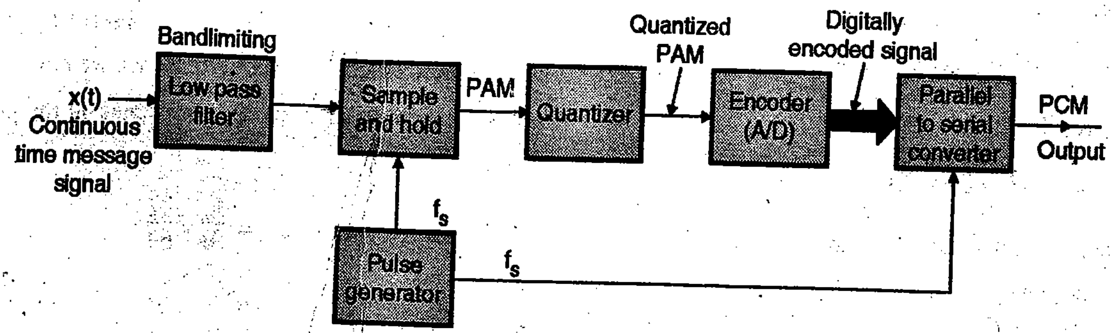
- Fig. 2.4.1 shows the simplified block diagram of a PCM system. It consists of a transmitter and receiver.
- The transmitter converts the message signal x (t) into a series of coded pulses and sends it over the communication channel.
- The transmitter is also called as an encoder.
- The receiver performs exactly in the reverse way as compared to the transmitter. It will convert the received encoded PCM pulses back into the message signal.



(D-771) **Fig. 2.4.1**

2.4.2 PCM Transmitter (Encoder) :

- Block diagram of the PCM transmitter is as shown in Fig. 2.4.2.



(L-221) **Fig. 2.4.2 : PCM transmitter (Encoder)**

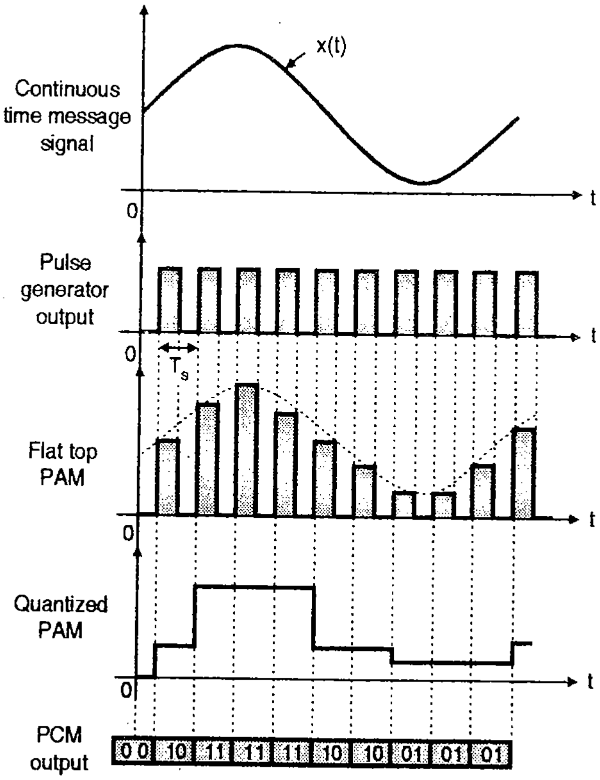
Operation of PCM transmitter :

Operation of the PCM transmitter is as follows :

- The analog signal x (t) is passed through a bandlimiting low pass filter, which has a cut-off frequency f_c = W Hz. This will ensure that x (t) will not have any frequency component higher than "W". This will eliminate the possibility of aliasing.
- The band limited analog signal is then applied to a sample and hold circuit where it is sampled at adequately high sampling rate. Output of sample and hold block is a flat topped PAM signal.
- These samples are then subjected to the operation called "Quantization" in the "Quantizer". Quantization process is the process of approximation as will be explained later on. The quantization is used to reduce the effect of noise. The combined effect of sampling and quantization produces the quantized PAM at the quantizer output.
- The quantized PAM pulses are applied to an encoder which is basically an A to D converter. Each quantized level is converted into an N bit digital word by the A to D converter. The value of N can be 8, 16, 32, 64 etc.
- The encoder output is converted into a stream of pulses by the parallel to serial converter block. Thus at the PCM transmitter output we get a train of digital pulses.
- A pulse generator produces a train of rectangular pulses with each pulse of duration "τ" seconds. The frequency of this signal is "f_s" Hz. This signal acts as a sampling signal for the sample and hold block. The same signal acts as "clock" signal for the parallel to serial converter. The frequency "f_s" is adjusted to satisfy the Nyquist criteria.

Waveforms :

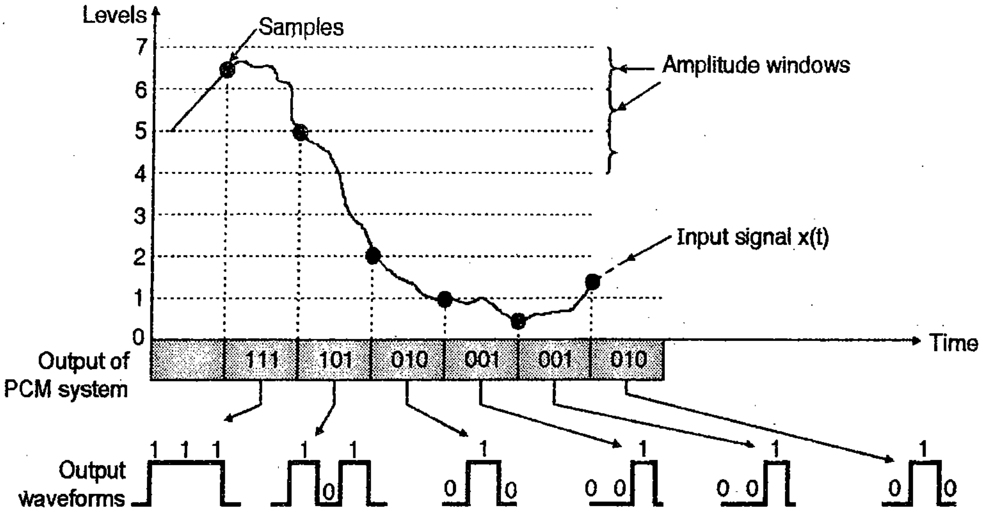
The waveforms at various points in the PCM transmitter are as shown in Fig. 2.4.3.



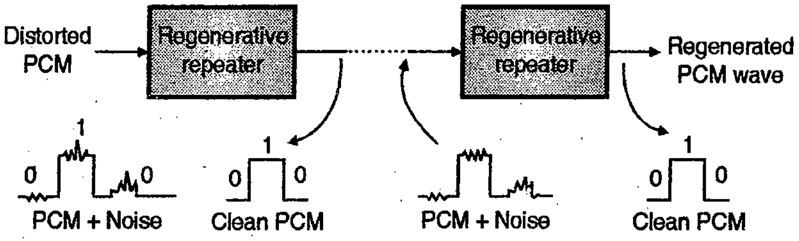
(L-222) **Fig. 2.4.3 : Waveforms at different points in PCM transmitter**

2.4.3 Shape of the PCM Signal :

- Fig. 2.4.4 shows input to and output of a PCM system. It is important to understand that the output is in the form of binary codes. Each transmitted binary code represents a particular amplitude of the input signal. Hence the "information" is contained in the "code" which is being transmitted.
- The range of input signal magnitudes is divided into 8-equal levels. Each level is denoted by a three bit digital word between 000 and 111.
- Input signal x (t) is sampled. If the sample is in the 5th - window of amplitude then a digital word 101 is transmitted. If the sample is in the 2nd - window then the transmitted word is 010 and so on.
- In this example we have converted the amplitudes into 3 bit codes, but in practice the number of bits per word can be as high as 8, 9 or 10.



(L-223) **Fig. 2.4.4 : Input and output waveforms of a PCM system**



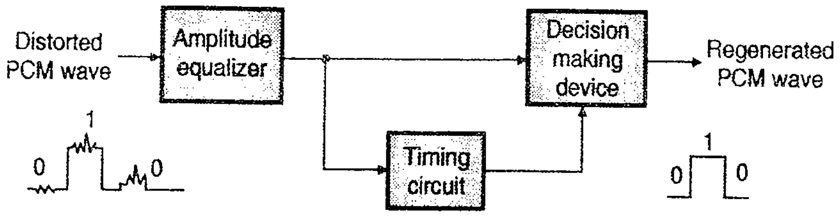
(D-472) **Fig. 2.4.5 : PCM transmission path**

2.4.4 PCM Transmission Path :

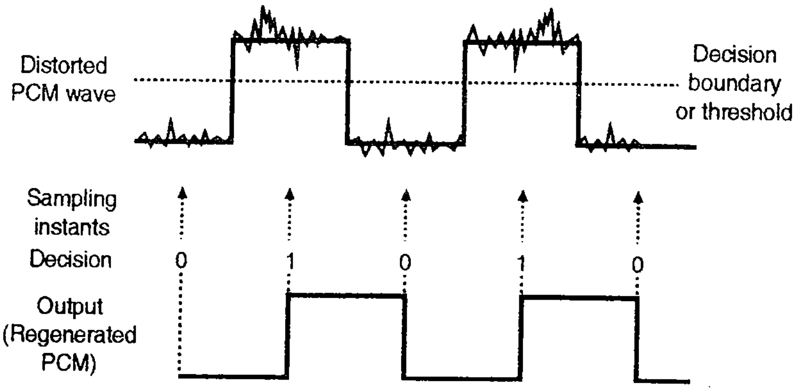
- The path between the PCM transmitter and PCM receiver over which the PCM signal travels is called as PCM transmission path and it is as shown in Fig. 2.4.5.
- The most important feature of PCM system lies in its ability to control the effects of distortion and noise when the PCM wave travels on the channel.
- PCM accomplishes this capacity by means of using a chain of regenerative repeaters as shown in Fig. 2.4.5.
- Such repeaters are spaced close enough to each other on the transmission path.
- The regenerative repeater performs three basic operations namely equalization, timing and decision making.
- So each repeater actually reproduces the clean noise free PCM signal from the PCM signal distorted by the channel noise. This improves the performance of PCM in presence of noise.

Block diagram of a repeater :

- The block diagram of a regenerative repeater is shown in Fig. 2.4.6.
- The amplitude equalizer shapes the distorted PCM wave so as to compensate for the effects of amplitude and phase distortions.
- The timing circuit produces a periodic pulse train that is derived from the input PCM pulses. This pulse train is then applied to the decision making device.
- The decision making device uses this pulse train for sampling the equalized PCM pulses. The sampling is carried out at the instants where the signal to noise ratio is maximum.
- The decision device makes a decision about whether the equalized PCM wave at its input has a 0 value or 1 value at the instant of sampling.
- Such a decision is made by comparing equalized PCM with a reference level called decision threshold as shown in Fig. 2.4.7.
- At the output of the decision device we get a clean PCM signal without any trace of noise.



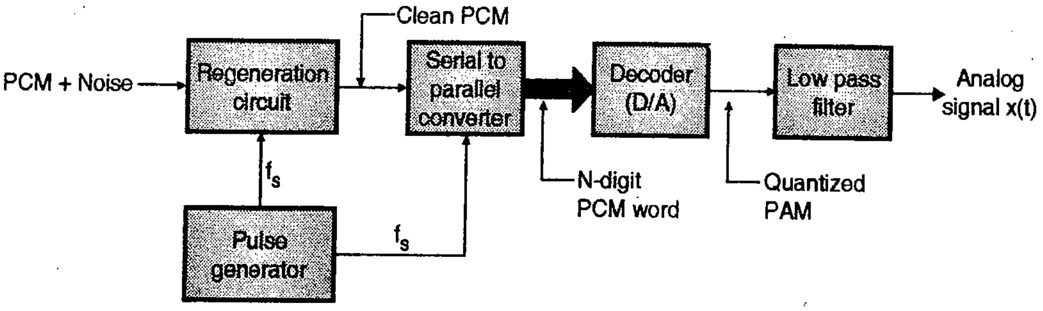
(E-667) **Fig. 2.4.6 : Block diagram of a regenerative repeater**



(E-668) **Fig. 2.4.7 : Waveforms of regenerative repeater**

2.4.5 PCM Receiver (Decoder) :

- Fig. 2.4.8 shows the block diagram of a PCM receiver.



(L-224) **Fig. 2.4.8 : PCM receiver (Decoder)**

Operation of PCM receiver :

- A PCM signal contaminated with noise is available at the receiver input.
- The regeneration circuit at the receiver will separate the PCM pulses from noise and will reconstruct the original PCM signal.
- The pulse generator has to operate in synchronization with that at the transmitter. Thus at the regeneration circuit output we get a "clean" PCM signal.
- The reconstruction of PCM signal is possible due to the digital nature of PCM signal. The reconstructed PCM signal is then passed through a serial to parallel converter.
- Output of this block is then applied to a decoder.
- The decoder is a D to A converter which performs exactly the opposite operation of the encoder.
- The decoder output is the sequence of a quantized multilevel pulses. The quantized PAM signal is thus obtained, at the output of the decoder.
- This quantized PAM signal is passed through a low pass filter to recover the analog signal, x (t).
- The low pass filter is called as the reconstruction filter and its cut off frequency is equal to the message bandwidth W.

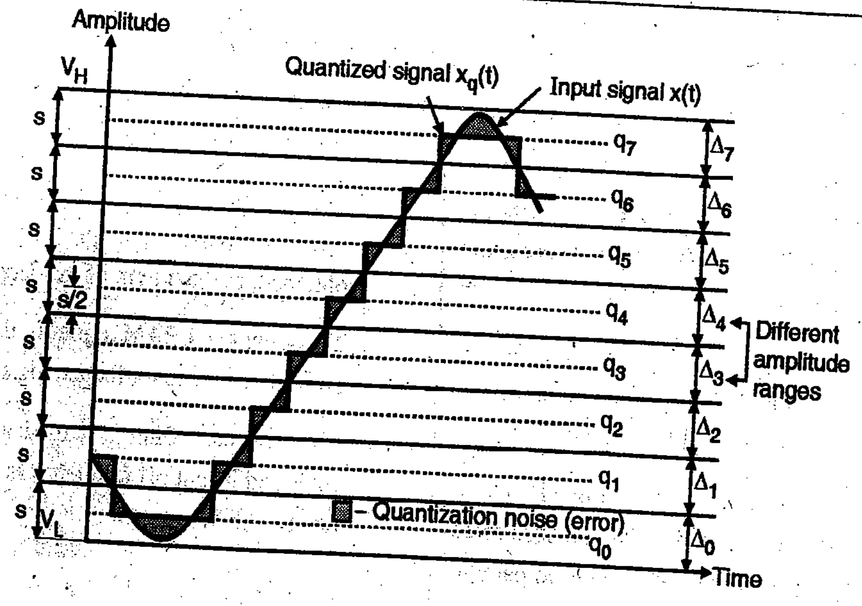
2.4.6 Quantization Process :

- Quantization is a process of approximation or rounding off. The sampled signal in PCM transmitted is applied to the quantizer block.
- Quantizer converts the sampled signal into an approximate quantized signal which consists of only a finite number of predecided voltage levels.
- Each sampled value at the input of the quantizer is approximated or rounded off to the nearest standard predecided voltage level.
- These standard levels are known as the "quantization levels". Refer to Fig. 2.4.9 to understand the process of quantization.
- The quantization process takes place as follows :
- The input signal x (t) is assumed to have a peak to peak swing of V_L to V_H volts. This entire voltage range has been divided into "Q" equal intervals each of size "s".
- "s" is called as the step size and its value is given as,

$$s = \frac{V_H - V_L}{Q} \qquad ...(2.4.1)$$

In Fig. 2.4.9, the value of Q = 8

- At the center of these ranges, the quantization levels q_0, q_1,...q_7 are placed. Thus the number of quantization levels is Q = 8. The quantization levels are also called as decision thresholds.
- x_q (t) represents the quantized version of x (t). We obtain x_q (t) at the output of the quantizer.
- When x (t) is in the range Δ_0, then corresponding to any value of x (t), the quantizer output will be equal to "q_0".
- Similarly for all the values of x (t) in the range Δ_1, the quantizer output is constant equal to "q_1".
- Thus in each range from Δ_0 to Δ_7, the signal x (t) is rounded off to the nearest quantization level and the quantized signal is produced.
- The quantized signal x_q (t) is thus an approximation of x (t). The difference between them is called as **quantization error or quantization noise.**
- This error should be as small as possible.
- To minimize the quantization error we need to reduce the step size "s" by increasing the number of quantization levels Q.



(L-225) **Fig. 2.4.9 : Process of quantization**

Why is quantization required ?

- If we do not use the quantizer block in the PCM transmitter, then we will have to convert each and every sampled value into a unique digital word.
- This will need a large number of bits per word (N). This will increase the bit rate and hence the bandwidth requirement of the channel.
- To avoid this, if we use a quantizer with only 256 quantization levels then all the sampled values will be finally approximated into only 256 distinct voltage levels.
- So we need only 8 bits per word to represent each quantized sampled value.
- Thus the number of bits per word can be reduced. This will eventually reduce the bit rate and bandwidth requirement.

Quantization error or quantization noise $\in$:

- The difference between the instantaneous values of the quantized signal and input signal is called as quantization error or quantization noise.

$$\in \;=\; x_q(t) - x(t) \qquad ...(2.4.2)$$

- The quantization error is shown by shaded portions of the waveform in Fig. 2.4.9.
- The maximum value of quantization error is ± s / 2 where s is step size.
- Therefore to reduce the quantization error we have to reduce the step size by increasing the number of quantization levels i.e. Q.
- The mean square value of the quantization is given by,

$$\text{Mean square value of quantization error} = \frac{s^2}{12} \qquad ...(2.4.3)$$

- The relation between the number of quantization levels Q and the number of bits per word (N) in the transmitted signal can be found as follows :
- Because each quantized level is to be converted into a unique N bit digital word, assuming a binary coded output signal,
- The number of quantization levels,

$$Q = \text{Number of combinations of bits/word.}$$

$$\therefore \quad Q = 2N \qquad ...(2.4.4)$$

- Thus if N = 4 i.e. 4 bits per word then the number of quantization levels will be 2^4 i.e.16.

Signal to quantization noise ratio (SNR_q) :

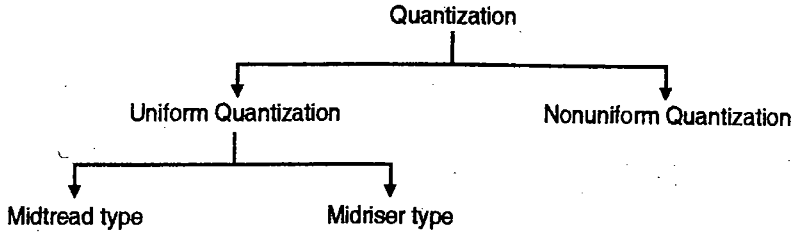
- This ratio is the figure of merit for the PCM systems. The signal to quantization noise ratio with a sinusoidal input signal to the PCM system is expressed as,

$$\frac{S_i}{N_q} = [1.8 + 6N]\ \text{dB} \quad \textbf{For a sinusoidal signal} \qquad ...(2.4.5)$$

- This equation shows that the signal to quantization noise ratio is solely dependent on the number of bits per word i.e. N.
- This ratio should be as high as possible, which can be achieved by increasing N. But this increases the bit rate and hence bandwidth of the PCM system.
- Therefore the number of bits per word is a compromise between high SNR_q and bandwidth requirements.

2.5 Types of Quantization :

- Fig. 2.5.1 shows the classification of quantization process.



(E-671) **Fig. 2.5.1 : Classification of quantization**

- The quantization process can be classified into two types as :

  1. Uniform quantization

  2. Non-uniform quantization.

- This classification is based on the step size "s" defined earlier.

2.5.1 Uniform Quantizer :

- A quantizer is called as a **uniform quantizer** if the step size (s) of the quantized signal remains constant throughout the input range.

- However if the step size varies depending on the input then the quantizer is known as th **non-uniform quantizer.**

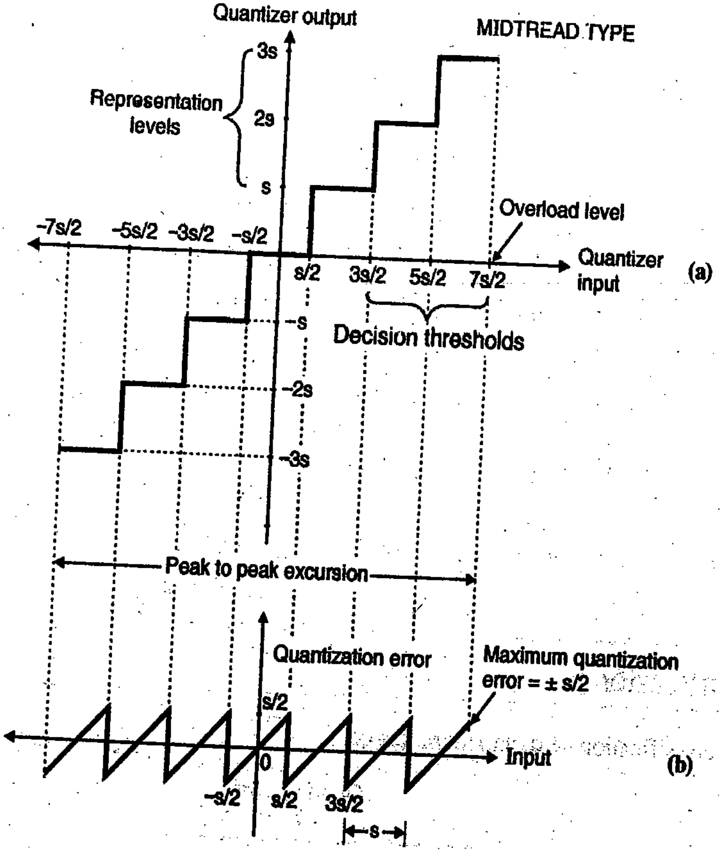
Types of uniform quantizer :

- There are two types of uniform quantizers :

  1. Symmetric quantizer of the midtread type.

  2. Symmetric quantizer of the midriser type.

- The transfer characteristics of the "midtread" type uniform quantizer is shown in Fig. 2.5.2(a) and the corresponding variation in the quantization error with input has been shown in Fig. 2.5.2(b).



(a) Transfer characteristic of quantizer of midtread type **(b) Variation of quantization error with input**

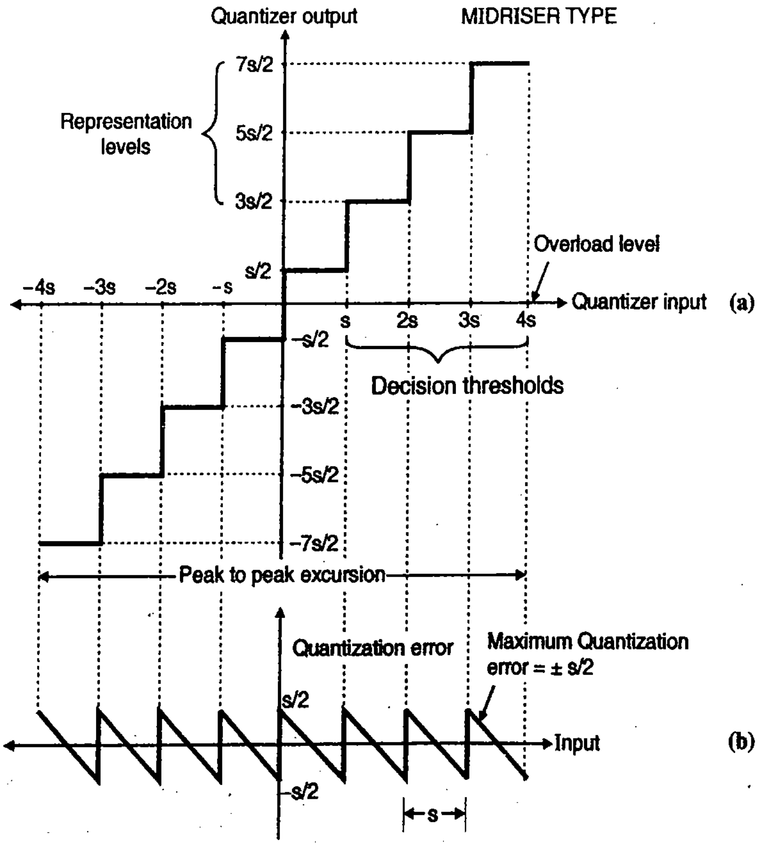
(E-672) **Fig. 2.5.2**

2.5.2 Symmetric Quantizer of Midtread Type :

- The meaning of graphical representation of the quantizing process is that a straight line representing the relation between the input and output of a linear analog system is to be replaced by a transfer characteristics of staircase type.
- The quantization process has a twofold effect as follows :
  1. The peak to peak range of the input signal voltage is divided into a finite set of decision levels or decision thresholds. These levels have been aligned with the "risers" of the staircase of Fig. 2.5.3(a).
  2. The output voltage of the quantizer is equal to a discrete value selected from a finite set of representation levels or reconstruction value. These levels are aligned with the "treads" of the staircase in Fig. 2.5.3(a).
- The separation between the decision thresholds and the separation between the representation levels is the same which is equal to step size "s".
- In Fig. 2.5.3(a), the decision thresholds are located at ± s/2, ± 3s/2, ± 5s/2 .... and the representation levels are located at 0, ± s, ± 2s ... where "s" is the step size.

2.5.3 Symmetric Quantizer of Midriser Type :

- The transfer characteristic of the "midriser" type uniform quantizer is as shown in Fig. 2.5.3(a) and the corresponding variation in the quantization error with input has been shown in Fig. 2.5.3(b).
- This is another staircase type transfer characteristics. The decision thresholds are located at 0, ± s, ± 2s etc. and the representation levels are located at ± s/2, ± 3s/2 ...
- This is called as the midriser type characteristics because in this case the origin lies in the middle of a riser of the staircase.



(a) Transfer characteristics of a quantizer of midriser type

(b) Variation of the quantization error with input

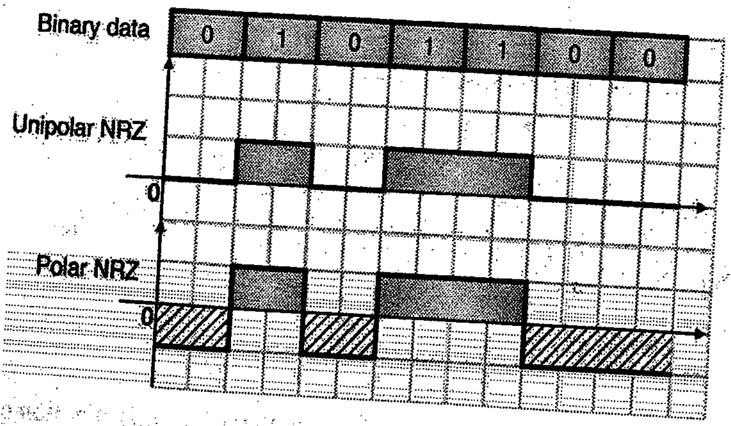
(E-673) **Fig. 2.5.3**

The important observations from Figs. 2.5.2(a) and 2.5.2(b) are as follows :

1. The quantizer of Fig. 2.5.2(a) is called as midtread type because the origin lies in the middle of a tread of the staircase. Similarly the quantizer of Fig. 2.5.3(a) is called as midriser type because the origin lies in the middle of the riser of the staircase.
2. Both these quantizers are "memoryless quantizers". This is because, the output is dependent only on the value of a corresponding input sample and it is not dependent on the previous or next analog sample applied at the input.
3. Overload level shown in Fig. 2.5.2(a) and Fig. 2.5.3(a), is equal to one half of the peak to peak range of the input sample values.
4. Fig. 2.5.2(b) and Fig. 2.5.3(b) shows the variation of quantization error with input. For both the types we observe that the maximum instantaneous value of this error is equal to half of one step size (s/2) and the total range of variation in the quantization error is from – s/2 to + s/2. This instantaneous variation in quantization error is plotted by subtracting the input from the output of the quantizer.

2.5.4 Encoding in PCM :

- We have seen that the process of encoding follows the sampling and quantizing processes.
- Encoding process converts the quantized samples into a code words.
- In a binary code each symbol is represented by either a 0 value or a 1 value.
- There are various formats (waveforms) used in digital communication for representing the binary sequence. They are called as line codes.
- Fig. 2.5.4 shows two of such line codes called unipolar NRZ and polar NRZ, where NRZ is stands for non-return to zero.



(E-674) Fig. 2.5.4 : Two binary formats

2.5.5 Multiplexing in PCM Systems :

- It is possible to multiplex the PCM signals using the Time Division Multiplexing (TDM) principle.
- With increase in number of independent message sources the time duration allotted to each source has to be reduced in order to accommodate all the sources.
- This reduces the duration of each binary bit in a PCM code word. This increases the bandwidth requirement of the system.
- If the pulses (bits) become too short then the impairments in the transmission medium will not allow the proper operation of the system to take place.
- Therefore in practice it is necessary to restrict the number of message sources.

2.5.6 Synchronization in PCM :

- For a PCM system with TDM, it is necessary to synchronize the transmitter and receiver for proper operation of the system.
- For synchronization, it is necessary to synchronize the clocks at the transmitter and receivers.
- One way of synchronizing the transmitter and receiver clocks is to send a code element (one bit) or a pulse at the end of each frame which can be used by the receiver to synchronize its clock.
- The receiver will have a circuit which searches for this code element and thereby carry out the synchronization.

2.5.7 Quantizer Saturation :

- The quantizer allocates Q levels of approximations. The range of input voltage for which the difference between input and output is small is known as **operating range.**
- If the input voltage goes beyond this range, the difference between the quantized output and input becomes large and it is said that **quantizer saturation** has taken place.
- Objectionable errors are introduced due to quantizer saturation. We can avoid quantizer saturation by using the automatic gain control (AGC). This effectively increases the operating range of the converter.

2.6 Derivation of Expression for the Quantization Error : SPPU : Dec. 08

University Questions

Q. 1 Derive the expression for quantization error and signal to quantization noise ratio for a non-sinusoidal PCM system. **(Dec. 08, 6 Marks)**

- The input signal x(t) varies between the voltage levels V_H and V_L. Therefore the total variation in amplitude is given by,

$$\text{Total variation in amplitude} = V_H - V_L \quad ...(2.6.1)$$

- If we assume that

$$V_H = V \text{ and } V_L = -V \text{ then}$$

$$\text{Total change in signal amplitude} = 2V \text{ Volts} \quad ...(2.6.2)$$

- If this range is divided into "Q" levels of quantization then the step size is given by,

$$s = \frac{V_H - V_L}{Q} = \frac{2V}{Q} \quad ...(2.6.3)$$

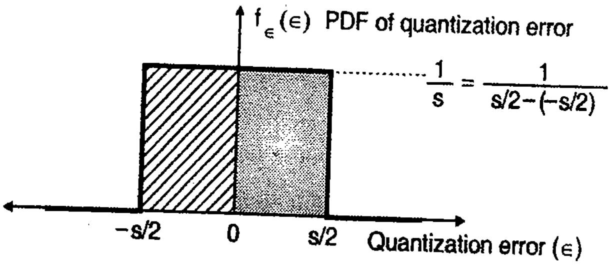
If we assume that $V_H = +1$ Volt and

$V_L = -1$ Volt then

$$s = 2/Q \quad ...(2.6.4)$$

- If the step size is assumed to be sufficiently small then we can assume that the quantization error is distributed uniformly and we can say that the quantization error is a random variable with "uniform distribution".
- As already seen the maximum quantization error is "$\pm s/2$". Therefore we can say that over the range $+s/2$ to $-s/2$, quantization error is a uniformly distributed random variable.
- The uniform distribution of quantization error is as shown in Fig. 2.6.1. The probability density function (PDF) for the quantization error "$\in$" is defined as,

$$\begin{aligned} f_\in(\in) &= 0 && \text{for } \in \leq s/2 \\ &= 1/s && \text{for } -s/2 \leq \in \leq s/2 \\ &= 0 && \text{for } \in > s/2 \end{aligned} \quad ...(2.6.5)$$



(E-675) **Fig. 2.6.1 : Uniform distribution for quantization error**

- The mean value or average value of the quantization error is zero.

The noise power is given by,

$$\text{Noise power} = \frac{V_n^2}{R} \quad ...(2.6.6)$$

where V_n^2 = Mean square noise voltage.

- We have defined the quantization noise as a random variable, with a probability density function (PDF) equal to $f_\in(\in)$, we can find the mean square value of noise voltage as,

$$\text{Mean square value} = E[\in^2] = \in^2 \quad ...(2.6.7)$$

$$= \int_{-\infty}^{\infty} \in^2 f_\in(\in)\, d\in \quad ...(2.6.8)$$

- But from Fig. 2.6.1 it is clear that the PDF $f_\epsilon(\epsilon)$ exists only over the range $-s/2$ to $+s/2$. Also substitute $f_\epsilon(\epsilon) = 1/s$ and change the limits of integration to get,

$$\overline{\epsilon^2} = \int_{-s/2}^{s/2} \epsilon^2 \times \frac{1}{s} d\epsilon \qquad ...(2.6.9)$$

$$= \frac{1}{s}[\epsilon^3/3]_{-s/2}^{s/2} = \frac{1}{s}\left[\frac{s^3}{24}+\frac{s^3}{24}\right] = \frac{s^2}{12} \qquad ...(2.6.10)$$

Thus, $\overline{V_n^2}$ = Mean square value of noise voltage $= \frac{s^2}{12}$

- If we substitute R = 1 ohm in Equation (2.6.6), then the noise power is called "normalized" noise power.

∴ Normalized noise power $= N_q$

$$= \frac{\overline{V_n^2}}{1} = \frac{s^2}{12}$$

Normalized quantization noise power :

$$N_q = \frac{s^2}{12} : \text{For linear quantization} \qquad ...(2.6.11)$$

This is the required expression.

2.7 Expression for the Maximum Signal to Quantization Noise Ratio[S/ N_q] :

SPPU : Dec. 08

University Questions

Q. 1 Derive the expression for quantization error and signal to quantization noise ratio for a non-sinusoidal PCM system. **(Dec. 08, 6 Marks)**

- The signal to quantization noise ratio can be defined as follows :

$$\left[\frac{P}{N_q}\right] = \frac{\text{Normalized signal power}}{\text{Normalized noise power}} \qquad ...(2.7.1)$$

- We have already obtained the value of normalized noise power (N_q).

$$N_q = \frac{s^2}{12} \qquad ...(2.7.2)$$

- Now we must obtain the expression for the normalized signal power.

Let us assume that the input signal x (t) is a sinusoidal signal with peak to peak voltage of "2V" Volts as shown in Fig. 2.7.1.

- The power dissipated in a resistance R by the signal voltage x (t) is given by,

$$P = \frac{(\text{Rms voltage})^2}{R} \qquad ...(2.7.3)$$

- The rms voltage of x (t) shown in Fig. 2.7.1 will be,

$$\text{rms voltage of x (t)} = \frac{V}{\sqrt{2}} \qquad ...(2.7.4)$$

- Substitute this value in Equation (2.7.3) to get,

$$\text{Signal power P} = \frac{(V/\sqrt{2})^2}{R}$$

$$\therefore \quad P = \frac{V^2}{2R} \qquad ...(2.7.5)$$

- Normalized signal power can be obtained by substituting R = 1 in Equation (2.7.5).

$$\therefore \textbf{ Normalized signal power, } P = \frac{V^2}{2} \qquad ...(2.7.6)$$

- Now substitute this value in Equation (2.6.11) along with the expression for normalized noise power "N_q" to get,

$$P/N_q = \frac{V^2/2}{s^2/12}$$

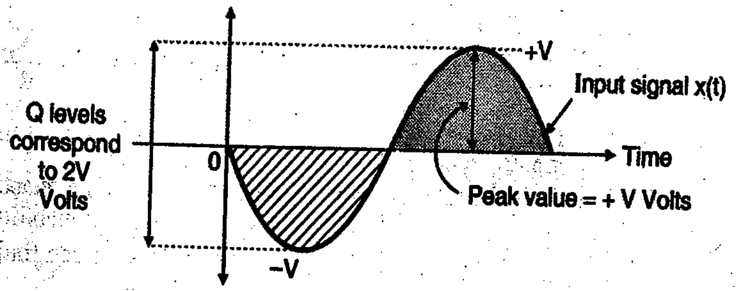
$$\therefore \quad [P/N_q] = \frac{6V^2}{s^2} \qquad ...(2.7.7)$$

But $s = \frac{2V}{Q}$ as already proved (Equation (2.6.3)).

$$\therefore \quad \left[\frac{P}{N_q}\right] = \frac{3}{2}Q^2 \qquad ...(2.7.8)$$

However, quantization levels $Q = 2^N$ where N is number of bits per word in PCM.

$$\therefore \quad \frac{P}{N_q} = \frac{3}{2}[2^N]^2 = \frac{3}{2}[2^{2N}] \qquad ...(2.7.9)$$



(E-676) Fig. 2.7.1 : Signal x(t)

Thus maximum signal to quantization noise ratio :

$$\frac{P}{N_q} = \frac{3}{2} \cdot 2^{2N} \qquad ...(2.7.10)$$

∴ Signal to quantization noise ratio in dB is given as,

$$SNR_q\ dB = \left[\frac{P}{N_q}\right]_{dB}$$

$$= 10 \log_{10}\left[\frac{3}{2} \times 2^{2N}\right]$$

$$= 10 \log_{10} (3/2) + 10 \log_{10} 2^{2N}$$

$$= 1.76 + 6N$$

$$\therefore\ SNR_q\ dB_{(max)} \approx (1.8 + 6\ N)\ dB$$

... For sinusoidal input signal ...(2.7.11)

This is the expression for the maximum signal to quantization noise ratio in dB when the signal is sinusoidal having a peak to peak voltage "2V" Volts.

Conclusion : Equation (2.7.11) indicates that the signal to quantization noise ratio SNR_q increases by 6 dB for every 1 bit increase in the number of bits in a PCM codeword.

2.8 Signaling Rate (Data Transfer Rate) and Transmission Bandwidth of PCM :

We know that, $Q = 2^N$ ...(2.8.1)

where, Q = Number of quantization levels

N = Number of bits per word

- The input signal x (t) is sampled at the sampling rate f_s, i.e. there are f_s number of samples per second. Each of these samples is then converted into an N bit digital word.

∴ Number of bits/sec. = Number of samples/sec × Number of bits/sample

$$= f_s \times N \qquad ...(2.8.2)$$

- But signaling rate is nothing but the number of bits per second.

∴ Signaling rate of PCM = $N\ f_s$ ...(2.8.3)

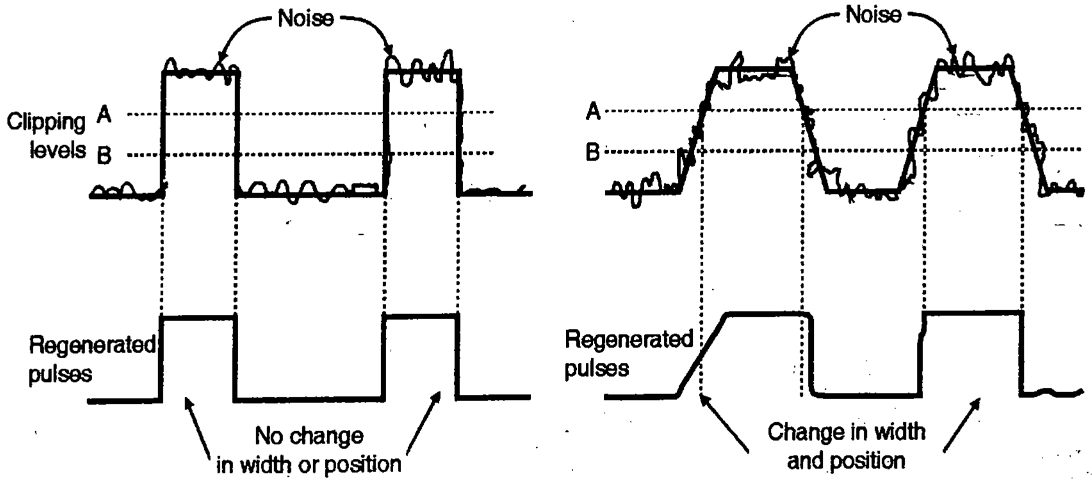
- The transmission bandwidth of PCM is equal to half the signaling rate.

∴ Transmission bandwidth of PCM $= \frac{1}{2}\ N\ f_s$ ...(2.8.4)

- This expression indicates that with increase in the value of N (number of bits per codeword in PCM) the required bandwidth also increases even though the signal to quantization noise ratio is decreasing.

2.9 Effect of Noise on the PCM System :

- Look at the two Figs. 2.9.1(a) and 2.9.1(b) which illustrate the effect of noise on the transmitted pulses.
- Consider Fig. 2.9.1(a) first. Due to the noise superimposed on the pulses, only the PAM system will be affected.
- However the PWM, PPM and PCM systems will remain unaffected. The regeneration of the pulses is achieved by using a clipper circuit with reference levels A and B.
- Now consider Fig. 2.9.1(b). Here the sides of the transmitted pulse are not perfectly vertical. In practice the transmitted pulses usually have slightly sloping sides (edges).
- As the noise is superimposed on them, the width and the position of the regenerated pulses is changed.
- Now this is going to distort the information contents in the PWM and PPM signals.
- But PCM is still unaffected as it does not contain any information in the width or the position of the pulses.
- Thus PCM has much better noise immunity as compared to PAM, PWM and PPM systems.



(a) (b)

(L-229) Fig. 2.9.1 : Effect of noise on PCM

2.9.1 Idle Channel Noise :

- The idle channel noise is the coding noise measured at the receiver output when the transmission input is equal to zero.
- The zero input condition situations can arise corresponding to pauses in speech.
- The average power corresponding to idle channel noise is dependent on the type of quantizer used.
- In a quantizer of midriser type, the zero input is quantized to ± s/2. If these two levels are assumed to be equiprobable, then the idle channel noise will have a zero mean and an average power of $s^2/4$.
- But if the quantizer is of midtread type, then the output is zero for zero input and hence the idle channel noise will be zero ideally.
- However for a practical midtread type quantizer, the output is never exactly equal to zero. Accordingly it is found that the average power of idle channel noise in a midtread type quantizer is also approximately equal to but slightly less than $s^2 / 4$.

2.10 Robust Quantization :

- In the previous section for uniform quantizer with a step size "s" it was shown that the variance of quantization noise or the normalized quantization noise power $N_q = \in^2 = s^2/12$.
- Thus the quantization noise is independent of the size of input signal. It is constant.
- As a result of this the signal to quantization noise ratio SNR_q decreases with decrease in the input signal power level. This is highly objectionable and unacceptable.
- In certain applications where PCM is used for the transmission of speech or music signals, this problem is very serious.
- Because the same quantizer has to accommodate the input signals of varying power levels. This happens because the range of voltages covered by a speech signals from maximum to minimum has a ratio of the order of 1000 : 1.
- Therefore the weak speech signals will have a small value of SNR_q and hence the PCM performance will degrade.
- Therefore it is desirable that SNR_q should remain essentially constant over a wide range of input power level.
- A quantizer that satisfies all these requirements is called as a **Robust Quantizer.**
- Such a robust performance can be obtained by using a **nonuniform quantization.**

2.10.1 Nonuniform Quantization :

SPPU : Dec. 08, May 09

University Questions

Q. 1 Explain the Non uniform quantization with help of neat diagram. **(Dec. 08, 8 Marks)**

Q. 2 Explain the need for non uniform quantization. **(May 09, 2 Marks)**

- If the quantizer characteristics is nonlinear and the step size is not constant instead if it is kept variable, dependent on the amplitude of input signal then the quantization is called as non uniform quantization.
- In non-uniform quantization, the step size is reduced with reduction in signal level. For weak signals (P < 1), the step size is small, therefore the quantization noise reduces, to improve the signal to quantization noise ratio for weak signals.
- The step size is thus varied according to the signal level to keep the signal to noise ratio adequately high. This is non-uniform quantization.
- The non-uniform quantization is practically achieved through a process called "companding". We will discuss companding in the next section.

Need of non-uniform quantization for speech signals :

- Non-uniform quantization is generally used for the speech and music signals.
- To understand the need of non-uniform quantization for the speech and music signals it is necessary to define an important parameter called "crest factor".
- Crest factor is defined as the ratio of peak amplitude to the rms amplitude of a signal.

$$\therefore \text{ Crest factor} = \frac{\text{Peak value}}{\text{rms value}} \quad ...(2.10.1)$$

- The value of crest factor is very high for the speech and music signals. Now let us see the effect of this high crest factor on the normalized power P.
- The destination signal power P is defined as,

$$P = \frac{\text{Mean square value of the signal}}{R}$$

$$\therefore \quad P = \frac{\overline{x^2(t)}}{R} \quad ...(2.10.2)$$

where, $\overline{x^2(t)}$ = Mean square value of the signal.

- The normalized signal power is obtained by substituting R = 1 in Equation (2.10.2).

∴ Normalized signal power

$$P = \overline{x^2(t)} \quad ...(2.10.3)$$

- The crest factor $= \frac{\text{Peak value}}{\text{rms value}}$

$$= \frac{x_{max}}{[\overline{x^2(t)}]^{1/2}} \quad ...(2.10.4)$$

- But $\overline{x^2(t)} = P$

$$\therefore \quad CF = \frac{x_{max}}{\sqrt{P}} \quad ...(2.10.5)$$

- Now if we normalize the signal i.e. if $x_{max} = 1$, then

$$CF = \frac{1}{\sqrt{P}}$$

$$\text{or} \quad P = \frac{1}{CF^2} \quad ...(2.10.6)$$

- The maximum possible value of the normalized power P is 1. Equation (2.10.6) shows that the normalized power P for the speech and music signal will be much less than 1 (which is its maximum possible value). This happens due to the high value of the crest factor.
- Equation (2.11.2) states that the signal to quantization noise ratio, for nonsinusoidal signals is given by,

$$\frac{S}{N_q} = 3 \times 2^{2N} \times P$$

- Hence if P <<< 1 then the signal to quantization noise ratio will reduce drastically. Thus for the speech and music signal having high crest factor, the signal to quantization noise ratio is poor which leads to degradation in the quality of sound.
- This problem can be overcome by use of non-uniform quantization. This is because in non-uniform quantization, the step size reduced with reduction in signal level.
- For weak signals (P << 1), the step size is small, therefore the quantization noise reduces, to improve the signal to quantization noise ratio for weak signals.
- The step size is thus varied according to the signal level to keep the signal to noise ratio adequately high. This is non-uniform quantization.
- The non-uniform quantization is practically achieved through a process called "companding". We will discuss companding in the next section.

2.11 Companding (Companded PCM) :

SPPU : Dec. 08, Dec. 12

University Questions

Q. 1 Explain the Non uniform quantization with help of neat diagram. **(Dec. 08, 8 Marks)**

Q. 2 What is the necessity of companding ? Explain the A law and μ law of companding graphically with expression. **(Dec. 12, 8 Marks)**

- Companding is non-uniform quantization. It is required to be implemented to improve the signal to quantization noise ratio of weak signals.
- The quantization noise is given by,

$$N_q = s^2 / 12$$

- This shows that in the uniform quantization once the step size is fixed, the quantization noise power remains constant.
- But the signal power is not constant. It is proportional to the square of signal amplitude. Hence signal power will be small for weak signals, but quantization noise power is constant.
- Therefore the signal to quantization noise ratio for the weak signals is very poor. This will affect the quality of signal. The remedy is to use **companding.**
- Companding is a term derived from two words, compression and expansion.

Companding = Compressing + Expanding

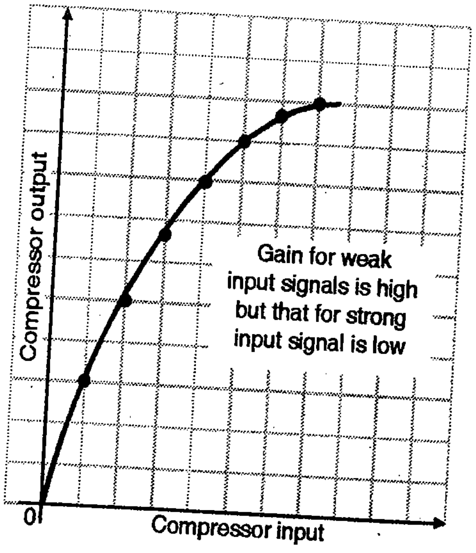
- Practically it is difficult to implement the non-uniform quantization because it is not known in advance about the changes in the signal level.
- Therefore a trick is used. The weak signals are amplified and strong signals are attenuated before applying them to a uniform quantizer.
- This process is called as "compression" and the block that provides it is called as a "compressor".
- At the receiver exactly opposite process is followed which is called expansion. The circuit used for providing expansion is called as an "expander".
- The compression of signal at the transmitter and expansion at the receiver is combined to be called as "companding".
- The process of companding is shown in the block diagram form in Fig. 2.11.1.

(D-479) **Fig. 2.11.1 : Model of companding**

2.11.1 Compressor Characteristics :

Fig. 2.11.2 shows the compressor characteristics.

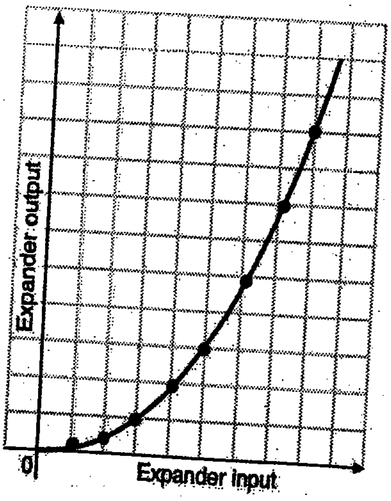
- As shown in Fig. 2.11.2, the compressor provides a higher gain to the weak signals and smaller gain to the strong input signals.
- Thus weak signals are artificially boosted to improve the signal to quantization noise ratio.
- Note that this compressor characteristics has been shown only for the positive input signal but we can draw it even for the negative input signals using the same principle.
- The compressor is included at the PCM transmitter.



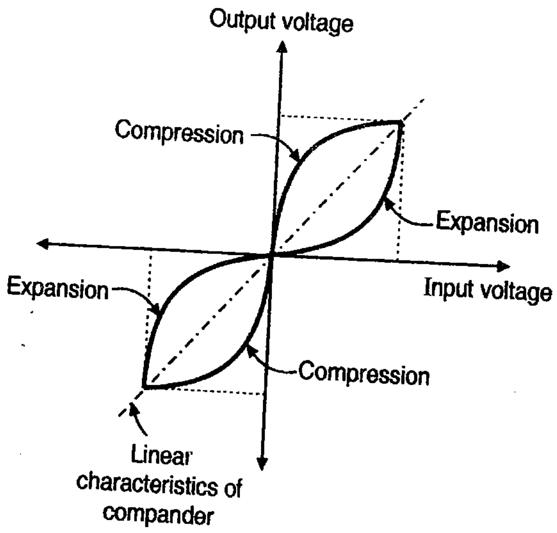
(D-480) **Fig. 2.11.2 : Compressor characteristics**

2.11.2 Expander Characteristics :

- The expander characteristics is shown in Fig. 2.11.3.



(D-481) **Fig. 2.11.3 (a) : Expander characteristics**



(D-481) **Fig. 2.11.3(b) : Companding curves for PCM**

- This characteristics is exactly the inverse of the compressor characteristics. It provides small gain of weak input signals and large gains for strong input signals.
- This ensures that all the artificially boosted signals by the compressor are brought back to their original amplitudes at the receiver.

2.11.3 Compander Characteristics :

- Fig. 2.11.3(b) shows the compander characteristics which is the combination of the compressor and expander characteristics.
- Due to the inverse nature of compressor and expander charactristics, the overall characteristics of the compander is a straight line (dotted line in Fig. 2.11.3(b)).
- This indicates that all the boosted signals are brought back to their original amplitudes.

Ex. 2.11.1 : **A voice signal bandlimited to 3.4 kHz is to be transmitted using PCM system. The signaling rate of the PCM is not to exceed 36000 bits/sec. Find :**

(a) Approximate value of f_s

(b) The number of quantization levels Q

(c) Number of digits (bits) per word N.

Soln. :

1. It is given that signaling rate

$$r \leq 36000$$

$$\therefore \; N f_s \leq 36000 \qquad \text{...(1)}$$

Minimum sampling frequency

$$f_{s\,(min)} = 2 \times f_M = 2 \times 3.4 \text{ kHz}$$

$$\therefore \; f_{s\,(min)} = 6.8 \text{ kHz} \qquad \text{...(2)}$$

2. Substitute this value of f_s in Equation (1) to get,

$$N \leq \frac{36000}{6.8 \times 10^3}$$

$$\therefore \; N \leq 10.29$$

So let us select $N = 5$ ...**Ans.**

The number of quantization levels

$$= \quad Q = 2^N = 2^5 = 32 \qquad \text{...Ans.}$$

3. Now with N = 5 let us calculate the maximum allowable value of sampling frequency $f_{s\,(max)}$.

$$f_{s\,(max)} = \frac{36000}{N}$$

$$= \frac{36000}{5} = 7.2 \text{ kHz} \qquad \text{...Ans.}$$

(a) f_s should be between 6.8 kHz and 7.2 kHz

(b) Number of quantization levels = 32

(c) N = 5.

Ex. 2.11.2 : In a binary PCM system, the output signal to quantization noise ratio is to be held to a minimum of 40 dB. First calculate the number of binary digits per word, necessary to meet this requirement and then find the actual value of the output signal to quantization noise ratio.

Soln. : Assuming the signal to be sinusoidal, we can use Equation (2.7.11) to calculate the number of digits per word.

$$\text{SNR}_q \text{ dB} = 1.8 + 6N$$

The minimum value of signal to quantization noise ratio is 40 dB

$$\therefore \quad \text{SNR}_q \geq 40 \text{ dB}$$

$$\therefore \quad 1.8 + 6\,N \geq 40$$

$$\therefore \quad N \geq 6.36$$

Hence let us have N = 7 ...Ans.

$$\therefore \quad \text{Actual value of} \left[\frac{S}{N_q}\right]_{dB} = 1.8 + (6 \times 7) = 43.8 \text{ dB}$$

...Ans.

Ex. 2.11.3 : An audio signal has spectral components present in the range of 300 Hz to 3300 Hz. A PCM signal is generated by sampling this audio signal at f_s = 8 kHz. The minimum value of signal to quantization noise ratio is 30 dB. Calculate :

(a) The minimum number of quantization levels Q and number of binary digits per word N.

(b) Signaling rate r.

(c) Minimum transmission bandwidth.

May 04. 6 Marks. Dec. 06. 8 Marks

Soln. :

$$\text{Given that} \quad f_s = 8 \text{ kHz and} \left[\frac{S}{N_q}\right] \geq 30 \text{ dB}$$

(a) To calculate Q and N :

Assuming that the input signal is sinusoidal we write,

$$\text{SNR}_q = 1.8 + 6\,N$$

$$\text{As} \quad \text{SNR}_q \geq 30 \text{ dB}$$

$$1.8 + 6\,N \geq 30 \text{ dB}$$

$$\therefore \quad N \geq 4.7$$

$\therefore$ Number of digits per word,

$$N = 5 \qquad \text{...Ans.}$$

Number of quantization levels

$$Q = 2^N = 2^5 = 32 \qquad \text{...Ans.}$$

(b) Signaling rate (r) :

$$(r) = N\,f_s = 5 \times 8 \text{ kHz} = 40 \text{ Kbits/sec.} \qquad \text{...Ans.}$$

(c) Transmission bandwidth :

$$B_T = \frac{1}{2} r = 20 \text{ kHz} \qquad \text{...Ans.}$$

Ex. 2.11.4 : A PCM system uses a uniform quantizer followed by a 7 bit encoder. The system bit rate is 50 Mbits/sec. Calculate the maximum bandwidth of the message signal for which this system operates satisfactorily.

Soln. : It has been given that : Bit rate r = 50 Mbits/sec and N = 7

We know that bit rate

$$r = N\,f_s$$

$$\therefore \quad f_s = \frac{r}{N} = \frac{50 \times 10^6}{7} = 7.14 \text{ MHz}$$

$\therefore$ Maximum signal bandwidth

$$BW = f_s/2 = \frac{7.14 \text{ MHz}}{2}$$

$$\therefore \quad BW = 3.57 \text{ MHz} \qquad \text{...Ans.}$$

Ex. 2.11.5 : The bandwidth of a video signal is 4.5 MHz. This signal is to be transmitted using PCM with the number of quantization levels Q = 1024. The sampling rate should be 20% higher than the Nyquist rate. Calculate the system bit rate.

May 98. Dec. 99. Dec. 01. 8 Marks

Soln. :

$$\text{Bandwidth } W = 4.5 \text{ MHz}$$

$$\therefore \quad \text{As per Nyquist rate } f_s = 2W = 9 \text{ MHz}$$

But f_s should be 20% higher than Nyquist rate

$$\therefore \quad f_s = 1.2 \times 9 \text{ MHz}$$

$$= 10.8 \text{ MHz} \qquad \text{...(1)}$$

We know that,

$$Q = 2^N,$$

$$\therefore \quad 1024 = 2^N$$

$$\therefore \quad N = 10 \qquad \text{...(2)}$$

$$\therefore \quad \text{System bit rate } r = N\,f_s$$

$$= 10 \times 10.8 \text{ MHz}$$

Ex. 2.11.6 : Derive the expression for the signal to quantization noise ratio of PCM system employing uniform quantization technique. Assume that input signal is of nonsinusoidal nature.

Soln. :

Expression for N_q : The signal to quantization noise ratio is defined as,

$$\frac{S}{N_q} = \frac{\text{Normalized signal power}}{\text{Normalized noise power}} \qquad ...(1)$$

In section 2.7 we have already derived the expression for the normalized quantization noise power. From Equation (2.6.11),

$$\text{Normalized noise power} = N_q = s^2/12 \qquad ...(2)$$

$$\text{But} \quad s = \frac{\text{Peak to peak signal amplitude}}{Q}$$

$$\text{Let, } 2x_{max} = \text{Peak to peak signal amplitude}$$

$$\therefore \quad s = \frac{2x_{max}}{Q}$$

$$\text{But } Q = 2^N$$

$$\therefore \quad s = \frac{2x_{max}}{2^N} \qquad ...(3)$$

Substitute Equation (3) into Equation (2) to get,

$$\therefore \quad N_q = \left[\frac{2x_{max}}{2^N}\right]^2 / 12 = \frac{4x_{max}^2}{2^{2N} \times 12}$$

$$\therefore \quad N_q = \frac{x_{max}^2}{3 \times 2^{2N}} \qquad ...(4)$$

Expression for S_i/N_q :

Let the signal power at the destination be represented by "P".

∴ Equation (1) can be written as,

$$\frac{S}{N_q} = \frac{S}{\left[x_{max}^2/3 \times 2^{2N}\right]} = \frac{3S \times 2^{2N}}{x_{max}^2}$$

∴ Signal to quantization noise ratio for nonsinusoidal

$$\text{signal} = \frac{3S}{x_{max}^2} \cdot 2^{2N} \qquad ...(2.11.1)$$

Now if the input signal x (t) is normalized i.e. $x_{max} = 1$ then Equation (2.11.1) gets modified to,

$$\frac{S}{N_q} = 3 \times 2^{2N} \times P \qquad ...(2.11.2)$$

And if the signal power "P" at the destination is also normalized,

$$\text{i.e. if} \quad P \leq 1 \qquad ...(2.11.3)$$

Then Equation (2.11.2) gets modified to,

$$SNR_q \leq 3 \times 2^{2N} \qquad ...(2.11.4)$$

Convert this equation in decibels as follows :

$$SNR_q \text{ (dB)} = 10 \log_{10}\left[\frac{S}{N_q}\right]$$

$$\text{as } \frac{S}{N_q} \text{ is a power ratio}$$

$$\therefore \quad SNR_q \text{ (dB)} \leq 10 \log_{10} \left[3 \times 2^{2N}\right]$$

$$SNR_q \text{ (dB)} \leq \left[10 \log_{10} 3 + 20\, N \log_{10} 2\right]$$

$$\therefore \quad SNR_q \text{ (dB)} \leq [4.8 + 6N] \qquad ...(2.11.5)$$

Hence,

Maximum signal to quantization noise ratio for normalized power P and input amplitude x (t) :

$$SNR_q = (4.8 + 6\,N) \text{ dB} \qquad ...(2.11.6)$$

Ex. 2.11.7 : An audio signal with highest frequency component 3300 Hz is pulse code modulated with a sampling rate of 8000 samples/sec. The required signal-to-quantization noise ratio is 40 dB.

1. What is the minimum number of uniform quantizing levels needed ?
2. What is the minimum number of bits per sample needed ?
3. Calculate the minimum number of bits per sample needed.

Dec. 16, 8 Marks

Soln. :

Given : $f_s = 8000$ Hz, $[S/N_q] = 40$ dB

1. To calculate N :

Assuming the input signal sinusoidal we write,

$$SNR_q = 1.8 + 6\,N$$

$$\therefore \quad 40 = 1.8 + 6\,N$$

$$\therefore \quad N = 6.37$$

$$\therefore \quad N = 7 \text{ bits/sample} \qquad \text{...Ans.}$$

2. To calculate Q :

$$\text{Quantization levels } Q = 2^N = 2^7$$

$$\therefore \quad Q = 128 \qquad \text{...Ans.}$$

2.11.4 Different Types of Compressor Characteristics :

SPPU : Dec. 11, Dec. 12

University Questions

Q. 1 Explain how companding improves the signal to noise ratio of PCM system with respect to µ-law ? **(Dec. 11, 8 Marks)**

Q. 2 What is the necessity of companding ? Explain the A law and µ law of companding graphically with expression. **(Dec. 12, 8 Marks)**

Ideally we need a linear compressor characteristic for small amplitudes of the input signal and a logarithmic characteristic elsewhere. Practically this is achieved by using two methods :

1. µ - law companding
2. A - law companding.

1. µ - Law companding :

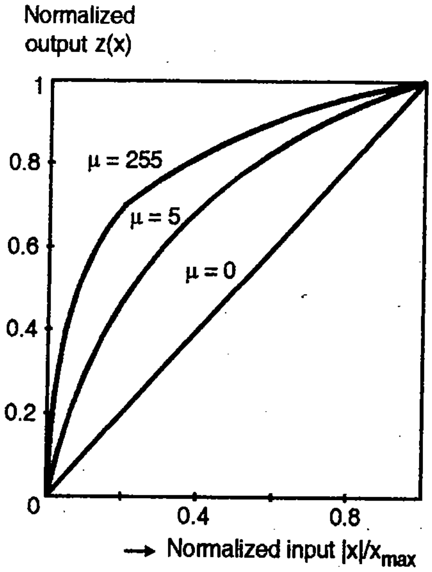
- In the µ-law companding, the compressor characteristic is continuous. It is approximately linear for smaller values of input levels and logarithmic for high levels of input signal.

- The μ-law compressor characteristic is mathematically expressed as,

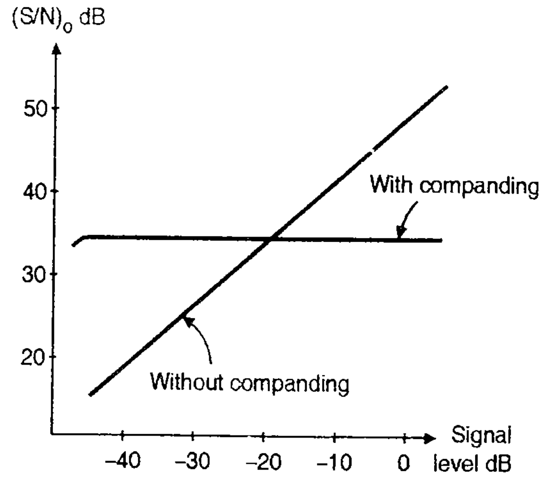
$$z(x) = (\text{sgn } x)\frac{ln(1+\mu|x|/x_{max})}{ln(1+\mu)} \quad ...(2.11.7)$$

where $0 \leq |x|/x_{max} \leq 1$.

- Here z (x) represents the output and x is the input to the compressor. | x | / xmax represents the normalized value of input with respect to the maximum value xmax. The (sgn x) term represents ± 1 i.e. positive and negative values of input and output.
- The μ -law compressor characteristics for different values of μ are as shown in Fig. 2.11.4(a). The practically used value of μ is 255.
- The characteristic corresponding to μ = 0 corresponds to the uniform quantization. It is a straight line. The μ-law companding is used for speech and music signals. It is used for PCM telephone systems in United States, Canada and Japan.
- Fig. 2.11.4(b) shows the variation of signal to quantization noise ratio with respect to signal level, with and without companding. It is clearly seen that SNR is almost constant at all the signal levels when companding is used.



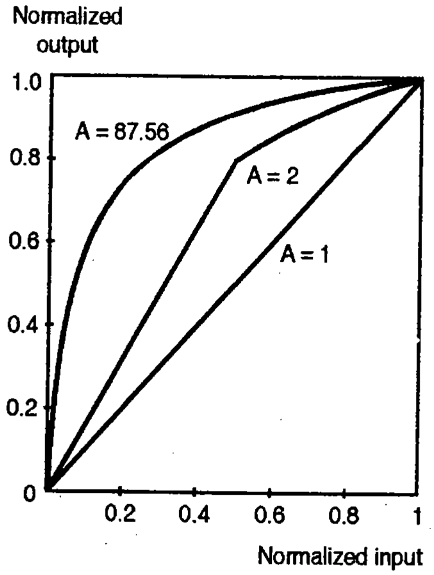
(D-482) **Fig. 2.11.4(a) : Compressor characteristic of a μ-law compressor**



(D-482) **Fig. 2.11.4(b) : PCM performance with μ-law companding**

2. A - Law companding :

- In the A-law companding, the compressor characteristic is of piecewise nature, made up of a linear segment for low level inputs and a logarithmic segment for high level inputs.
- Fig. 2.11.5 shows the A-law compressor characteristics for different values of A. Corresponding to A = 1 we observe that the characteristic is linear which corresponds to a uniform quantization.
- The practically used value of "A" is 87.56. The A-law companding is used for PCM telephone systems in Europe.



(D-483) **Fig. 2.11.5 : Compressor characteristics of A-law compressor**

- The linear segment of the characteristics is for low level inputs whereas the logarithmic segment is for high level input. It is mathematically expressed as,

$$\frac{z(x)}{x_{max}} = \begin{cases} \dfrac{A|x|/x_{max}}{1+\log_e A} & 0 \le \dfrac{|x|}{x_{max}} \le 1 \\ \dfrac{1+\log_e [A|x|/x_{max}]}{1+\log_e A} & \dfrac{1}{A} \le \dfrac{|x|}{x_{max}} \le 1 \end{cases} \qquad \text{...(2.11.8)}$$

2.11.5 Effect of Companding :

- Refer Fig. 2.11.6 which shows the curves of output signal to noise ratio (SNR_o) versus input signal power for the uniform and nonuniform quantizers.
- The curve for nonuniform quantizer corresponds to the μ law companding with μ = 255.
- The curves in Fig. 2.11.6 have been plotted with the following assumptions.

1. Value of Q is 256.
2. Parameter $\mu = 255$
3. Let X denote the random variable which represents the input. Let it has the **Laplacian distribution** which is represented mathematically as follows.

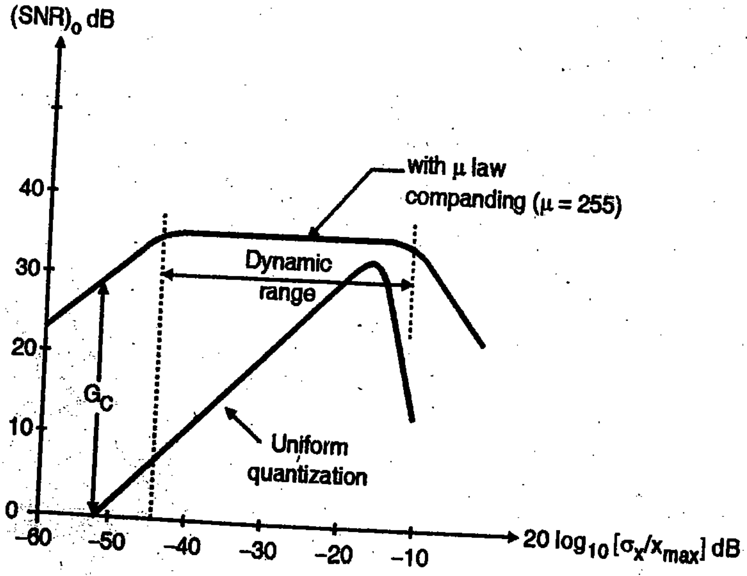
$$f_X(x) = \frac{1}{\sqrt{2}\,\sigma_x} e^{-2|x|/\sigma_x} \qquad \text{...(2.11.9)}$$

where σ_x^2 is the variance of X.

4. The input signal ranges between $-x_{max}$ to $+x_{max}$.

Dynamic range :

- Fig. 2.11.6 indicates that the performance of uniform quantizer is highly dependent on the input, whereas the μ law compander has a dynamic range of about 30 dB (from – 15 dB to – 45 dB).
- Dynamic range is defined as the range of input signal over which the output SNR remains within 3 dB of the maximum value of 38 dB.



(E-683) Fig. 2.11.6 : Output SNR characteristics for uniform and nonuniform quantization

2.11.6 Companding Gain :

- This is another important parameter used for assessing the improvement in the PCM performance due to the companding process.
- It is denoted by G_c and defined as,

$$G_c = \left. \frac{dz\,(x)}{dx} \right|_{x \to 0} \qquad ...(2.11.10)$$

where z (x) represents the compressor characteristics.

- The companding gain for a µ - law compressor is given by,

$$G_c = \frac{\mu}{\log_e (1 + \mu)} \qquad ...(2.11.11)$$

- If µ = 255, then

$$G_c = \frac{255}{\log_e (256)}$$

$$= 45.98 \qquad ...(2.11.12)$$

- Converting into dB, we get

$$G_c\,dB = 20 \log_{10} G_c = 33.3 \text{ dB}$$

- The value of companding gain should be as high as possible.
- The effect of the companding gain of 33.3 dB has been shown in Fig. 2.11.6. It shows that due to companding, the smallest step size with companding is smaller than the step size of a uniform quantizer by a factor equal to $G_c = 32$.

What is the difference between µ -law and A-law companding ?

The most important difference between the two types of compressors is that the A-law compressor has a midriser at the origin whereas the µ-law compressor has a midtread at the origin. Thus the A-law compressor has no zero value.

Ex. 2.11.8 : Plot the characteristics of a µ-law compressor. **May 97. 4 Marks**

Soln. :

To plot the characteristics of µ-law compressor :

The expression for the normalized output of a µ-law compressor is given by,

$$Z(x) = \pm \frac{ln\,(1 + \mu\,|x|)}{ln\,(1 + \mu)}$$

$$..... x \leq 1$$

Let $\mu = 255$.

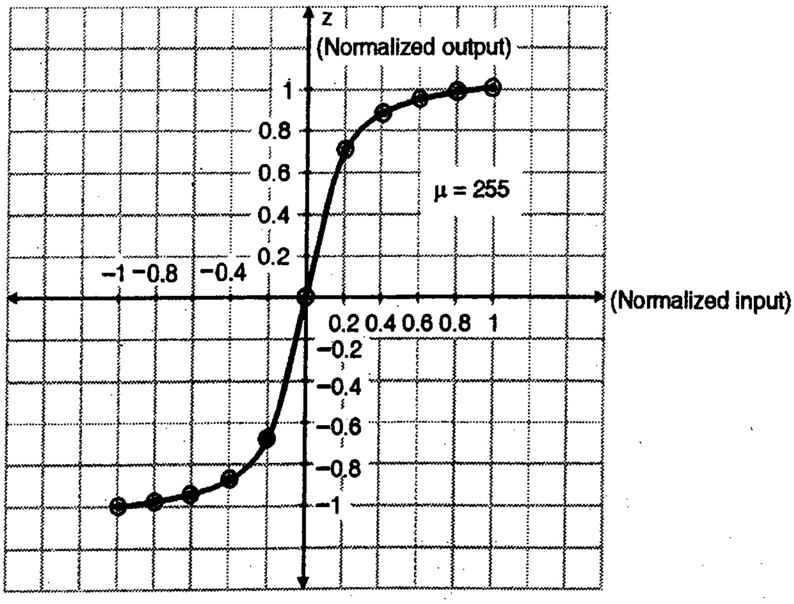
Note that instead of $|x| / x_{max}$ we have written only $|x|$ and restricted the values of x only upto 1. This has the same effect as that of normalizing.

$$\therefore \quad Z(x) = \pm \frac{ln\,(1 + 255\,|x|)}{ln\,256} \qquad ...(1)$$

Substitute x = 0, 0.2, 0.4, 0.6, 0.8 and 1 in Equation (1) to get the corresponding values of Z as shown in the following table.

|x|	0	0.2	0.4	0.6	0.8	1
Z	0	± 0.71	± 0.84	± 0.9	± 0.96	± 1

Therefore the compressor characteristics is as shown in Fig. P. 2.11.8.



(E-684) Fig. P. 2.11.8 : µ-law compressor characteristics

2.12 PCM with Noise : SPPU : Dec. 07, May 08

University Questions

Q. 1 Explain decoding noise in PCM. Derive its $\left(\frac{S}{N}\right)_D$ expression. **(Dec. 07, 8 Marks)**

Q. 2 Explain the following term with relevant mathematical expression and diagram : Noise performance of PCM. **(May 08, 5 Marks)**

- In order to judge the performance of the PCM system we have to consider two major sources of noise as follows :

  1. Channel noise 2. Quantization noise.

1. **Channel noise** may get introduced anywhere along the transmission path. It is also called as decoding noise.
2. **Quantization noise** as we have defined earlier, is introduced at the transmitter and is carried from the transmitter to the receiver output.

- Both of them are present simultaneously but we will consider them one by one to find their effect on the PCM system.

2.12.1 Channel Noise and its Effect :

- The major effect of channel noise is that it introduces transmission errors at the receiver when the PCM signal is being reconstructed.
- Due to such errors the receiver will make mistake in making the decision about whether a 0 was received or a 1 was received.
- A 0 may be mistaken as 1 and a 1 may be mistaken as 0. Such errors should be minimized so as to improve the fidelity of PCM system.

Error rate or probability of error (P_e) :

The quality of a PCM system in presence of channel noise can be measured in terms of error rate or probability of error. The probability of error or error rate is defined as the probability that the symbol at the receiver output is different from the one transmitted.

Expression for the error probability (P_e) :

- To obtain the expression for error probability (P_e) we will use the matched filter shown in Fig. 2.12.1 and assume that the type of noise is AWGN i.e. additive white gaussian noise.
- The matched filter and its performance has been discussed in chapter 5.
- Here we are going to only state the expression for the error probability (P_e) of PCM system as follows,

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{1}{2}\sqrt{\frac{E_{max}}{N_0}}\right] \quad ...(2.12.1)$$

Where E_{max} = Peak signal energy

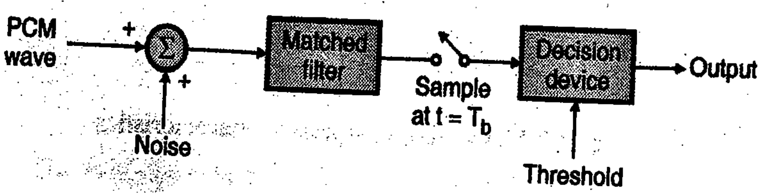
N_0 = Noise spectral density

- We can substitute $E_{max} = P_{max}\,T_b$ where P_{max} is the maximum or peak signal power and T_b is the bit duration. Hence the error probability in terms of power is as follows

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{1}{2}\sqrt{\frac{P_{max}\times T_b}{N_0}}\right]$$

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{1}{2}\sqrt{\frac{P_{max}}{N_0/T_b}}\right] \quad ...(2.12.2)$$

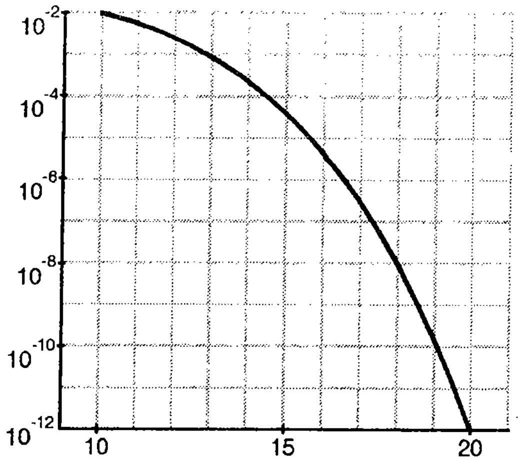
- The ratio N_0 / T_b can be seen as the average noise power contained in a transmission bandwidth equal to the bit rate ($1/T_b$).
- Hence E_{max} / N_0 may be viewed as the peak signal to noise power ratio.



(E-685) **Fig. 2.12.1 : Matched filter receiver for PCM**

2.12.2 Conclusions from Equation (2.12.2) :

1. The average probability of error in PCM receiver depends only on the ratio of peak signal energy E_{max} to the noise power spectral density N_0 measured at the receiver input.
2. The complementary error function "erfc" is a monotonically decreasing function. So erfc $\sqrt{\frac{E_{max}}{N_0}}$ will decrease with increase in the ratio (E_{max} / N_0) as shown in Fig. 2.12.2.



(E-686) **Fig. 2.12.2 : Probability of error in a PCM system**

2.12.3 Error Threshold :

- From Fig. 2.12.2, it is seen that the error probability decreases very rapidly as the value of this ratio E_{max} / N_0 is increased.

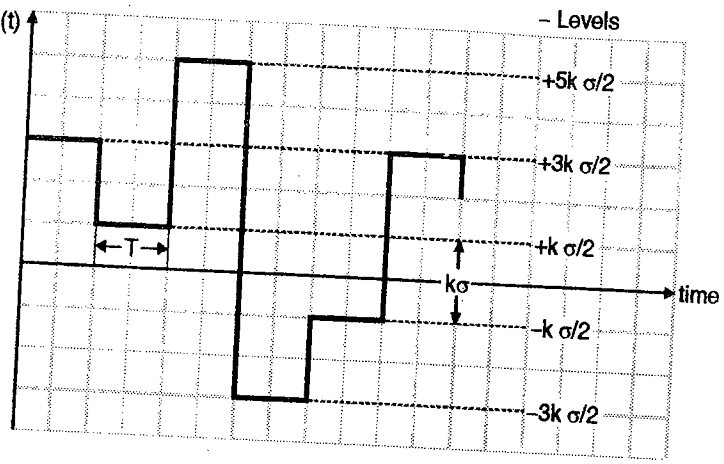
Table 2.12.1 : Effect of E_{max} / N_0 on probability of error

E_{max} / N_0	Probability of error P_e	For a bit rate of 10^5 bits per second this is about 1 error every
10.3 dB	10^{-2}	1 msec
14.4 dB	10^{-4}	0.1 sec
16.6 dB	10^{-6}	10 sec
18 dB	10^{-8}	20 min
19 dB	10^{-10}	1 day
20 dB	10^{-12}	3 months

- A very small increase in transmitted signal energy or power will make the reception of binary pulses almost error free.
- The effect of increase in the ratio E_{max} / N_0 has been demonstarted in Table 2.12.1.
- Table 2.12.1 shows that there is an error threshold at about 17 dB.
- That means when the value of E_{max}/ N_0 is higher than 17 dB, the error probability is very low. Whereas below this threshold the error probability is high and therefore the effect of noise is significant.
- The effect of channel noise can be reduced by using the regenerative repeaters.
- Another important characteristics of PCM system is its **ruggedness to interference.** For an on off signalling, the effect of noise is not seen unless the peak amplitude is greater than half the pulse height.
- Thus if adequate margin over the error threshold is provided, then the system can perform well even in presence of large amount of noise and interference.
- So we can say that PCM is a **noise resistant or rugged** system.

2.12.4 Average Transmitted Power for the Provision of Noise Margin :

- Consider an M-ary PCM system. It uses a codeword consisting of N code elements each one having M possible discrete amplitude values. Hence its name is M-ary PCM.
- In order to provide adequate noise margin as well as maintain a small error rate, there has to be a certain separation between these M discrete amplitude levels.
- Let the separation between the adjacent amplitude levels be k σ as shown in Fig. 2.12.3 where k is constant and $\sigma^2 = N_0 B$ i.e. the noise variance measured in a channel bandwidth B.
- Let us calculate the signal power P corresponding to this signal as follows,
- All the M messages are equally likely, therefore every message i.e. every level in the Fig. 2.12.3 is likely to occur atleast once.



(E-687) **Fig. 2.12.3**

- Therefore the average signal power can be obtained as follows :

$$\text{Average signal power (P)} = \frac{\text{Sum of the powers corresponding to individual messages}}{\text{Total number of messages}}$$

$$\therefore \ P = \frac{[k\,\sigma/2]^2 + [-k\,\sigma/2]^2 + [3\,k\,\sigma/2]^2 + [-3\,k\,\sigma/2]^2 + \ldots + \left[\frac{M-1}{2}k\,\sigma\right]^2 + \left[-\frac{M-1}{2}k\,\sigma\right]^2}{M} \qquad \ldots(2.12.3)$$

- In the above expression, the powers of individual messages are the "normalized" powers.
- The normalized power of the first message level $(k\,\sigma/2)$ is the power consumed by a $1\ \Omega$ resistance hence it would be $\frac{(k\,\sigma/2)^2}{1}$ or $(k\sigma/2)^2$. Similarly the normalized powers can be obtained for all the messages.

$$\therefore \ P = \frac{2}{M}\left\{\left[\frac{k\,\sigma}{2}\right]^2 + \left[\frac{3\,k\,\sigma}{2}\right]^2 + \ldots + \left[\frac{(M-1)\,k\,\sigma}{2}\right]^2\right\} \qquad \ldots(2.12.4)$$

$$\therefore \ P = \frac{M^2-1}{12}(k\,\sigma)^2 \qquad \ldots(2.12.5)$$

Conclusion :

We can draw two important observations from Equation (2.12.5) as follows :

1. For a prescribed noise variance σ^2 (constant) the average transmitted power P (required to operate above the error threshold) increases rapidly with increase in the number of levels M. Higher the number of levels M, higher the value of P required to achieve the same performance. This is a drawback.
2. If M = 2 corresponding to NRZ polar signaling we get

$$P = \frac{(2)^2-1}{12}k^2\sigma^2$$

$$\therefore \ P = \frac{k^2\sigma^2}{4} \qquad \ldots(2.12.6)$$

Thus for the same noise margin, the use of NRZ polar signaling needs one half of the average transmitted power required for NRZ unipolar signaling.

2.12.5 M-ary PCM Performance :

- In this section we will discuss the performance of M-ary PCM system by using the Shannon's channel capacity theorem.
- In order to obtain the expression for channel capacity we have to first obtain the expression for M i.e. the number of levels.
- Rearrange the Equation (2.12.5) for M to get,

$$M^2 = 1 + \frac{12\,P}{(k\,\sigma)^2}$$

$$\therefore \ M = \left[1 + \frac{12\,P}{(k\,\sigma)^2}\right]^{1/2} \qquad \ldots(2.12.7)$$

- The rms noise voltage is equal to σ. Hence the normalized noise power (i.e. in 1Ω resistance) will be $N = \sigma^2/1 = \sigma^2$.
- Therefore substitute $\sigma^2 = N$ i.e. noise power in the Equation (2.12.7) to get,

$$M = \left[1 + \frac{12\,P}{k^2 N}\right]^{1/2} \qquad \ldots(2.12.8)$$

Thus we have obtained the expression for M.

- Now substitute $N = N_0\,B$ to get

$$M = \left[1 + \frac{12\,P}{k^2 N_0\,B}\right]^{1/2}$$

where B is the bandwidth.

- The rate of information transmission system is given by

$$R = 2W \times N \log_2 M \text{ bits/s} \qquad \text{...(2.12.9)}$$

- But WN = B/2

$$\therefore \quad R = 2 \times \frac{B}{2} \log_2 M$$

$$\therefore \quad R = B \log_2 M \qquad \text{...(2.12.10)}$$

- Substituting the expression of M from Equation (2.12.7) we get,

$$\therefore \quad R = B \log_2 \left[1 + \frac{12}{k^2} \frac{P}{N_0 B} \right] \text{ bits/sec} \qquad \text{...(2.12.11)}$$

- From Shannon's theorem, for acceptable probability of error,

$$C \geq R$$

- That means the lowest value of C = R. Using this identity we get,

$$C = R = B \log_2 \left[1 + \frac{12}{k^2} \frac{P}{N_0 B} \right] \qquad \text{...(2.12.12)}$$

  This is the required result.

- The ideal system is described by Shannon's channel capacity theorem. It states the channel capacity C is given by,

$$C = B \log_2 \left[1 + \frac{P}{N_0 B} \right] \qquad \text{...(2.12.13)}$$

- Now compare Equations (2.12.12) and (2.12.13). If they are to be equal then the average transmitted power of the PCM system should be increased by a factor $k^2/12$ as compare to the ideal system.

Trade off between bandwidth and SNR :

- By Shannon Hartley theorem we get the channel capacity as,

$$C = B \log_2 \left[1 + \frac{P}{N} \right]$$

- Let us try to find out the maximum possible value of C. From the equation for "C" it is evident that it depends on two factors, which are the bandwidth "B" and the P/N ratio. Let us find their effect on "C" one by one.

Effect of S/N on C :

If the communication channel is noiseless then N = 0. Therefore $(P/N) \to \infty$ and so "C" also will tend to ∞. Thus the noiseless channel will have an infinite capacity.

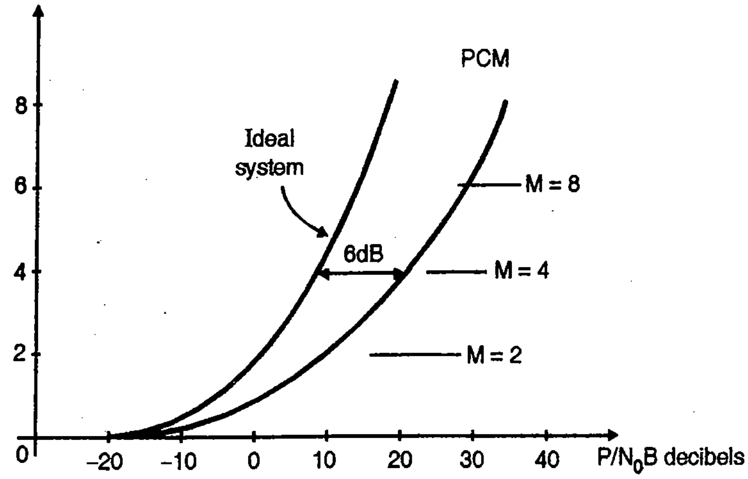
Effect of bandwidth B on C :

- Now consider that some white Gaussian noise is present hence (P/N) is not infinite. Now as the bandwidth approaches infinity the channel capacity does not become infinite since $N = N_0 B$ will also increase with the bandwidth B.
- This will reduce the value of (P/N) with increase in B, assuming the signal power P to be constant.
- Thus the ideal system operates under two constraints namely, limited bandwidth B and limited power P.
- However for the M-ary PCM we have to consider an additional constraint of a finite number of amplitude levels M.
- Fig. 2.12.4 is plotted as input signal to noise ratio P/N_0 B dB versus bandwidth efficiency R/B bits per second per hertz.
- In Fig. 2.12.4 we have substituted R = C and for an M-ary PCM system for different values of M, k has been substituted equal to 7.

- The constraint on M, in M-ary PCM tends to drive the M-ary PCM system into saturation when the bandwidth efficiency satisfies the following condition

$$\frac{R}{B} \leq 2 \log_2 M$$

- This expression tells us that with increase in the value of M, the saturation level goes on increasing.
- When a PCM system enters into saturations the error probability reaches its limiting value of 1.



(E-688) **Fig. 2.12.4 : Comparison of Mary PCM with the ideal system**

2.12.6 PCM versus Analog Modulation :

SPPU : Dec. 07, May 08, Dec. 08

University Questions

Q. 1 Explain decoding noise in PCM. Derive its $\left(\frac{S}{N}\right)_D$ expression. **(Dec. 07, 8 Marks)**

Q. 2 Explain the comparison of PCM with analog modulation with relevant methematical expression and diagram. **(May 08, 5 Marks)**

Q. 3 Explain the performance comparison of PCM and analog modulation with help of neat diagram. **(Dec. 08, 4 Marks)**

- The threshold effect in PCM is similar to a property of analog modulation methods such as FM or PPM.
- The property is that, these systems tend to reduce the wideband noise above their threshold levels.
- The PCM also provides the wideband noise reduction if it is operated above its threshold which is given by,

$$(S/N)_D = 3\,Q^2\,S_x$$

where $Q = 2^N$ for binary PCM and

$Q = M^N$ for M-ary PCM

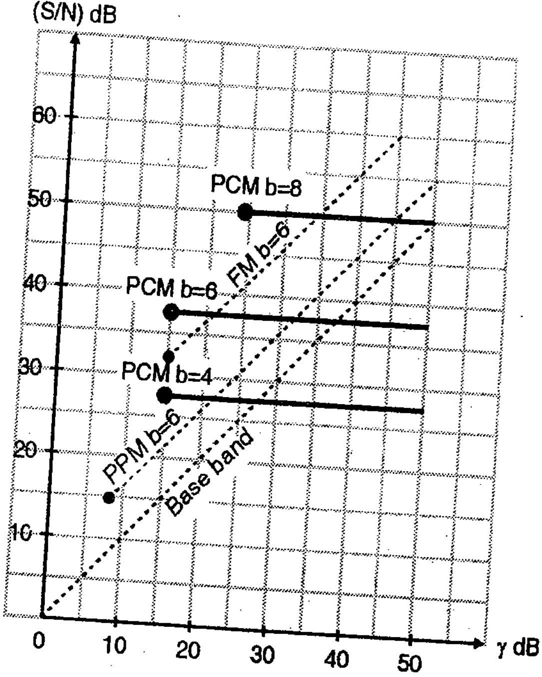
- We assume that the sampling frequency is close to the Nyquist rate and $B_T = NW$ Hz
- Then $Q = M^N = M^b$ where $b = B_T/W$ is called as the bandwidth ratio.

$$\therefore \quad (S/N)_D = 3\times(M^N)^2\,S_x = 3\times M^{2N}\,S_x = 3\,M^{2b}\,S_x$$

Here $(S/N)_D$ = Signal to noise ratio at the destination

S_x = Signal power at the destination

- Note that here the signal to noise ratio $(S/N)_D$ is proportional to M^{2b} which is much higher than the $(S/N)_D$ of the wideband FM which is proportional to only b or b^2.
- Therefore PCM performs better than FM in presence of same amount of noise.
- Fig. 2.12.5 shows the performance of various modulation types as a function of γ.
- All the curves in Fig. 2.12.5 have been plotted for $S_x = 1/2$. The dots indicate the threshold points.
- The PCM curves are drawn for M = 2 and N = b.



(E-689) **Fig. 2.12.5 : Comparison of PCM and analog communication**

Observations :

Some of the important observations from Fig. 2.12.5 are as follows :

1. For PCM if "b" is constant, then even if we increase γ beyond its threshold value γ_{th} (corresponding to the threshold point) the $(S/N)_D$ does not increase at all. See the flat PCM curves in Fig. 2.12.5. So PCM should be operated just above the threshold.
2. Near threshold, the PCM does offer some advantage over FM and PPM, if the values of "b" and $(S/N)_D$ are same.
3. But the price paid to gain this advantage is more complicated and expensive circuitry.
4. The $(S/N)_D$ for FM and PPM increases linearly with increase in the value of γ and becomes better than that of PCM for the higher values of γ.

Benefits of PCM :

Fig. 2.12.5 does not reveal the following benefits of using the PCM.

1. PCM allows the use of regenerative repeaters. This will improve its noise performance.
2. PCM allows the transmission of analog signals in the form of digital signals.

Why is PCM not used for broadcasting ?

- In radio broadcasting a relatively large signal to noise ratio (typically of the order of 60 dB) is required.
- To get this level of $(S/N)_D$ the PCM with b > 8 is required.
- However we can obtain the same performance with an FM system with b = 6 and with much simpler transmitter and receiver circuits.

- So higher bandwidth requirement and complicated circuitry are the disadvantages of PCM which does not allow its use for the radio, TV broadcasting applications.

2.13 Virtues Limitations and Modifications of PCM :

- The PCM is considered to be the best modulation scheme to transmit the voice and video signals.
- All the advantages of PCM are due to the fact that it uses coded pulses for the transmission of information.

2.13.1 Applications of PCM :

Some of the applications of PCM are as follows :

1. In digital telephone systems.
2. In the space communication, space craft transmits signals to earth. Here the transmitted power is very low (10 to 15W) and the distances are huge (a few million km). Still due to the high noise immunity, only PCM systems can be used in such applications.

2.13.2 Virtues of PCM :

1. Very high noise immunity.
2. Due to digital nature of the signal, repeaters can be placed between the transmitter and the receivers. The repeaters actually regenerate the received PCM signal. This is not possible in analog systems. Repeaters further reduce the effect of noise.
3. It is possible to store the PCM signal due to its digital nature.
4. It is possible to use various coding techniques so that only the desired person can decode the received signal. This makes the communication secure.
5. The increased channel bandwidth requirement for PCM is balanced by the improved SNR.

2.13.3 Limitations of PCM :

1. The encoding, decoding and quantizing circuitry of PCM is complex.
2. PCM requires a large bandwidth as compared to the other systems.

2.13.4 Modifications in PCM :

- Even though PCM is complex, it is possible to implement it using the VLSI technology.
- Due to the improvements in VLSI technology, the use of PCM for digital transmission of analog signals is going to increase.
- But if the simplicity is a more important issue, then one should use the **Delta Modulation** in place of PCM.
- The requirement of large channel bandwidth for PCM is not a real problem now, due to the availability of wideband communication channels.
- Due to the liberation from bandwidth contraint it has become possible to use the communication satellites and optical fiber communication.
- It is possible to remove the **redundancy** in PCM by means of using the data compression **techniques.** This will reduce the bit rate of transmitted data without serious degradation in the contents.
- This will increase the complexity of PCM further.

Why is PCM not used for broadcasting ?

- In radio broadcasting a relatively large signal to noise ratio (typically of the order of 60 dB) is required.
- To get this level of $(S/N)_D$ the PCM with $b > 8$ is required, where $b = B_T / W$ i.e. ratio of transmission bandwidth to baseband bandwidth.
- However we can obtain the same performance with an FM system with $b = 6$ and with much simpler transmitter and receiver circuits.
- So higher bandwidth requirement and complicated circuitry are the disadvantages of PCM which do not allow its use for the radio, TV broadcasting applications.

Ex. 2.13.1 : A low pass signal of 3 kHz bandwidth and amplitude over – 5 Volts to + 5 Volts range is sampled at Nyquist rate and converted to 8-bit PCM using uniform quantization. The mean squared value of message signal is 2 Volts-squared.

Calculate :

1. The normalised power for quantization noise.
2. The bit transmission rate.
3. The signal to quantization noise ratio in dB.

Derive the expressions used in 1 and 3.

Dec. 95, Dec. 04, 12 Marks

Soln. :

Given : $W = 3\text{ kHz},\ V_L = -5\text{ V},\quad V_H = +5\text{V}, N = 8$

Uniform quantization used.

$$\overline{x^2(t)} = \text{Mean square value of message signal is 2 Volt}^2$$

1. **Normalized power for quantization noise (N_q) :**

$$N_q = \frac{S^2}{12} \qquad \text{where } S = \text{Step size} \qquad ...(1)$$

$$\text{But} \quad S = \frac{V_H - V_L}{Q} = \frac{V_H - V_L}{2^N}$$

$$\therefore \quad S = \frac{5-(-5)}{2^8} = \frac{10}{256}$$

$$= 39.06\text{ mV} \qquad ...(2)$$

Substitute in Equation (1) to get,

$$N_q = \frac{(39.06 \times 10^{-3})^2}{12} = 127.15 \times 10^{-6}\text{ W} \qquad \textbf{...Ans.}$$

2. The bit transmission rate (r) :

The bit transmission rate or signaling rate is the number of bits transmitted by the PCM system per second.

$$\therefore \quad r = N f_s$$

As the signal is sampled at Nyquist rate, $f_s = 2W$.

$$\therefore \quad r = 8 \times 2W$$

$$\therefore \quad r = 16 \times 3 \text{ kHz}$$

$$= 48 \text{ kbits/sec} \qquad \text{...Ans.}$$

3. The signal to quantization noise ratio in dB :

The normalized signal power P

$$= \frac{\text{Mean square value of signal}}{1\Omega}$$

$$\therefore \quad P = \frac{2 \text{ Volt}^2}{1\,\Omega} = 2\text{Watt} \qquad \text{...(3)}$$

$$\therefore \quad (\text{SNR})_q = \frac{P}{N_q} = \frac{2}{127.15 \times 10^{-6}}$$

$$= 15728.64$$

$$\text{And } (\text{SNR})_q \text{ in dB} = 10 \log_{10} (15728.64)$$

$$(\text{SNR})_q = 41.96 \text{ dB} \qquad \text{...Ans.}$$

Ex. 2.13.2 : For a full scale sinusoidal modulating signal with peak value A, show that, output signal to quantization noise ratio in binary PCM system is given by, S/N = 1.76 + 20 log M dB where M = Number of quantization levels. A compact disc recording system samples each of the two stereo signals with a 16 bit A/D converter at 44.1 kb/sec.

(a) Determine output S/N ratio for a full scale sinusoid.

(b) The bit stream of digitized data is augmented by addition of error correcting bits, clock extraction bits etc. and these additional bits represent 100% overhead. Determine output bit rate of CD system.

(c) The CD can record an hour's worth of music. Determine number of bits recorded on CD.

Soln. :

Given : There are two stereo channels.

$$N = 16, f_s = 44.1 \text{ kbits/sec.}$$

(a) Output signal to noise ratio for full scale sinusoid :

$$\left[\frac{S}{N_q}\right] = 1.76 + 6\,N = 1.76 + (6 \times 16)$$

$$= 97.76 \text{ dB} \qquad \text{...Ans.}$$

(b) Output bit rate of the CD system :

The bit rate for each of two stereo channels = $N f_s$

$\therefore$ The bit rate of two channels

$$= 2\,N\,f_s$$

$$= 2 \times 16 \times 44.1 \times 10^3$$

$$= 1.4112 \text{ Mbits/sec} \qquad \text{...(1)}$$

Including the additional 100% overhead, the output bit rate is,

$$2 \times 1.4112 \times 10^6 \text{ b/s} = 2.822 \text{ Mbits/sec} \qquad \text{...Ans.}$$

(c) Number of bits recorded on CD :

The CD can record an hour's worth of music.

$\therefore$ Number of bits recorded on CD

$$= \text{Bit rate} \times \text{Number of seconds/hour}$$

$$= 2.822 \times 10^6 \times 3600$$

$$= 10.16 \times 10^9 \text{ bits or } 10.16 \text{ gigabits} \qquad \text{...Ans.}$$

Ex. 2.13.3 : A compact disc (CD) records audio signals digitally by PCM. Assume audio signal's bandwidth to be 15 kHz. If signals are sampled at a rate 20% above Nyquist rate for practical reasons and the samples are quantised into 65,536 levels, determine bits/sec required to encode the signal and minimum bandwidth required to transmit encoded signal.

Dec. 98. 6 Marks. Dec. 13. 8 Marks

Soln. :

$$W = 15 \text{ kHz},$$

$$f_s = 1.2 \times 2\,W = 2.4 \times 15 \text{ kHz} = 36 \text{ kHz},$$

$$Q = 65{,}536.$$

1. Signaling rate (r) :

$$\text{We know that } Q = 2^N$$

$$\therefore \quad N = \log_2 Q$$

$$\therefore \quad N = \frac{\log_{10} (65{,}536)}{\log_{10} 2} = 16 \qquad \text{...(1)}$$

$$\text{Signaling rate } r = N f_s$$

$$= 16 \times 36 \text{ kHz}$$

$$= 576 \text{ kbits/sec.} \qquad \text{...Ans.}$$

Thus the signaling rate r is 576 kbits/sec.

2. Minimum bandwidth :

$$B_T = \frac{1}{2}$$

$$\text{Signaling rate} = \frac{576}{2} \text{ kbits/sec}$$

$$\therefore \text{ Minimum bandwidth } B_T = 288 \text{ kHz} \qquad \text{...Ans.}$$

Ex. 2.13.4 : The information in an analog signal voltage waveform is to be transmitted over a PCM system with an accuracy of ± 0.1 % full scale accuracy. The analog voltage waveform has a bandwidth of 100 Hz and an amplitude range of – 10 to + 10 Volts.

(a) Determine the minimum sampling rate required.

(b) Determine the number of bits in each PCM word.

(c) Determine the minimum bit rate required in the PCM system.

(d) Determine the minimum absolute channel bandwidth required for the transmission of the PCM signal.

May 99. 8 Marks

Soln. : It has been given that,

1. Accuracy of ± 0.1 % of full scale is expected.
2. W = 100 Hz and amplitude range is – 10 to + 10 V

(a) Sampling rate f_s :

By sampling theorem the minimum sampling rate is

$$f_{s\,(min)} = 2W$$

$$= 200 \text{ Hz} \qquad \text{...Ans.}$$

(b) Number of bits per word (N) :

As accuracy is expected to be ± 0.1 % of full scale, the maximum quantization error should be ± 0.1 % of full scale.

$$\therefore \quad \epsilon_{max} = \pm 0.1 \% \text{ of full scale}$$

$$= \pm 0.001 [10 - (-10)]$$

$$= \pm 0.001 \times 20$$

$$\therefore \quad \epsilon_{max} = \pm 0.02 \text{ Volts} \qquad \text{...(1)}$$

We know that the maximum value of the quantization error is

$$\epsilon_{max} = \pm S/2 \qquad \text{...(2)}$$

$$\therefore \quad \pm S/2 = \pm 0.02$$

$$\therefore \quad S = 0.04 \text{ Volt} \qquad \text{...(3)}$$

$$\text{But } S = \frac{V_H - V_L}{Q}$$

$$\text{where } V_H = 10 \text{ V}$$

$$\text{and } V_L = -10 \text{ V}$$

$$\therefore \quad Q = \frac{10 + 10}{0.04}$$

$$= \frac{20}{0.04} = 500 \qquad \text{...(4)}$$

$$\text{But } Q = 2^N$$

$$\therefore \quad N \log_{10} 2 = \log_{10} 500$$

$$\therefore \quad N = 8.96 \approx 9 \qquad \text{...Ans.}$$

(c) System bit rate :

$$\text{System bit rate (r)} = N f_s$$

$$= 9 \times 200$$

$$= 1800 \text{ bits/sec} \qquad \text{...Ans.}$$

(d) Transmission channel bandwidth (B_T) :

$$B_T \geq \frac{1}{2} N f_s$$

$$\therefore \quad B_T \geq 900 \text{ Hz} \qquad \text{...Ans.}$$

Ex. 2.13.5 : The information in an analog waveform with a maximum frequency f_m = 3 kHz is to be transmitted over an M level PCM system, where the number of pulse levels is M = 16. The quantization distortion is specified not to exceed 1% of peak to peak analog signal.

1. What is the maximum number of bits per sample that should be used in this PCM system ?
2. What is the minimum sampling rate and what is the resulting bit transmission rate ?

Dec. 99. 6 Marks

Soln. :

Given : f_m = 3 kHz,

Number of quantization levels M = 16.

1. Number of bits/sample (N) :

We know that, number of quantization levels $M = 2^N$.

$$\therefore \quad 2^N = 16$$

$$\therefore \quad N = 4 \qquad \text{...Ans.}$$

2. Minimum sampling rate :

By sampling theorem :

$$f_{s\,(min)} = 2 f_m$$

$$= 2 \times 3 \text{ kHz}$$

$$\therefore \quad f_{s\,(min)} = 6 \text{ kHz} \qquad \text{...Ans.}$$

3. Bit transmission rate :

$$r = N f_s = 4 \times 6 \text{ kHz}$$

$$\therefore \quad r = 24 \text{ kbits/sec.} \qquad \text{...Ans.}$$

∴ Number of quantization levels,

$$Q = 2^N = 2^7 = 128 \qquad \text{...Ans.}$$

Ex. 2.13.6 : A TV signal with a bandwidth of 4.2 MHz is transmitted using binary PCM. The number of quantization levels is 512. Calculate :

1. Code word length
2. Transmission bandwidth
3. Final bit rate
4. Output signal to quantization noise ratio.

May 09, Dec. 13, 8 Marks

Soln. :

Given : $f_m = 4.2$ MHz and $Q = 512$.

1. Code word length (N) :

$$Q = 2^N$$

$$\therefore \quad N = \frac{\log 512}{\log 2}$$

$$\therefore \quad N = 9 \text{ bits/word}$$ **...Ans.**

2. Transmission bandwidth :

$$B_T = \frac{1}{2} N f_s = \frac{1}{2} N (2 f_m)$$

$$\therefore \quad B_T = 9 \times 4.2 \text{ MHz}$$

$$= 37.8 \text{ MHz}$$ **...Ans.**

3. Final bit rate (r) :

$$r = N f_s = 9 \times 2 \times f_m = 18 \times 4.2 \text{ MHz}$$

$$\therefore \quad r = 75.6 \text{ Mb/s}$$ **...Ans.**

4. Signal to quantization noise ratio :

Since the TV signal is not a sinusoidal signal, let us use the general expression of signal to quantization noise ratio.

$$\left[\frac{S}{N_q}\right] = 4.8 + 6 \text{ N dB} = 4.8 + (6 \times 9)$$

$$\therefore \quad \frac{S}{N_q} = 58.8 \text{ dB}$$ **...Ans.**

This is the maximum signal to noise ratio that we are expected to get from this system.

Ex. 2.13.7 : For a typical quantizer in a PCM system, amplitude of input signal m (t) is confined to the range of $(-M_p, +M_p)$. Assuming this range, divided in L zones, each of step size Δ, derive the expression for quantization error in terms of M_p and L. Sketch input/output characteristics of this quantizer. What are the advantages of non-uniform quantizing over uniform quantizing ? State the laws followed regarding non-uniform quantizing in practice.

May 2000, 10 Marks

Soln. :

Given :

1. Amplitude of input signal is confined to the range $(-M_p, +M_p)$.
2. This range is divided into L zones, each of step size Δ. We are asked to obtain the expression for quantization error N_q in terms of M_p and L. For this derivation, refer to section 2.4.6. There the expression for quantization noise power has been derived. Equation (2.4.3) states that,

Normalized quantized noise power,

$$N_q = \frac{S^2}{12} \qquad ...(1)$$

where S = Step size.

Here step size is Δ.

$$\therefore \quad N_q = \Delta^2/12 \qquad ...(2)$$

$$\text{But step size } \Delta = \frac{\text{Peak to peak amplitude of signal m (t)}}{\text{Number of quantization levels}}$$

$$\therefore \quad \Delta = \frac{M_p - (-M_p)}{L}$$

$$= \frac{2 M_p}{L} \qquad ...(3)$$

Substituting Equation (3) into Equation (2) we get,

$$N_q = \frac{[2 M_p/L]^2}{12} = \frac{4 M_p^2}{12 L}$$

$$\therefore \quad N_q = \frac{M_p^2}{3L}$$ **...Ans.**

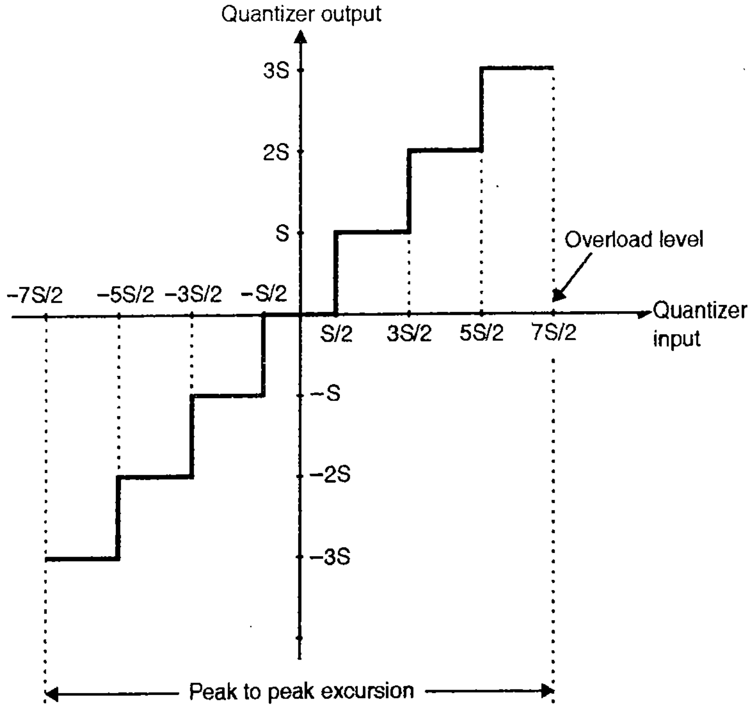
This is the expression for quantization noise.

This is a uniform quantizer and the input/output characteristic of the quantizer is as shown in Fig. P. 2.13.7.

Advantages of non-uniform quantizing over uniform quantizing are as follows :

1. Signal to quantization ratio almost remains constant as shown in Fig. P. 2.13.7 irrespective of the input signal strength.
2. Dynamic range increases. (Dynamic range is the range of input signal power in dB over which the output SNR is higher than about 30 dB).
3. Non-uniform quantizing is used for speech and music signals which have a high crest factor.

The laws followed for non-uniform quantization in practice are μ-law and A-law companding.



(E-723) **Fig. P. 2.13.7 : Input/output characteristic of the quantizer**

Ex. 2.13.8 : The output signal to noise ratio (SNR) of a 10 bit PCM was found to be 30 dB. The desired SNR is 42 dB. It was decided to increase the SNR to the desired value by increasing the number of quantization levels. Find fractional increase in the transmission bandwidth required for this SNR.

Dec. 2000. 6 Marks

Soln. :

Given : SNR = 30 dB, N = 10

Desired value of SNR = 42 dB.

1. To find "N" for SNR = 42 dB :

With increase in N by 1 bit the value of SNR increases by 6 dB. Therefore to increase the value of SNR by 12 dB it is necessary to increase N by 2.

$$\therefore \; N = 10 + 2 = 12 \quad \text{...Ans.}$$

2. Fractional increase in BW :

$$\text{BW of PCM system} = \frac{1}{2} N f_s$$

$$\therefore \; \text{BW with } N = 10 \text{ is given by,}$$

$$BW_{10} = \frac{1}{2} \times 10 f_s = 5 f_s$$

$$\text{And BW with } N = 12 \text{ is given by,}$$

$$BW_{12} = \frac{1}{2} \times 12 f_s = 6 f_s$$

$$\therefore \; \text{Change in BW} = \Delta BW$$

$$= 6 f_s - 5 f_s = f_s$$

$\therefore$ Fractional change in BW

$$= \frac{\Delta BW}{BW_{10}} = \frac{f_s}{5 f_s} \times 100 \%$$

$$= 20 \% \quad \text{...Ans.}$$

Ex. 2.13.9 : A television signal (video and audio) has a bandwidth of 4.5 MHz. This signal is sampled, quantized and binary coded to obtain a PCM signal.

1. Determine the sampling rate if the signal is to be sampled at a rate 20% above the Nyquist rate.
2. If the samples are quantized into 1024 levels, determine the number of binary

   Pulses required to encode each sample.
3. Determine the binary pulse rate (bits per second) of binary coded signal and the minimum bandwidth required to transmit the signal.

If above linear PCM system is converted to companded PCM, will the output bit rate changed ? Justify.

Dec. 01. 8 Marks

Soln. :

Given : $W = 4.5$ MHz

1. Sampling rate = 1.2 × Nyquist rate = 1.2 × 2 W

   $= 1.2 \times 2 \times 4.5 \times 10^6 = 10.8$ MHz ...Ans.

2. Given that number of quantization levels, Q = 1024

But $Q = 2^N$

$\therefore$ Number of binary pulse per word, $N = \log_2 Q$

$$\therefore \quad N = \log_2 1024$$

$$\therefore \quad N = 10 \qquad \text{...Ans.}$$

3. Binary pulse rate (bit rate)

$$= N f_s = 10 \times 10.8 \text{ MHz}$$

$$= 108 \text{ Mbps} \qquad \text{...Ans.}$$

$$\text{Bandwidth} = \frac{1}{2} \text{ bit rate} = 54 \text{ MHz} \qquad \text{...Ans.}$$

Ex. 2.13.10 : Express μ law of companding. For μ = 255 determine the maximum advantage over linear quantizer if the peak power to average power ratio is 9 and dynamic range of input signal is 30 dB and quantizer uses 256 levels. **Dec. 05, 10 Marks**

Soln. :

For μ law compounding refer section 2.11.4.

The advantage over linear quantization can be shown by calculating the output signal to noise ratio and the companding gain of the μ law compander.

1. **Signal to noise ratio :**

$$SNR_o = \frac{3\,Q^2}{[\log_e (1+\mu)]^2}$$

Where Q = Number of quantization levels.

$$\therefore \ SNR_o = \frac{3 \times (256)^2}{[\log_e (1+255)]^2}$$

$$= 6393.96 \text{ or } 38.05 \text{ dB}$$

2. **Companding gain :**

$$G_c = \frac{\mu}{\log_e (1+\mu)}$$

$$= \frac{255}{\log_e (1+255)} = 45.98$$

$$G_c \text{ (dB)} = 20 \log_{10} (45.98) = 33.25 \text{ dB.}$$

Ex. 2.13.11 : An analog waveform with bandwidth 15 kHz is to be quantized with 200 levels and transmitted via binary PCM signal. Find rate of transmission and bandwidth required. If 10 such signals are to be multiplexed find the bandwidth requirement. **May 06, 6 Marks**

Soln. :

Given : $f_m = 15$ kHz, $Q = 200$.

To find :

1. Rate of transmission 2. Bandwidth.

Step 1 : Find f_s :

$$f_s = 2f_m = 2 \times 15 \times 10^3 = 30 \times 10^3$$

Step 2 : Find N :

$$Q = 2^N$$

$$\text{Number of bits/sample } N = \log_2 Q = \frac{\log_{10} 200}{\log_{10} 2}$$

$$\therefore \ N = 7.6438 \approx 8$$

Step 3 : Calculate rate of transmission :

$$\text{Rate of transmission} = Nf_s$$

$$= 8 \times 30 \times 10^3 = 240 \text{ kbps}$$

Step 4 : Transmission bandwidth :

$$B = \frac{1}{2} Nf_s = \frac{1}{2} \times 240 = 120 \text{ kHz.}$$

Ex. 2.13.12 : A binary channel with bit rate 36 kbps is available for PCM voice transmission. Find appropriate values of number of quantization levels, number of bits per sample, and sampling frequency. Given that voice signal is band limited to 3.4 kHz. **May 06, 6 Marks**

Soln. :

Given : Bit rate = 36 kbps, $f_m = 3.4$ kHz

To find : 1. Q 2. N 3. f_s

1. Sampling frequency $f_s \geq 2f_m$

$$\therefore \ f_s \geq 2 \times 3.4 \text{ kHz}$$

$$\therefore \ f_s \geq 6.8 \text{ kHz}$$

For voice transmission the standard value of sampling frequency is $f_s = 8$ kHz.

2. Bit rate = Nf_s

$$\therefore \ 36 \times 10^3 = N \times 8 \times 10^3$$

$$\therefore \ N = 4.5 \approx 5$$

$$\therefore \ Q = 2^N = 2^5 = 32.$$

Ex. 2.13.13 : A signal of bandwidth 3.5 kHz is sampled and quantized and coded by a PCM system. The coded signal is then transmitted over a transmission channel with transmission rate 50 kbps. Calculate the maximum signal to Noise ratio that can be obtained by the system. The input signal has peak to peak value of 4 V and rms value of 0.2 V. **May 07, 8 Marks**

Soln. :

Given : f_m or W = 3.5 kHz, Bit rate = 50 kbps, $V_{PP} = 4$ V.

To find : $SNR_{q\,(max)}$.

- From the values of peak to peak voltage and rms voltage we conclude that the input signal is nonsinusoidal.
- So,

$$SNR_q \text{ (dB)}_{max} = 4.8 + 6\,N \qquad \text{...(1)}$$

$$\text{or } SNR_{q\,(max)} = 3 \times 2^{2N} \qquad \text{...(2)}$$

i.e. N = Number of bits per word.

Step 1 : Calculate N :

- Bit rate or transmission rate

$$r = N f_s$$

- For a voice signal $f_s = 8$ kHz is a standard value.

$$\therefore \quad 50 \times 10^3 = N \times 8 \times 10^3$$

$$\therefore \quad N = \frac{50}{8} = 6.25 \approx 7 \text{ bits}$$

Step 2 : Calculate SNR_q :

Substituting in Equation (1) we get,

$$SNR_q \, (dB)_{max} = 4.8 + (6 \times 7)$$

$$= 46.8 \text{ dB}$$

...Ans.

Ex. 2.13.14 : A signal band-limited to 1 MHz is sampled at a rate 50% higher than the Nyquist rate and quantized into 256 levels using a μ law quantizer with μ = 255.

1. Determine the signal to quantization noise ratio
2. The SNR found in part I was unsatisfactory. It must be increased by atleast 10 dB. Would you be able to obtain the desired SNR without increasing the transmission bandwidth. If it was found that a sampling rate 20% above the nyquist rate is adequate ? If so explain how. What is maximum SNR that can be realized in this way ?

Dec. 07, 10 Marks

Soln. :

f_m or $w = 1$ MHz, $f_s = 1.5 \times 2w = 3$MHz,

$Q = 256$, $\mu = 255$

1. **Signal to noise ratio :**

$$SNR_o = \frac{3Q^2}{[\log_e(1+\mu)]^2}$$

$$= \frac{3 \times (256)^2}{[\log_e(1+255)]^2}$$

$$= 6393.96 \text{ or } 38 \text{ dB}$$

2. The new sampling rate is $1.2 \times 2w = 2.4$ MHz.

The signal to noise ratio is dependent on Q and $Q = 2^N$ where N is number of bits per word.

If signal to noise ratio is to be increased by 10 dB then we have to increase Q to a new value Q′ as calculated below :

$$SNR_o' = 38 + 10 = 48 \text{ dB} = 63095.73$$

$$\therefore \quad 63095.73 = \frac{3\,(Q')^2}{[\log_e(256)]^2}$$

$$\therefore \quad Q' = 804.12$$

New value of number of bits/word = N′

$$= \log_2 Q' = \log_2 804.12$$

$$= 9.65 \approx 10$$

$$\therefore \quad \text{Bit rate} = N' f_s = 10 \times 2.4 \times 10^6$$

$$= 24 \text{ Mb/s}$$

$$\therefore \text{Transmission B.W.} = \frac{1}{2} \times \text{bit rate} = 12 \text{ Mb/sec}$$

$$\text{Old transmission B.W.} = \frac{1}{2} N f_{S1}$$

$$= \frac{1}{2} \times 8 \times 3 \text{ MHz} = 12 \text{ Mb/sec}$$

Conclusion : If we increase N from 8 bits to 10 bits then it is possible to increase SNR by 10 dB with increasing the transmission bandwidth.

$$\text{Maximum } SNR_o = \frac{2 \times (2^{10})^2}{[\log_e(256)]^2} = 68202.3 = 48.34 \text{ dB}$$

Ex. 2.13.15 : A signal m(t) bandlimited to 4 kHz is sampled at a rate 50% higher than Nyquist rate. The maximum acceptable error in the sample amplitude is 1% of peak amplitude. The quantized samples are binary coded. Find minimum bandwidth of a channel required to transmit the encoded binary signal.

Dec. 10, 8 Marks

Soln. :

Given : W = 4 Hz,

Maximum error in the sampled amplitude = 1% of peak amplitude

To find : Minimum channel bandwidth B_T.

1. **Sampling rate :**

$$f_s = 1.5 \times \text{Nyquist rate}$$

$$= 1.5 \times 2W$$

$$= 1.5 \times 2 \times 4 \text{ kHz}$$

$$= 12 \text{ kHz} \qquad ...(1)$$

2. **Number of bits per word :**

As the acceptable error is 1% is 0.01 of the peak amplitude, the maximum quantization error should be ± 1% of the peak.

$$\therefore \quad \epsilon_{max} = \pm 0.01 \, V_H$$

$$\text{Assume } V_H = 10 \text{ V}$$

$$\therefore \quad \epsilon_{max} = \pm 0.01 \times 10 = \pm 0.1 \text{ Volts}$$

$$\text{But } \epsilon_{max} = \pm S/2$$

$$\therefore \quad \pm 0.1 = \pm S/2$$

$$\therefore \quad S = 0.2 \text{ Volts}$$

$$\text{But } S = \frac{V_H - V_L}{Q}$$

$$\text{Let } V_L = -10 \text{ V}$$

$$\therefore \quad S = \frac{10-(-10)}{Q}$$

$$\therefore \quad Q = \frac{20}{0.2} = 100$$

$$\text{But } Q = 2^N$$

$$N \log_{10} 2 = \log_{10} Q$$

$$\therefore \quad 0.3010\,N = \log_{10} 100 = 2$$

$$\therefore \quad N = \frac{2}{0.3010} = 6.6445$$

Round off N to 7

∴ Number of bits per word i.e. $N = 7$ ...(2)

3. **System bit rate :**

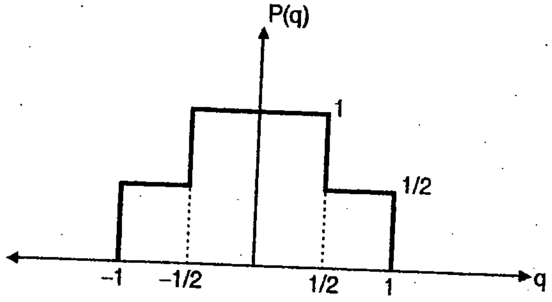
$$\text{System bit rate } r = N f_s = 7 \times 12 \text{ kHz} = 84 \text{ kbps} \quad ...(3)$$

4. **Minimum channel bandwidth :**

$$B_{T(\min)} = \frac{1}{2} \times r = \frac{1}{2} \times 84 \text{ kHz} = 42 \text{ kHz}$$ ...Ans.

Ex. 2.13.16 : In a 8 bit PCM scheme, the voice signal is sampled at a rate of 8 kHz. The maximum signal amplitude is 1V, voice signal bandwidth is 3.5 kHz. The quantization noise signal amplitude is uniformly distributed as shown in Fig. P. 2.13.16. Calculate the signal to noise ratio of the system.

May 11. 8 Marks



(E-1343) **Fig. P. 2.13.16**

Soln. :

Given : PCM system, $f_s = 8$ kHz, $V = 1V$, W or $f_m = 3.5$ kHz.

To find : Signal to noise ratio.

Step 1 : Calculate the noise power :

Mean square value of noise

$$= \int_{-1}^{-1/2} \in^2 \times \frac{1}{2} \times d\in + \int_{-1/2}^{1.2} \in^2 \times 1 \times d\in + \int_{1/2}^{1} \in^2 \times \frac{1}{2} \times d\in$$

$$= \left[\frac{\in^3}{3}\right]_{-1}^{-1/2} \times \frac{1}{2} + \left[\frac{\in^3}{3}\right]_{-1/2}^{1/2} + \frac{1}{2}\left[\frac{\in^3}{3}\right]_{1/2}^{1}$$

$$= \frac{1}{6}\left[-\frac{1}{8} + 1\right] + \frac{1}{3}\left[\frac{1}{8} - \left(-\frac{1}{8}\right)\right] + \frac{1}{6}\left[1 - \frac{1}{8}\right]$$

$$= 0.1458 + 0.08333 + 0.1458$$

$$\therefore V_n^2 = 0.375$$

$$\therefore \text{Normalized noise power } N_q = \frac{V_n^2}{1} = 0.375 \text{ W} \quad ...(1)$$

Step 2 : Calculate the signal power :

$$\text{Normalized signal power} = P = \frac{V_n^2}{1} = \frac{(1)^2}{2} = 0.5 \text{ W} \quad ...(2)$$

Step 3 : Signal to noise ratio :

$$(SNR)_q = \frac{P}{N_q} = \frac{0.5}{0.375} = 1.333$$ ...Ans.

$$(SNR)_q = 10 \log_{10}(1.333) = 1.25 \text{ dB}$$ ...Ans.

Ex. 2.13.17 : The information in an analog waveform, whose maximum frequency $f_m = 4000$ Hz is to be transmitted using a 16-level PAM system. The quantization distortion must not exceed ± 1% of the peak to peak analog signal.

(a) What is the minimum number of bits per sample that should be used in the transmission system ?

(b) What is the minimum required sampling rate and bit rate of the system.

(c) What is the 16-ary PAM symbol transmission rate ?

Dec. 11. 8 Marks

Soln. :

Given : $W = 4$ kHz, $Q = 16$,

$\in_{\max} = \pm 1$ % of peak to peak amplitude

Let peak amplitude of analog signal be 10 Volts.

(a) Number of bits/sample (N) :

Number of quantization levels $Q = 16$

But $Q = 2^N$ where N = Number of bits per sample

$$\therefore 16 = 2^N$$

$$\therefore N = 4 \text{ bits/sample}$$ ...Ans.

(b) Minimum sampling rate and bit rate of the system :

$$\text{Minimum sampling rate } f_{s(\min)} = 2W = 2 \times 4 \text{ kHz} = 8 \text{ kHz}$$ ...Ans.

$$\text{Bit rate of system } (r) = N \times f_{s(\min)} = 4 \times 8 \text{ kHz} = 32 \text{ kbits/sec}$$ ...Ans.

(c) Symbol transmission rate :

$$1 \text{ symbol} \equiv N \text{ bits}$$

$$\therefore \text{Symbol transmission rate} = \frac{\text{Bit rate (r)}}{N} = \frac{N f_s}{N} = f_s \text{ symbols/sec.} = 8000 \text{ symbols/sec}$$

Ex. 2.13.18 : A signal m (t) band-limited to 3kHz is sampled at a rate $33\frac{1}{3}\%$ higher than the Nyquist rate. The maximum acceptable error in the sample amplitude (the maximum quantization error) is 1% of peak amplitude m_p. The quantized samples are binary coded. Find the minimum bandwidth of a channel required to transmit the encoded binary signal. If 24 such signals are time-division-multiplexed, determine the minimum transmission bandwidth required to transmit the multiplexed signal. **Dec. 12, 8 Marks**

Soln. :

Given : W = 3 kHz,
Maximum quantization error = 1% of peak amplitude

To find : Minimum channel bandwidth B_T

(I)

1. **Sampling rate :**

f_s should be $33\frac{1}{3}\%$ higher than Nyquist rate

$$\begin{aligned} f_s &= 1.33 \times \text{Nyquist rate} \\ &= 1.33 \times 2\,W \\ &= 1.33 \times 2 \times 3 \text{ kHz} \\ &= 7.98 \text{ kHz} \quad ...(1) \end{aligned}$$

2. **Number of bits per word :**

Maximum quantization error is ± 1 % of peak

$$\therefore \in_{max} = \pm 0.01\,V_H$$

$$\text{Assume } V_H = 10\text{ V}$$

$$\therefore \in_{max} = \pm 0.01 \times 10 = \pm 0.1\text{ V}$$

$$\text{But } \in_{max} = \pm S/2$$

$$\therefore \pm 0.1 = \pm S/2$$

$$\therefore S = 0.2\text{ V}$$

$$\text{But } S = \frac{V_H - V_L}{Q}$$

$$\text{Let } V_L = -10\text{ V}$$

$$\therefore S = \frac{10-(-10)}{Q}$$

$$\therefore Q = \frac{20}{0.2} = 100$$

$$\text{But } Q = 2^N$$

$$N \log_{10} 2 = \log_{10} Q$$

$$0.3010\,N = \log_{10} 100 = 2$$

$$\therefore N = \frac{2}{0.3010} = 6.6445$$

$\therefore$ Number of bits per word i.e. N = 7

3. **System bit rate :**

System bit rate

$$\begin{aligned} r &= N f_s \\ &= 7 \times 7.98 \text{ kHz} \\ &= 55.86 \text{ kbps} \end{aligned}$$

4. **Minimum channel bandwidth :**

$$B_{T(min)} = \frac{1}{2} \times r = \frac{1}{2} \times 55.86 \text{ kHz}$$

$$= 27.93 \text{ kHz} \quad \text{...Ans.}$$

(II)

$$\text{Number of signals} = 24$$

$$N = 24$$

$$\text{Bit rate} = N f_s = 24 \times 7.98 \text{ kHz} = 191.52 \text{ kbps}$$

$$\text{Bandwidth} = \frac{1}{2}\text{ bit rate}$$

$$= 95.76 \text{ kHz} \quad \text{...Ans.}$$

2.14 Linear Delta Modulation (D.M.) :

SPPU : Dec. 09

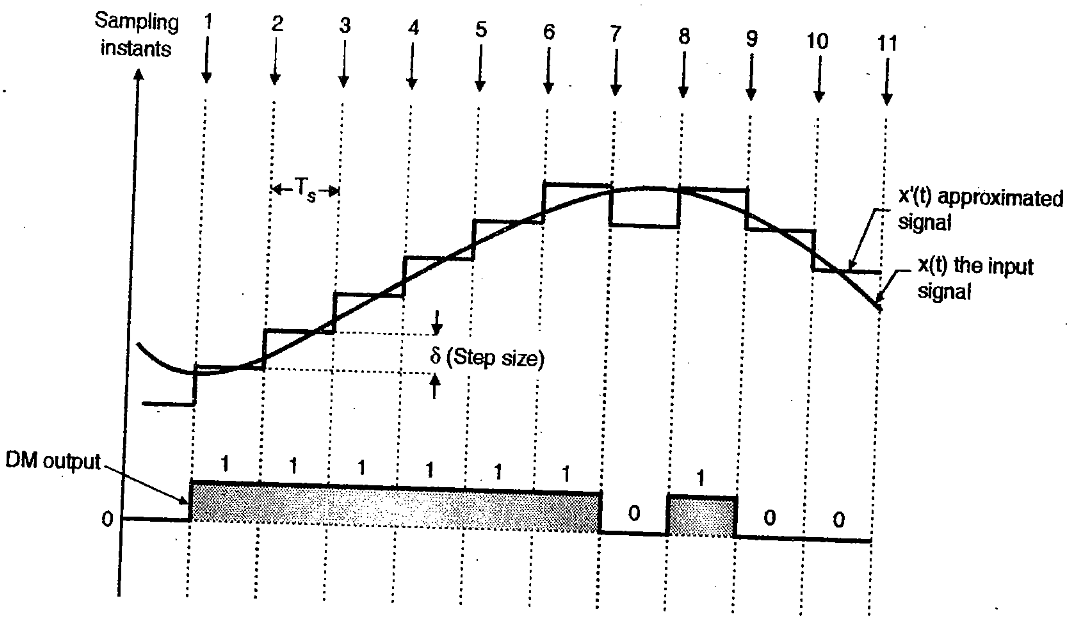
University Questions

Q. 1 Explain in detail principle of Delta Modulation system with block schematic and supporting waveforms. Derive expression for quantization noise in the same. **(Dec. 09, 8 Marks)**

- In PCM system, N number of binary digits are transmitted per quantized sample. Hence the signaling rate and transmission channel bandwidth of the PCM system are very large.
- These disadvantages can be overcome by using the delta modulation.

Principle of operation :

- Delta modulation transmits only one bit per sample instead of N bits transmitted in PCM. This reduces its signaling rate and bandwidth requirement to a great extent.



(L-235) Fig. 2.14.1 : D.M. Waveforms

- In the basic or linear D.M., a staircase approximated version of the sampled input signal is produced as shown in Fig. 2.14.1.
- The original signal and its staircase representation are then compared to produce a difference signal.
- And this difference signal is quantized into only two levels namely ± δ corresponding to positive and negative difference respectively.
- That means if the approximated signal x′ (t) lies below x (t) at the sampling instant, then the approximated signal is increased by "δ". (See instants 1, 2, 3, 4, 5 and 6 in Fig. 2.14.1.)
- Whereas if x′ (t) is greater than x (t) at the sampling instant, then x′ (t) is decreased by "δ" (see instants 7, 9 and 10 in Fig. 2.14.1.)

D.M. output :

- As shown in Fig. 2.14.1, the D.M. output is 1 if the staircase signal x′ (t) is increased by "δ" i.e. at sampling instants 1, 2, 3, 4, 5 and 6.
- Whereas D.M. output is 0 if x′ (t) is decreased by "δ" i.e. at sampling instants 7, 9 and 10.
- In delta modulation, the present sample value x (t) is compared with the approximate value x′ (t) and the result of this comparison is transmitted.
- Thus we are sending the information of whether the present sample value is higher than or lower than the approximate value. Note that the actual sampled value is not being transmitted.

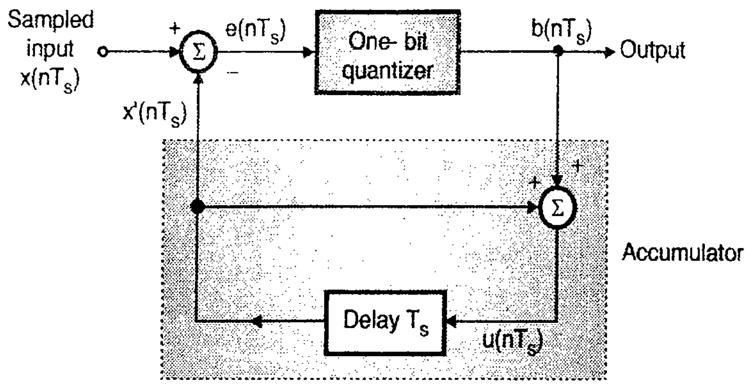
2.14.1 D.M. Transmitter :

SPPU : Dec. 10

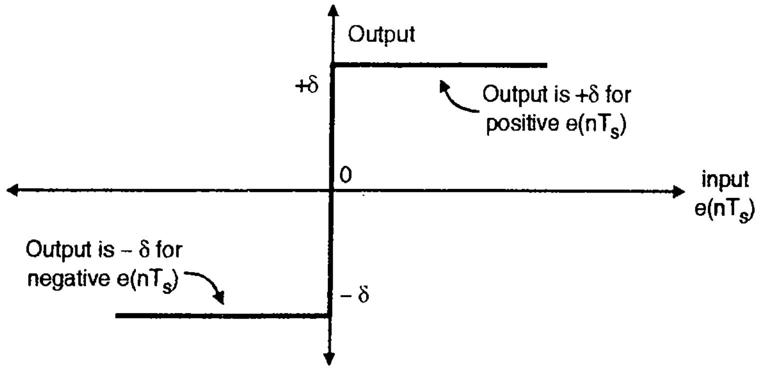
University Questions

Q. 1 Draw the block diagram of DM transmitter and explain its working. Comment on the drawbacks of DM. Explain how the drawback of accumulation of noise is eliminated by Delta-Sigma modulator ? **(Dec. 10, 10 Marks)**

- The block diagram of a D.M. transmitter is as shown in Fig. 2.14.2.



(E-693) **Fig. 2.14.2 : D.M. transmitter**



(E-694) **Fig. 2.14.3 : Input-output characteristic of two level quantizer**

- The sampled input signal x (n T_s) and its approximated version x' (n T_s) are compared with each other.
- The difference between them is produced as e (n T_s) i.e. error which is given by,

$$e(nT_s) = x(nT_s) - x'(nT_s)$$
$$= x(nT_s) - u(nT_s - T_s)$$

- Because x' (n T_s) = u (n T_s − T_s). The delay unit output is thus the previous value of x (n T_s).
- This is applied to a one bit quantizer that quantizes this difference e (n· T_s) into a two bit signal b (n T_s).

$$b(nT_s) = \delta \, \text{sgn}[e(nT_s)] \quad ...(2.14.1)$$

- If e (n T_s) is positive then sgn [e (n T_s)] is +1 and if e(nT_s) is negative then sgn [e (n T_s)] is − 1.

$$\therefore \quad b(nT_s) = \pm\delta$$

- The input-output characteristic of the two-level quantizer is shown in Fig. 2.14.3.
- The input to the delay unit in the transmitter is given by,

$$u(nT_s) = u(nT_s - T_s) + b(nT_s)$$

where u (n T_s − T_s) = x' (n T_s) and T_s is the sampling period.

- e (n T_s) is the prediction error that represents the difference between the present sample x (n T_s) and its previous approximate value x' (n · T_s) where $x'(nT_s) = u(nT_s - T_s)$.
- The binary quantity b (n T_s) at the output of D.M. transmitter is the algebraic sign of error e (n T_s) except for the scaling factor δ.
- Thus b (n T_s) is a single bit word transmitted by the D.M. system.
- The **accumulator** in the D.M. transmitter circuit is initially set to zero. The accumulator output is given by,

$$u(nT_s) = \delta \sum_{i=1}^{n} \text{sgn}[e(iT_s)] \quad ...(2.14.2)$$

But $\text{sgn}[e(iT_s)] = \pm 1$

$$\therefore \quad u(nT_s) = \delta \Sigma \pm 1 \quad ...(2.14.3)$$

Summary of operation of D.M. transmitter :

1. Sampled input x (n T_s) is applied at the input.
2. Accumulator produces a staircase approximated version x' (n T_s) of the previous sample value.
3. The prediction error is produced by taking the difference between x (n T_s) and x' (n T_s).

$$e(nT_s) = x(nT_s) - x'(nT_s)$$

4. The prediction error is quantized into a single bit (0 or 1) by the one bit quantizer.
5. The quantizer output is transmitted as D.M. signal. The accumulator uses the same signal to produce the delayed approximate signal x' (t).

2.14.2 Delta Modulator Transmitter (Another Method) : SPPU : Dec. 10, Dec. 15

University Questions

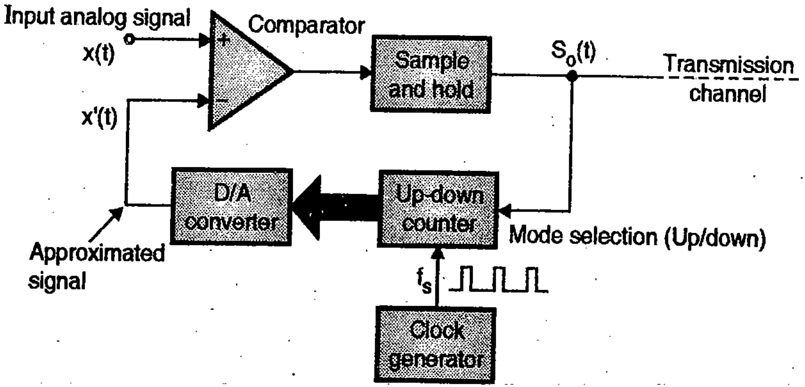
Q. 1 Draw the block diagram of DM transmitter and explain its working. Comment on the drawbacks of DM. Explain how the drawback of accumulation of noise is eliminated by Delta-Sigma modulator ? **(Dec. 10, 10 Marks)**

Q. 2 Draw the block diagram of DM transmitter and explain its working. Comment on the drawbacks of DM. **(Dec. 15, 6 Marks)**

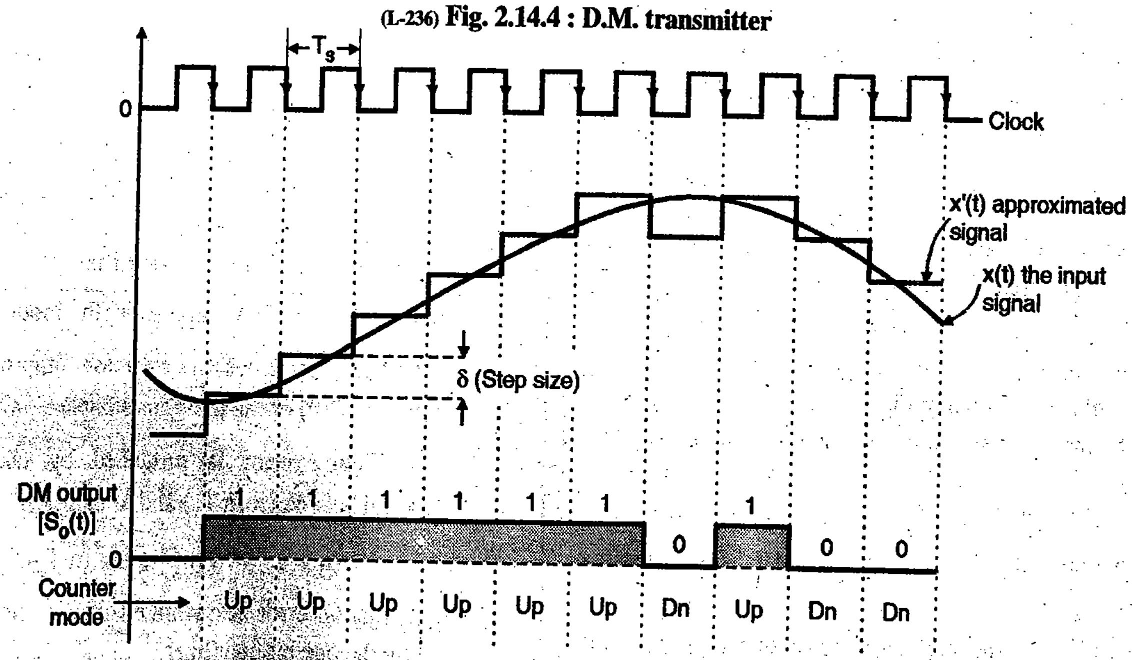
The block diagram of a delta modulator transmitter is as shown in the Fig. 2.14.4.

The operation of the circuit is as follows :

- x (t) is the analog input signal and x′ (t) is the quantized (approximated) version of x (t). Both these signals are applied to a comparator.
- The comparator output goes high if x (t) > x′ (t) and it goes low if x (t) < x′ (t). Thus the comparator output is either 1 or 0. The sample and hold circuit will hold this level (0 or 1) for the entire clock cycle period.
- The output of the sample and hold circuit is transmitted as the output of the DM system. Thus in DM, the information which is transmitted is only whether x (t) > x′ (t) or vice versa. Also note that one bit per clock cycle is being sent. This will reduce the bit rate and hence the BW.
- The transmitted signal is also used to decide the mode of operation of an up/down counter. The counter output increments by 1 if S_0 (t) = 1 and it decrements by 1 if S_0 (t) = 0, at the falling edge of each clock pulse. This is as shown in the waveform in the Fig. 2.14.5.
- The counter output is converted into analog signal by a D to A converter. Thus we get the approximated signal x′ (t) at the output of the D to A converter.



(L-236) **Fig. 2.14.4 : D.M. transmitter**



(L-237) **Fig. 2.14.5 : D.M. waveforms**

2.14.3 D.M. Receiver :

- The block schematic of D.M. receiver is shown in Fig. 2.14.6.
- The D.M. signal is passed through the accumulator to produce the staircase approximation in a manner similar to that used at the transmitter.
- The accumulator output is then applied to a low pass filter to produce the original signal.

2.14.4 D.M. Receiver (Alternate Method) :

- The block diagram of the D.M. receiver is as shown in Fig. 2.14.7.
- Compare it with the transmitter block diagram, you will find that it is identical to the chain of blocks producing the signal x′ (t) i.e. the approximated signal.
- The original modulating signal can be recovered back by passing this signal through a low pass filter.

2.14.5 Comparison of D.M. and DPCM :

- The comparison of D.M. and DPCM systems revels that except for an output low pass filter, they are identical 60 D.M. is actually a special case of DPCM.

2.14.6 Features of D.M. :

- The output codeword consist of only one bit. Hence no need of framing.
- Simplicity of design for transmitter and receiver.
- Less bit rate and lower bandwidth.

2.14.7 Applications of D.M. :

- For some types of digital communications.
- For digital voice storage.

2.14.8 Quantization Noise (Distortions) in the DM System :

SPPU: Dec. 09, Dec. 15

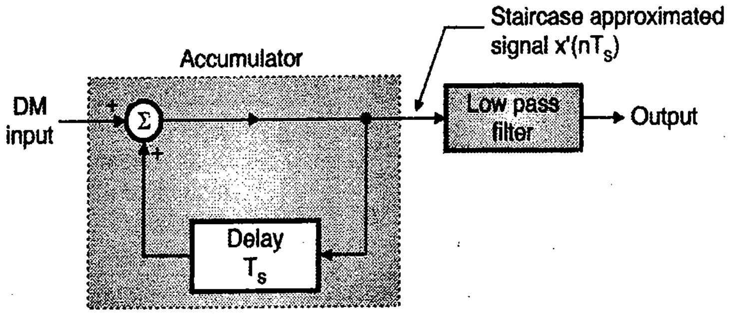
University Questions

Q. 1 Explain in detail principle of Delta Modulation system with block schematic and supporting waveforms. Derive expression for quantization noise in the same. **(Dec. 09, 8 Marks)**

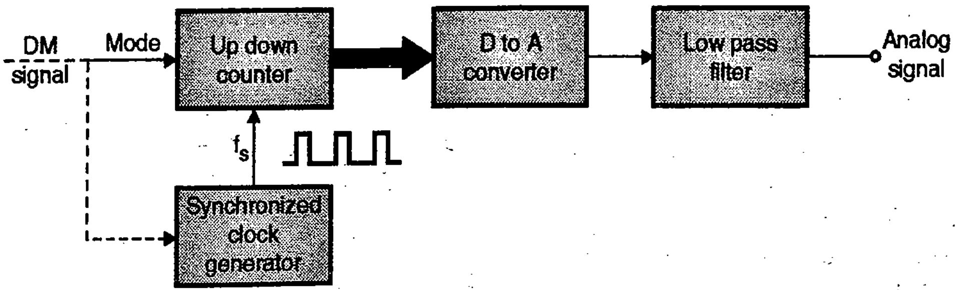
Q. 2 Draw the block diagram of DM transmitter and explain its working. Comment on the drawbacks of DM. **(Dec. 15, 6 Marks)**

The DM system is subjected to two types of quantization error or distortions :

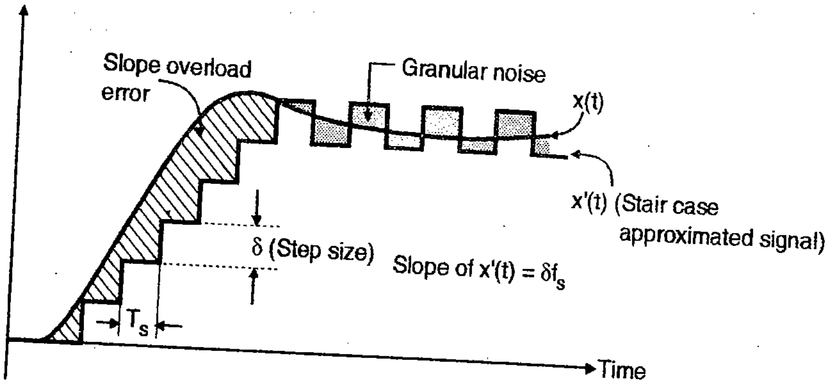
1. Slope overload distortion and
2. Granular noise.



(E-697) **Fig. 2.14.6 : D.M. receiver**



(L-238) **Fig. 2.14.7 : D.M. receiver**



(L-239) **Fig. 2.14.8 : Distortions in D.M.**

1. Slope overload distortion :

- Look at the Fig. 2.14.8 Due to small step size (δ), the slope of the approximated signal x′ (t) will be small.

The slope of x′ (t) $= \frac{\delta}{T_s} = \delta f_s$ ...(2.14.4)

- If slope of the input analog signal x (t) is much higher than that of x′ (t) over a long duration then x′ (t) will not be able to follow the variations in x (t), at all.
- The difference between x (t) and x′ (t) is called as the slope overload distortion.
- Thus the slope overload error occurs when slope of x (t) is much larger than slope of x′ (t).
- The slope overload error can be reduced by increasing slope of the approximated signal x′ (t).
- Slope of x′ (t) can be increased and hence the slope overload error can be reduced by either increasing the step size "δ" or by increasing the sampling frequency fs.
- However with increase in δ the granular noise increases and if f_s is increased, signaling rate and bandwidth requirements will go up. Thus reducing the slope overload error is not easy.

2. Granular noise :

- When the input signal x (t) is relatively constant in amplitude, the approximated signal x′ (t) will fluctuate above and below x (t) as shown in Fig. 2.14.8. The difference between x (t) and x′ (t) is called as granular noise.
- The granular noise is similar to the quantization noise in the PCM system.
- It increases with increase in the step size δ. To reduce the granular noise, the step size should be as small as possible.
- However this will increase the slope overload distortion.
- In the linear delta modulator the step size δ is not variable. If it is made variable then the slope overload distortion and granular noise both can be controlled.
- A system with a variable step size is known as the adaptive delta modulator (ADM).

D.M. bit rate (signaling rate) and bandwidth :

- D.M. bit rate (r)
  = Number of bits transmitted / second
  = Number of samples/sec × Number of bits/sample
  $= f_s \times 1 = f_s$ ...(2.14.5)
- Thus the D.M. bit rate is (1/N) times the bit rate of a PCM system, where N is the number of bits per transmitted PCM codeword.
- Hence the channel bandwidth for the D.M. system is reduced to a great extent as compared to that for the PCM system.
- The bandwidth of D.M. system $= \frac{1}{2} \times$ Bit rate $= \frac{1}{2} f_s$

2.14.9 Advantages of Delta Modulation :

SPPU : May 12

University Questions

Q. 1 What are the advantages of DM over PCM in terms of signalling rate and bandwidth requirement ? Derive the output S/N ratio of delta modulator. Brief the condition to avoid slope overload error. **(May 12, 8 Marks)**

1. Low signaling rate and low transmission channel bandwidth, because in delta modulation, only one bit is transmitted per sample.
2. The delta modulator transmitter and receiver are less complicated to implement as compared to PCM.

2.14.10 Disadvantages of Delta Modulation :

SPPU : Dec. 07, Dec. 15, May 16

University Questions

Q. 1 Draw the block diagram of DM transmitter and explain its working. Comment on the drawbacks of DM. **(Dec. 15, 6 Marks)**

Q. 2 What are the limitations of delta modulation ? How are they overcome in delta sigma modulation and adaptive delta modulation ? Explain with necessary diagrams. **(Dec. 07, 8 Marks, May 16, 7 Marks)**

1. The two distortions discussed earlier i.e. slope overload error and granular noise are present.
2. Practically the signaling rate with no slope overload error will be much higher than that of PCM.

The slope overload error can be reduced by using another type of delta modulation, called as Adaptive Delta Modulation (ADM).

2.14.11 Condition for Avoiding the Slope Overload Error :

SPPU : May 12

University Questions

Q. 1 What are the advantages of DM over PCM in terms of signalling rate and bandwidth requirement ? Derive 'the output S/N ratio of delta modulator. Brief the condition to avoid slope overload error. **(May 12, 8 Marks)**

Refer the following example to derive the condition for avoiding the slope overload error.

Ex. 2.14.1 : Consider a sinusoidal signal $x(t) = A\cos(\omega_m t)$ applied to a delta modulator with a step size δ. Show that the slope overload distortion will occur if $A > \frac{\delta}{\omega_m T_s} = \frac{\delta}{2\pi}\left(\frac{f_s}{f_m}\right)$ where T_s is the sampling period.

May 09, Dec. 12, 8 Marks, May 16, 7 Marks

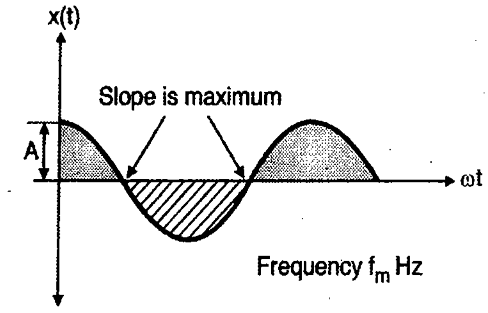
Soln. :

- Let the input signal be sinusoidal with amplitude A volts and frequency f_m Hz as shown in Fig. P. 2.14.1.
- The given signal is $x(t) = A\cos\omega_m t$
- The slope of this signal will be maximum when derivative of x (t) with respect to time is maximum.

$$\therefore \quad \text{Slope of } x(t) = \frac{dx(t)}{dt} = -A\,\omega_m \sin\omega_m t$$

The maximum value of the slope of $x(t) = -A\,\omega_m$ ...(1)

Slope of the staircase approximated signal $x'(t) = \frac{\delta}{T_s}$ ...(2)



(E-700) **Fig. P. 2.14.1 : Input signal x (t)**

- To avoid the slope overload distortion, it is necessary that the maximum slope of x (t) be less than the slope of $x'(t)$.

$$\therefore \quad \left|\frac{dx(t)}{dt}\right|_{\max} \le \frac{\delta}{T_s}$$

$$\therefore \quad A\,\omega_m \le \frac{\delta}{T_s}$$

$$\therefore \quad A \le \frac{\delta}{\omega_m T_s}$$

This is the condition for avoiding the slope overload distortion. Therefore the slope overload distortion will occur if this condition is not satisfied i.e.,

$$\text{If } A > \frac{\delta}{\omega_m T_s} \qquad \text{...(2.14.6)}$$

2.14.12 Maximum Output Signal to Noise Ratio :

SPPU : May 07, May 12

University Questions

Q. 1 Derive an expression for signal to quantization noise power ratio for delta modulation. Assume no slope overload distortion exists. **(May 07, 8 Marks)**

Q. 2 What are the advantages of DM over PCM in terms of signalling rate and bandwidth requirement ? Derive 'the output S/N ratio of delta modulator. Brief the condition to avoid slope overload error. **(May 12, 8 Marks)**

- It can be proved that the maximum signal to noise ratio of a D.M. system is given by,

$$\frac{S}{N_q} = \frac{3}{8\pi^2 f_m^2 f_M T_s^3} \qquad \text{...(2.14.7)}$$

where f_M = Cutoff frequency of the low pass filter in the D.M. receiver.

Ex. 2.14.2 : For a sinusoidal modulating signal

$m(t) = A\cos\omega_m t \quad \omega_m = 2\pi f_m$

show that the maximum output signal to quantization noise ratio in a DM system with no slope overload distortion is given by,

$$\left(\frac{S}{N_q}\right) = \frac{3}{8\pi^2 f_m^2 f_M T_s^3}$$

where f_s = Sampling frequency and f_M = Cutoff frequency of a low pass filter at the output of a receiver.

Soln. :

- From Equation (2.14.6), the condition for avoiding the slope overload error i.e.

$$A < \frac{\delta}{\omega_m T_s} = \frac{\delta}{2\pi}\left(\frac{f_s}{f_m}\right) \qquad \text{...(1)}$$

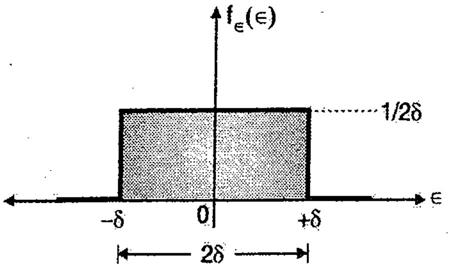
- Therefore the maximum value of the output signal power is given by,

$$P_{\max} = \left[\frac{A}{\sqrt{2}}\right]^2$$

(As P is proportional to square of rms value).

$$\therefore \quad P_{\max} = \frac{A^2}{2} = \frac{\delta^2 f_s^2}{8\pi^2 f_m^2} \qquad \text{...(2)}$$

- Now we need to obtain the expression for quantization noise power. The quantizing error in D.M. is equally likely to lie anywhere in the interval ($-\delta$, δ). i.e. the maximum quantization error $\epsilon_{max} = \pm\delta$.
- This error can be assumed to be uniformly distributed as shown in Fig. P. 2.14.2(a).



(E-701) **Fig. P. 2.14.2(a) : PDF of quantization error for delta modulation**

- The PDF is thus uniform distribution defined as follows :

$$f_\epsilon(\epsilon) = \frac{1}{2\delta} \quad \ldots \; -\delta \le f_\epsilon(\epsilon) \le +\delta$$

$$= 0 \quad \ldots \text{ elsewhere}$$

- The mean square value or the variance of the quantization noise is given by,

$$\overline{\epsilon^2} = \int_{-\delta}^{\delta} \epsilon^2 f_\epsilon(\epsilon)\, d\epsilon$$

$$= \int_{-\delta}^{\delta} \epsilon^2 \frac{1}{2\delta}\, d\epsilon = \frac{1}{2\delta}\left[\frac{\epsilon^3}{3}\right]_{-\delta}^{\delta}$$

$$= \frac{1}{2\delta}\left[\frac{\delta^3}{3} + \frac{\delta^3}{3}\right] = \frac{1}{2\delta} \times \frac{2\delta^3}{3}$$

$$\therefore \quad \overline{\epsilon^2} = \frac{\delta^2}{3} \qquad \ldots(3)$$

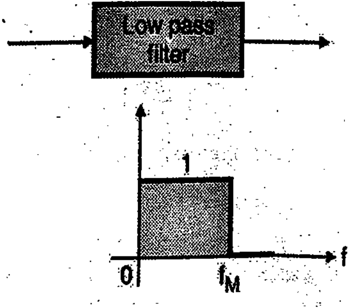
$\therefore$ Normalized quantization noise power,

$$N_q = \frac{\overline{\epsilon^2}}{1} = \frac{\delta^2}{3} \qquad \ldots(4)$$

The DM signal is passed through a reconstruction low pass filter at the output of a DM receiver. The bandwidth of this low pass filter is f_M such that,

$f_M \ge f_m$ and $f_M << f_s$

- The arrangement of filter is shown in Fig. P. 2.14.2(b).



(E-702) **Fig. P. 2.14.2(b) : Low pass filter at D.M. receiver**

- Now assuming that the quantization noise power N_q is distributed uniformly over the frequency band upto f_s, the output quantization noise power within the bandwidth f_M is given by,

Normalized noise power at the filter output,

$$N_q' = \frac{\delta^2}{3} \times \frac{f_M}{f_s} \qquad \ldots(5)$$

- So substituting the values from Equations (2) and (4) we get the expression for output signal to quantization noise ratio as,

$$\left[\frac{S}{N_q}\right]_o = \frac{P_{max}}{N_q'} = \frac{\delta^2 f_s^2}{8\pi^2 f_m^2} \times \frac{3 f_s}{\delta^2 f_M}$$

$$\left[\frac{S}{N_q}\right]_o = \frac{3 f_s^3}{8\pi^2 f_m^2 f_M} \qquad \ldots(6)$$

- But $f_s = 1/T_s$

$$\therefore \quad \left[\frac{S}{N_q}\right]_o = \frac{3}{8\pi^2 f_M f_m^2 T_s^3} \qquad \ldots(7)$$

This is the desired expression for the output signal to quantization noise ratio.

2.14.13 Examples on D.M. :

Ex. 2.14.3 : A sinusoidal voice signal x (t) = cos (6000 π t) is to be transmitted using either PCM or DM. The sampling rate for PCM system is 8 kHz and for the transmission with DM, the step size δ is decided to be of 31.25 mV. The slope overload error is to be avoided. Assume that the number quantization levels for a PCM system is 64. Calculate the signaling rates of both these systems and comment on the result.

Soln. :

1. **Signaling rate of a PCM system :**

$$r = N f_s$$

$$\text{But } Q = 2^N$$

$$\therefore \quad N = \log_2 Q$$

$$= \log_2 64 = 6$$

$$\therefore \quad \text{Signaling rate of PCM} = r$$

$$= 6 \times 8 \text{ kHz}$$

$$= 48 \text{ kHz} \qquad \ldots\textbf{Ans.}$$

2. Signaling rate of DM system :

- The signaling rate of a DM system is equal to its sampling rate f_s because in DM only one bit is transmitted per sample. We know that the condition to avoid the slope overload distortion is given by,

$$A \le \frac{\delta}{\omega_m T_s} \quad \text{or} \quad A \le \frac{\delta f_s}{2 \pi f_m}$$

- We want to calculate f_s

$$\therefore \quad f_s \ge \frac{2 \pi f_m A}{\delta}$$

- Substitute values to get

$$f_s \ge \frac{2 \pi \times 3 \times 10^3 \times 1}{31.25 \times 10^{-3}}$$

$$\therefore \quad f_s \ge 603.18 \text{ kHz}$$

$\therefore$ Signaling rate of DM $\ge$ 603.18 kHz ...**Ans.**

Comment : To transmit the same voice signal, the DM needs a very large signaling rate as compared to PCM. This is the biggest disadvantage of DM, which makes it an impractical system.

Ex. 2.14.4 : Determine the output signal to noise ratio of a linear delta modulation system for a 2 kHz sinusoidal input signal sampled at 64 kHz. Slope overload error is not present and the post reconstruction filter has a bandwidth of 4 kHz.

Soln. :

$$(SNR)_o = \frac{3 f_s^3}{8 \pi^2 f_m^2 f_M}$$

Here, $f_s = 64$ kHz, $f_m = 2$ kHz and $f_M = 4$ kHz

$$\therefore \quad (SNR)_o = \frac{3 \times [64 \times 10^3]^3}{8 \pi^2 \times (2 \times 10^3)^2 \times 4 \times 10^3}$$

$$\therefore \quad (SNR)_o = 622.51$$
$$= 27.94 \text{ dB}$$ ...**Ans.**

Ex. 2.14.5 : For the same sinusoidal input of Ex. 2.14.4 calculate the signal to quantization noise ratio of a PCM system which has the same data rate of 64 Kbits/s. The sampling frequency is 8 kHz and the number of bits per sample is N = 8. Comment on the result.

Soln. :

The signal to noise ratio of a PCM system is given by,

$$(SNR)_q = (1.8 + 6 N) \text{ dB} = 1.8 + (6 \times 8)$$
$$= 49.8 \text{ dB}$$ ...**Ans.**

Comment : The SNR of a DM system is 27.94 dB which is to poor as compared to 49.8 dB of an 8 bit PCM system. Thus for all the simplicity of DM, it cannot perform as well as an 8 bit PCM.

Ex. 2.14.6 : A delta modulator systems is designed to operate at 5 times the Nyquist rate for a signal with 3 kHz bandwidth. Determine the maximum amplitude of 1.2 kHz input sinusoid for which the delta modulator does not have slope overload. Quantising step size is 250 mV. Derive the expression used.

Dec. 2000, 6 Marks

Soln. :

It has been given that,

$W = 3$ kHz, $f_m = 1.2$ kHz,

$f_s = 5 \times 2W = 30$ kHz, $\delta = 250$ mV.

Let the maximum amplitude of 1.2 kHz input sinusoid for no slope overload error be "A". We have already derived the condition to avoid slope overload as,

$$A \le \frac{\delta}{\omega_m T_s} \quad \text{...(1)}$$

$$\therefore \quad \text{Maximum value of } A = \frac{\delta}{\omega_m T_s} = \frac{\delta f_s}{2 \pi f_m}$$

Substituting the values we get,

$$A = \frac{250 \times 10^{-3} \times 30 \times 10^3}{2 \pi \times 1.2 \times 10^3}$$

$$\therefore \quad A = 0.994 \text{ Volts}$$ ...**Ans.**

For the derivation of the expression stated in Equation (1), refer to Ex. 2.14.1.

Ex. 2.14.7 : If a voice frequency signal is sampled at the rate of 32,000 samples/sec and characterized by peak value of 2 Volts, determine the value of step size to avoid slope overload. What is quantization noise power N_q and corresponding SNR ? Assume bandwidth of signal as 4 kHz.

May 98, 6 Marks

Soln. :

Given : $f_s = 32{,}000$ samples/sec.

Peak value of the signal A = 2V.

Bandwidth B = 4 kHz.

1. Step size δ to avoid slope overload :

To avoid slope overload the following condition should be satisfied.

$$A \leq \frac{\delta}{2\pi f_m T_s} = \frac{\delta f_s}{2\pi f_m}$$

Substituting the values we get,

$$2 \leq \frac{\delta \times 32000}{2\pi \times 4 \times 10^3}$$

$$\therefore \quad \delta \geq \frac{2 \times 2\pi \times 4 \times 10^3}{32000}$$

$$\therefore \quad \delta \geq 1.57 \text{ Volts}$$ ...Ans.

2. Quantization noise power (N_q) :

The quantization noise power for a delta modulator is given by,

$$N_q = \frac{\delta^2}{3} = \frac{(1.57)^2}{3}$$

$$\therefore \quad N_q = 0.822 \text{ W}$$ ...Ans.

3. Signal to noise ratio :

$$SNR = \frac{3 f_s^3}{8 \pi^2 f_m^2 B}$$

$$= \frac{3 \times (32 \times 10^3)^3}{8 \pi^2 \times (4 \times 10^3)^2 \times 4 \times 10^3}$$

$$= 19.45$$

Ex. 2.14.8 : In a single integration DM scheme, the voice signal is sampled at a rate of 64 kHz. The maximum signal amplitude is 1 Volt.

1. Determine the minimum value of step size to avoid slope overload.
2. Determine granular noise power N_0, if the voice signal bandwidth is 3.5 kHz.
3. Assuming signal to be sinusoidal, calculate signal power S_o and signal to noise ratio (SNR).
4. Assuming that the voice signal amplitude is uniformly distributed in the range, (–1, 1), determine S_o and SNR.

Dec. 2000, 8 Marks

Soln. :

Given : $f_s = 64$ kHz, $A = 1$ Volt

1. Minimum step size to avoid slope overload :

$$A \leq \frac{\delta f_s}{2\pi f_m}$$

$$\therefore \quad \delta_{min} = \frac{2\pi f_m A}{f_s}$$

$$= \frac{2\pi \times 3.5 \times 10^3 \times 1}{64 \times 10^3}$$

$$\therefore \quad \delta_{min} = 0.3436 \text{ Volts}$$ ...Ans.

2. Granular noise power :

$$N_q = \frac{\delta^2}{3} \times \frac{f_m}{f_s} = \frac{(0.3436)^2}{3} \times \frac{3.5}{64}$$

$$\therefore \quad N_q = 2.15 \times 10^{-3} \text{ W}$$ ...Ans.

3. Signal power S_o and SNR_o :

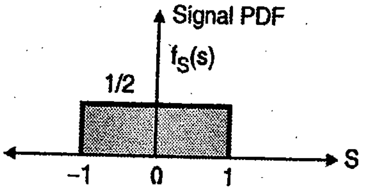
As the signal is sinusoidal, the normalized output signal power

$$S_o = [A/\sqrt{2}]^2$$
$$= A/2$$
$$= 1/2 \text{ Watts.}$$ ...Ans.

$$\therefore \quad SNR_o = \frac{S_o}{N_q} = \frac{0.5}{2.15 \times 10^{-3}}$$
$$= 232.3 \text{ or } 23.66 \text{ dB.}$$

4. Signal power for uniformly distributed signal :

The signal PDF for a uniformly distributed signal is as shown in Fig. P. 2.14.8.



(E-722) **Fig. P. 2.14.8**

∴ Mean square value of the signal,

$$E[S^2] = \int_{-1}^{1} S^2 \cdot f_s(S)\, dS$$

$$= \frac{1}{2} \int_{-1}^{1} S^2\, dS = \frac{1}{2} [S^3/3]_{-1}^{1}$$

$$= 1/3$$

Assuming R = 1.

Normalized signal power S_o

= Mean square value

= 1/3 W ...Ans.

Signal to noise ratio $= SNR = \dfrac{1/3 \text{ W}}{2.15 \times 10^{-3} \text{ W}}$

= 155.03 or 21.9 dB ...Ans.

Ex. 2.14.9 : In a single integration DM scheme the voice signal is sampled at a rate of 64 kHz. The maximum signal amplitude is 2 Volts. Voice signal bandwidth is 3.5 kHz. Determine the minimum value of step size to avoid slope overload and granular noise power.

Dec. 01, 4 Marks

Soln. :

Given : $f_s = 64$ kHz, $A_{max} = 2V$, $f_m = 3.5$ kHz

1. **Minimum step size to avoid slope overload :**

We know that

$$A_{max} = \frac{\delta_{min} f_s}{2\pi f_m}$$

$$\therefore \quad \delta_{min} = \frac{2\pi f_m A_{max}}{f_s} = \frac{2\pi \times 3.5 \times 10^3 \times 2}{64 \times 10^3}$$

$$\therefore \quad \delta_{min} = 0.6872 \text{ Volts} \qquad \text{...Ans.}$$

2. **Granular noise power :**

$$N_q = \frac{\delta^2}{3} \times \frac{f_m}{f_s} = \frac{(0.6872)^2}{3} \times \frac{3.5}{64}$$

$$\therefore \quad N_q = 8.6 \times 10^{-3} \text{ W} \qquad \text{...Ans.}$$

Ex. 2.14.10 : Delta modulator (DM) gives output pulses + p (t) of – p (t). The output of DM is + p (t) when instantaneous sample is larger than previous sample value and is – p(t) when instantaneous sample is smaller than previous sample value (last sample). The p(t) has 2 microsecond duration and 628 mV amplitude and repeats every 10 microsecond. Plot the input and output of DM on graph paper one below other to same scale if the input to DM is 1 Volt sine wave of frequency 10 kHz for one cycle of input wave. What is the maximum frequency with 1 Volt amplitude that can be used in this system without slope overload distortion.

Dec. 05, 6 Marks

Soln. :

Part I : To plot the DM signal :

The D. M. Signal is as shown in Fig. P. 2.14.10.

Part II : Maximum frequency :

Given :

$A = 1V$, $f_m = 10$ kHz,

$f_s = 1/T_s = 1/10\ \mu S = 100$ kHz. $\delta = 265$ mV.

The condition for avoiding the slope overload is

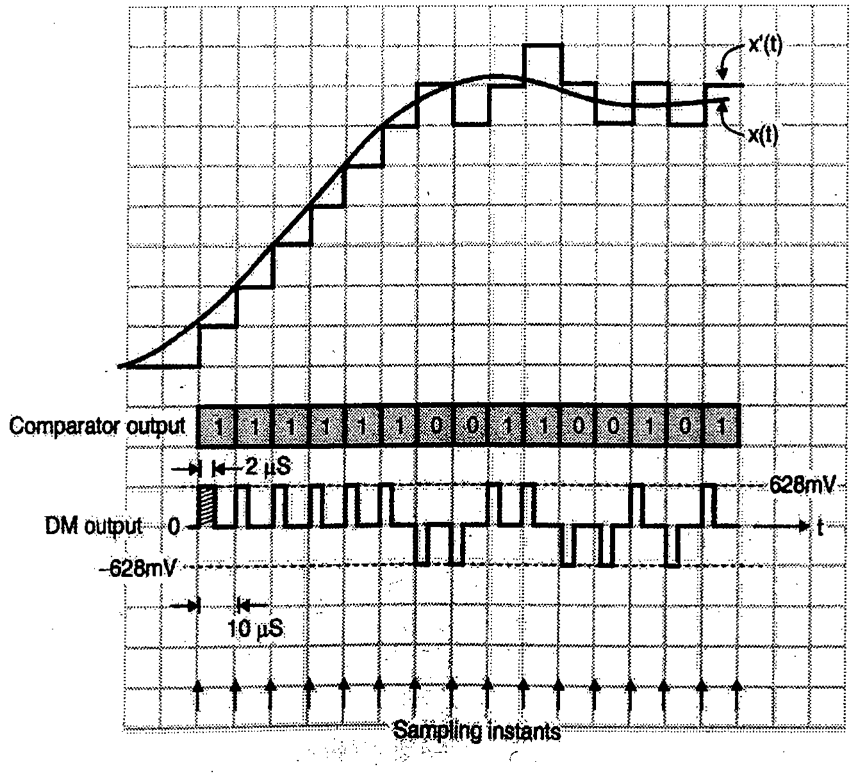
$$A > \frac{\delta}{\omega_m T_s}$$

$$\therefore \quad 1 = \frac{256 \times 10^{-3}}{\omega_m \times 10 \times 10^{-6}}$$

$$\therefore \quad \omega_n = 25.6 \times 10^3 \text{ rad /sec.}$$

$$\therefore \quad f_{max} = 4074.36 \text{ Hz} \qquad \text{...Ans.}$$

This is the maximum input frequency without introducing any slope overload distortion.



(E-724) Fig. P. 2.14.10 : DM signal

Ex. 2.14.11 : A speech signal band limited to 3.4 kHz having maximum amplitude of 1 V is to be delta modulated at 20 kbps. What is appropriate step size for the same ? Derive the formula used. **Dec. 06, 8 Marks**

Soln. :

Given : $f_m = 3.4$ kHz, A = 1 V, Delta modulation,
$f_s = 20$ kHz.

To find : Step size δ.

The amplitude required for avoiding the slope overload is given by,

$$A = \frac{\delta}{2\pi}\left[\frac{f_s}{f_m}\right]$$

$$\therefore \quad 1 = \frac{\delta}{2\pi}\left(\frac{20 \times 10^3}{3.4 \times 10^3}\right)$$

$$\therefore \quad \delta = 1.068 \text{ Volts} \qquad \text{...Ans.}$$

For the derivation of the formula used, refer section 2.14.11.

Ex. 2.14.12 : A 1 kHz signal of voice channel is sampled at 4 kHz using 12-bit PCM and a DM system. If 25 cycles of voice signal are digitized find in each case :

1. Signaling rate
2. Bandwidth required
3. Number of bits required to be transmitted
4. Comment on results.

Dec. 09, 8 Marks

Soln. :

Step 1 : Signaling rate :

1. **For PCM :**

N = 12 and $f_s = 4$ kHz.

$\therefore$ Signaling rate $= Nf_s = 12 \times 4 \times 10^3$
$= 48$ kbps ...**Ans.**

2. **For DM :**

Signaling rate $= f_s = 4$ kbps ...**Ans.**

Step 2 : Bandwidth :

1. For PCM

$$BW \geq \frac{1}{2} N f_s$$

$$\therefore \quad BW \geq 24 \text{ kHz} \qquad \text{...Ans.}$$

2. For DM

$$BW \geq \frac{1}{2} \times f_s$$

$$\therefore \quad BW \geq 2 \text{ kHz} \qquad \text{...Ans.}$$

Step 3 : Number of bits required to be transmitted :

Number of cycles to be digitized = 25

1 cycle $\equiv$ 1/1 kHz = 1 mS

$\therefore$ 25 cycles $\equiv$ 25 mS

1. For PCM, bit rate = 48 kbps

$\therefore$ Number of bits required to be transmitted

$= 25 \times 10^{-3} \times 48 \times 10^3$

$= 1200$ bits ...**Ans.**

2. For DM, bit rate = 2kbps

$\therefore$ Number of bits required to be transmitted

$= 25 \times 10^{-3} \times 2 \times 10^3$

$= 50$ bits ...**Ans.**

Step 4 : Comments :

In order to send the same voice signal, the D. M. system needs less bandwidth and less number of bits as compared to the PCM system.

Ex. 2.14.13 : A signal having bandwidth 3 kHz is to be encoded using :

1. 8 bit PCM system 2. DM system.

If 10 cycles of the signal are digitized, state how many bits will be digitized, output in each case if sampling frequency is 10 kHz. Also find bandwidth required in each case.

Dec. 10, 8 Marks

Soln. :

Step 1 : Signaling rate :

1. **For PCM :** N = 8 bits and f_m or W = 3 kHz,
$f_s = 10$ kHz.

$\therefore$ Signaling rate $= Nf_s = 8 \times 10 = 80$ kbps

2. **For DM :**

Signaling rate $= f_s = 10$ kbps

Step 2 : Number of bits to be digitized :

Number of cycles digitized = 10

1 cycle = 1/3 kHz = 0.33 mS

10 cycles $\equiv$ 3.3 mS

1. For PCM, bit rate = 80 kbps.

Number of bits to be digitized in 10 cycles

$= 3.3 \times 10^{-3} \times 80 \times 10^3$

$= 264$ bits. ...**Ans.**

2. For DM, bit rate = 10 kbps

$\therefore$ Number of bits to be digitized

$= 3.3 \times 10^{-3} \times 10 \times 10^3$

$= 33$ bits ...**Ans.**

Step 3 : Bandwidth :

1. For PCM $\quad BW \geq \frac{1}{2} Nf_s$

$\therefore$ BW $\geq$ 40 kHz. ...**Ans.**

2. For DM $\quad BW \geq \frac{1}{2} f_s$

$\therefore$ BW $\geq$ 5 kHz. ...**Ans.**

Ex. 2.14.14 : Consider a DM system designed to accommodate analog message signals limited to bandwidth W = 5 kHz. A sinusoidal test signal of amplitude A = 1 volt and frequency f_m = 1 kHz is applied to the system. The sampling rate of the system is 50 kHz.

1. Calculate the minimum step size Δ required to minimize slope overload.
2. Calculate signal-to (quantization) noise ratio of the system for the specified sinusoidal test signal.

May 13, 8 Marks

Soln. :

Given : W = 5 kHz, A = 1 Volt, f_m = 1 kHz, f_s = 50 kHz

Step 1 : Calculate minimum step size required to minimize slope overload :

$$A \le \frac{\delta f_s}{2\pi f_m}$$

$$\therefore \delta = \frac{2\pi A f_m}{f_s}$$

$$= \frac{2\pi \times 1 \times 1 \times 10^3}{50 \times 10^3}$$

$$= 0.1256 \text{ Volts.} \qquad \text{...Ans.}$$

Step 2 : Calculate signal to noise ratio :

$$\text{SNR} = \frac{3 f_s^3}{8 \pi^2 f_m^2 \times W}$$

$$= \frac{3 \times (50 \times 10^3)^3}{8 \times \pi^2 \times (1 \times 10^3)^2 \times 5 \times 10^3}$$

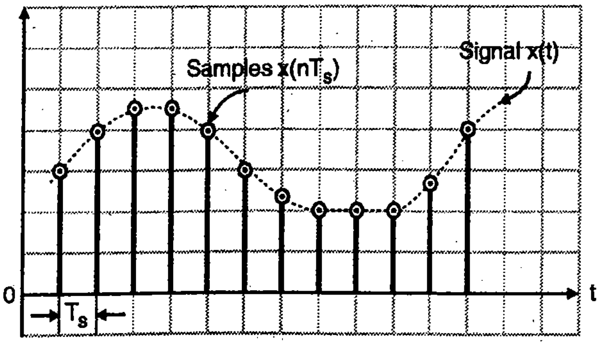
$$\therefore \text{SNR} = 949.88 \qquad \text{...Ans.}$$

2.15 Differential Pulse Code Modulation (DPCM) :

- In a PCM system, the signal x (t) is sampled at a rate which is slightly higher than the Nyquist rate. It is observed that the resulting sampled signal has a high correlation between the adjacent samples.
- That means there is a correlation between the adjacent samples. It is observed that generally the signal x (t) does not change rapidly from one sample to the next.
- Therefore the difference in amplitudes of adjacent samples is very small, as shown in Fig. 2.15.1.
- When these highly correlated samples are encoded using a standard PCM system, the resulting encoded PCM signal contains redundant information. Redundant bits do not contain any new information.
- By removing this redundancy before encoding, we can obtain a more efficiently coded signal. The DPCM system operates on this principle. In DPCM system a special circuit called "predictor" is used.
- The "predictor" can actually predict the values of the future samples of x (t). This helps in reducing the redundancy.

2.15.1 Role of a Predictor :

- It is observed that if the sampling takes place at a rate which is higher than the Nyquist rate, then there is a correlation between successive samples of the signal x (t).
- Hence we can predict the range of next required increment or decrement in x(t) at the predictor output, if we know the past sample value or the difference.
- This reduces the difference or error between x (t) and $\overline{x}$ (t). Therefore to encode this small value of error the DPCM system requires less number of bits which will ultimately reduce the bit rate. This is the role predictor in DPCM system.



(E-707) **Fig. 2.15.1 : Samples have a correlation between them**

2.15.2 DPCM Transmitter : SPPU : Dec. 14

University Questions

Q. 1 Explain with neat schematic and mathematical analysis, a transmitter and receiver for DPCM. **(Dec. 14, 8 Marks)**

The DPCM transmitter is shown in Fig. 2.15.2.

Operation :

- Suppose that a baseband signal x (t) is sampled at a rate $f_s = 1/\ T_s$ to produce the sampled signal $x(n\ T_s)$. This signal acts as the input signal to the DPCM transmitter.
- Let the sequence of such samples be denoted by $\{x(n\,T_s)\}$. Where n is an integer.
- Let the predictor produce a predicted version of the sampled input and let the predictor output be denoted by $\hat{x}(n\,T_s)$.
- The predictor output is subtracted from the sampled input to obtain a difference signal e ($n\ T_s$) as follows :

$$e(n\,T_s) = x(n\ T_s) - \hat{x}(n\ T_s) \qquad ...(2.15.1)$$

- The predictor value $\hat{x}(n\ T_s)$ is produced by the predictor whose input consists of quantized version of input signal $x(n\,T_s)$.
- The difference signal $e(n\,T_s)$ is called as **prediction error,** because it represents the difference between the sample and its predicted value.
- The quantizer output $v(n\ T_s)$ is encoded to obtain the digital pulses i.e. DPCM signal.
- Let the input output characteristics of the quantizer be denoted by a nonlinear function Q (·).
- So referring to Fig. 2.15.2 we get the quantizer output as,

$$v(n\,T_s) = Q[e(n\,T_s)]$$

$$= e(n\,T_s) + q(n\,T_s) \qquad ...(2.15.2)$$

where $q(n\,T_s)$ is the quantization error.

- Referring to Fig. 2.15.2 the predictor input is given by,

$$u(n\,T_s) = \hat{x}(n\,T_s) + v(n\,T_s) \qquad ...(2.15.3)$$

- Substituting the expression for $v(n\,T_s)$ we get

$$u(n\,T_s) = \hat{x}(n\,T_s) + e(n\,T_s) + q(n\,T_s) \qquad ...(2.15.4)$$

- But $\hat{x}(n\,T_s) + e(n\,T_s) = x(n\,T_s)$

$$\therefore\ u(n\,T_s) = x(n\,T_s) + q(n\,T_s) \qquad ...(2.15.5)$$

This is nothing but the quantized version of input $x(n\,T_s)$.

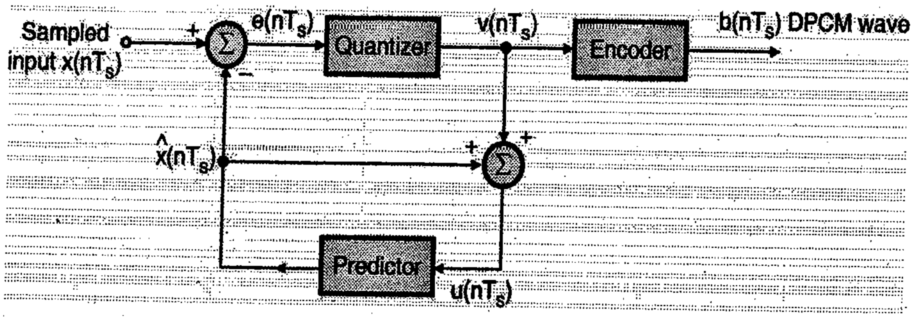
Thus the quantized signal $u(n\ T_s)$ at the predictor input differs from the original input signal by $q(n\,T_s)$ i.e. the quantization error.

2.15.3 DPCM Receiver : SPPU : Dec. 14

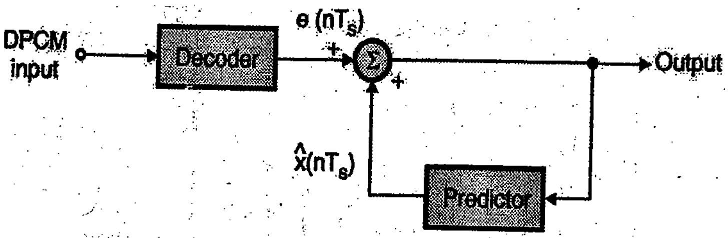
University Questions

Q. 1 Explain with neat schematic and mathematical analysis, a transmitter and receiver for DPCM. **(Dec. 14, 8 Marks)**

- The block diagram of a DPCM receiver is shown in Fig. 2.15.3.



(E-708) Fig. 2.15.2 : DPCM transmitter



(E-709) Fig. 2.15.3 : DPCM receiver

- The DPCM signal is applied to the decoder for reconstructing the quantized version of the input.
- The decoder output is actually the reconstructed quantized error signal.
- This signal is then added to the predictor output to produce the original signal.
- The predictor used at the receiver is same as that at the transmitter.

$$\text{Receiver output} = e(n\,T_s) + \hat{x}(n\,T_s)$$

$$= x(n\,T_s) \qquad ...(2.15.6)$$

2.15.4 Output Signal to Noise Ratio :

- The output signal to quantization noise ratio for a DPCM system can be defined in a similar way as that for a PCM system.

$$\therefore (SNR) = \frac{\text{Mean square value of signal}}{\text{Mean square value of quantization noise}}$$

- But mean square value is equal to the variance.
- Hence,

$$(SNR) = \frac{\sigma_X^2}{\sigma_Q^2} \qquad ...(2.15.7)$$

Where σ_X^2 is the variance of original input signal $x(n\,T_s)$ and σ_Q^2 is the variance of the quantization error $q(n\,T_s)$.

- We can rearrange the above expression as follows,

$$SNR = \frac{\sigma_X^2}{\sigma_E^2} \times \frac{\sigma_E^2}{\sigma_Q^2} \qquad ...(2.15.8)$$

Where σ_E^2 is the variance of the prediction error $e(n\,T_s)$

$$SNR = G_p\,(SNR)_p \qquad ...(2.15.9)$$

Where $G_p = (\sigma_X^2/\sigma_E^2)$ and called as prediction gain and $(SNR)_p$

$= (\sigma_E^2/\sigma_Q^2)$ is the prediction error-to-quantization noise ratio.

Importance of prediction gain :

- The prediction gain G_p is defined as follows :

$$G_p = \frac{\sigma_X^2}{\sigma_E^2} \qquad ...(2.15.10)$$

- The prediction gain should be as high as possible. For a given baseband signal, the variance σ_X^2 is fixed hence in order to maximize G_p, we have to minimize the variance σ_E^2 of the prediction error $e(n\,T_s)$.
- The predictor should be designed accordingly.

2.15.5 Types of Predictors :

The predictors used for DPCM are :

1. One-tap predictors
2. N-tap predictors

2.15.6 Advantage of DPCM :

- As the difference between $x(n\,T_s)$ and $\bar{x}(n\,T_s)$ is being encoded and transmitted by the PCM technique, a small difference voltage is to be quantized and encoded.
- This will need less number of quantization levels and hence less number of bits to represent them.
- Thus signalling rate and bandwidth of a DPCM system will be less than that of PCM.

2.15.7 Disadvantages :

1. High bit rate.
2. Needs the predictor circuit to be used which is very complex.
3. Practical usage is limited.

2.16 Adaptive Delta Modulation (ADM) :

SPPU : Dec. 07, May 16

University Questions

Q. 1 What are the limitations of Delta modulation ? How are they overcomed in Delta sigma modulation and Adaptive Delta modulation ? Explain with necessary diagrams. **(Dec. 07, 8 Marks, May 16, 7 Marks)**

- In the ADM system, the step size is not constant. Rather when the slope overload occurs the step size becomes progressive larger and therefore $x'(t)$ will catch up with $x(t)$ more rapidly.
- Whenever the slope of input signal is large, the step size of the staircase approximated signal $x'(t)$ is increased.
- On the other hand when the input signal is varying slowly the step size is reduced.
- Thus the step size is adapted as per the level of input signal.

2.16.1 Types of ADM :

There are various types of ADM systems available depending on the type of scheme used for adjusting the step size. In one type a discrete set of values is provided for the step size whereas in another type a continuous range of step size variation is provided. We will discuss the first type here.

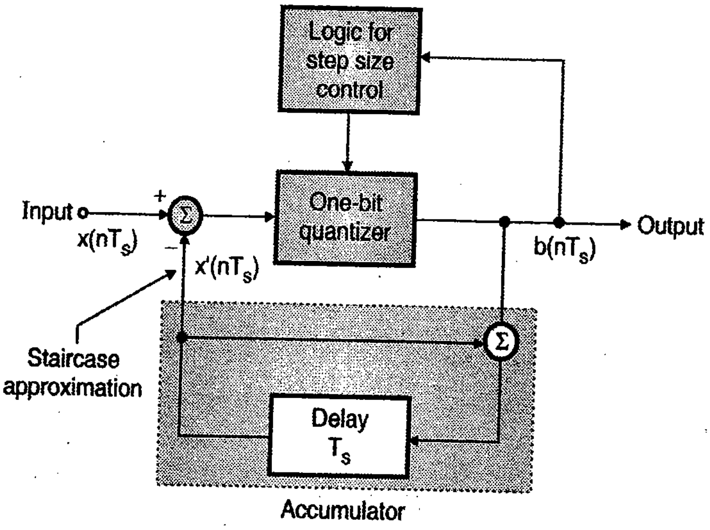
2.16.2 ADM Transmitter :

SPPU : Dec. 16

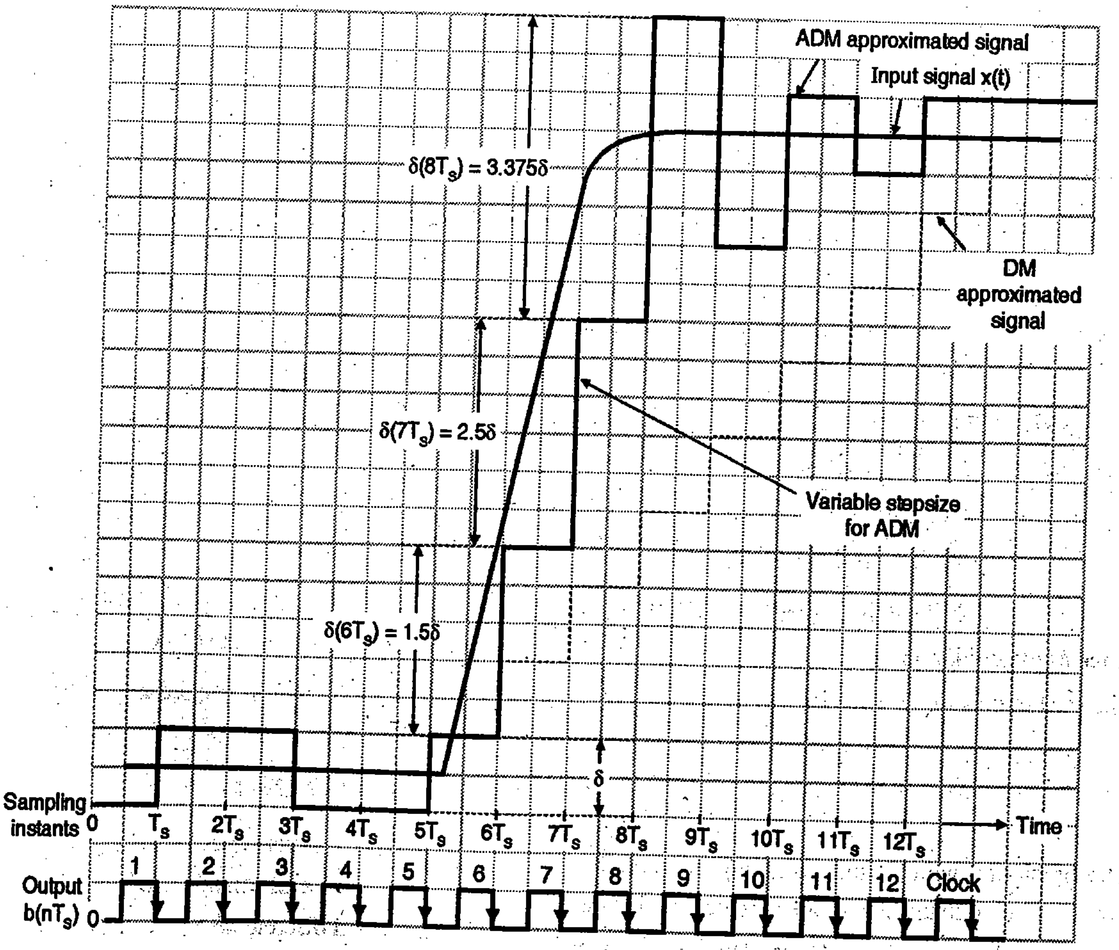
University Questions

Q. 1 Draw block diagram of adaptive delta modulator and explain the same. What are the advantages of adaptive delta modulator over delta modulator ? **(Dec. 16, 8 Marks)**

- Fig. 2.16.1 shows the ADM transmitter block diagram.
- If you compare this block diagram with that of the linear D.M., then you will find that the block named "logic for step size control" is the only addition. The remaining block schematic is same as that of linear D.M.
- Fig. 2.16.2 shows the staircase approximated waveform for an ADM system.



(E-710) Fig. 2.16.1 : ADM transmitter



(E-711(a)) Fig. 2.16.2 : Waveforms of ADM

- Let the stepsize at various sampling instants be denoted by $\delta(n\,T_s)$ with "n" an integer. So from Fig. 2.16.2, the stepsize at n = 1 is $\delta(T_s) = \delta$ and the step size at n = 6 is $\delta(6\,T_s) = 2\,\delta$. Similarly $\delta(7\,T_s) = 3\delta$ and so on.

Stepsize adjustment :

The step size at any given sampling instant "n Ts" is obtained by using the following adaptation rule.

$$\delta(n\,T_s) = g(n\,T_s)\,\delta(n\,T_s - T_s) \qquad ...(2.16.1)$$

where $g(n\,T_s)$ is the time varying multiplier and its value depends on the present output $b\cdot(n\,T_s)$ and the previous output $b(n\,T_s - T_s)$ as follows :

$$g(n\,T_s) = K$$

$$...\text{ if } b(n\,T_s) = b(n\,T_s - T_s) \qquad ...(2.16.2)$$

$$\text{and} \quad g(n\,T_s) = K^{-1}$$

$$...\text{ if } b(n\,T_s) \neq b(n\,T_s - T_s) \qquad ...(2.16.3)$$

This adaptation algorithm is known as a constant factor ADM with one bit memory because we are utilizing only one previous bit $b(n\,T_s - T_s)$ to obtain the value of $g(n\,T_s)$.

Illustration :

- Let us illustrate now the use of Equation (2.16.1) in calculating the step size.
- Refer the waveforms of Fig. 2.16.2. Take n = 6 i.e. the 6th sampling instant, $t = 6\,T_s$. The corresponding step size is given by,

$$\delta(6\,T_s) = g(6\,T_s)\,\delta(6\,T_s - T_s)$$

$$= g(6\,T_s)\,\delta(5\,T_s) \qquad ...(2.16.4)$$

- But $g(6\,T_s) = K$ ...since $b(6\,T_s) = b(5\,T_s)$

$$\therefore \quad \delta(6\,T_s) = K\,\delta(5\,T_s) \qquad ...(2.16.5)$$

- But $\delta(5\,T_s) = \delta$ from Fig. 2.16.2

$$\therefore \quad \delta(6\,T_s) = k\,\delta \qquad ...(2.16.6)$$

- The value of k is adjusted between 1 and 2. Practically it is observed that k = 1.5 yields the most desired results.

$$\therefore \quad \delta(6\,T_s) = 1.5\,\delta \qquad ...(2.16.7)$$

- Similarly we can prove that

$$\delta(7\,T_s) = 1.5\,\delta(6\,T_s)$$

$$= 1.5 \times 1.5\,\delta = 2.25\,\delta$$

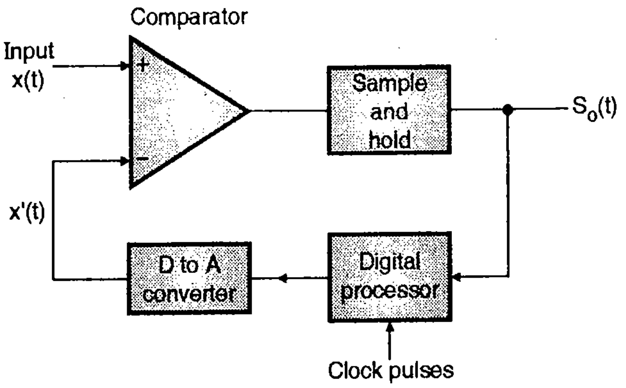
$$\text{and } \delta(8\,T_s) = k\,\delta(7\,T_s) = 1.5 \times 2.25\,\delta$$

$$= 3.375\,\delta$$

- As shown in Fig. 2.16.2, due to variable step size, the slope overload error is reduced. But quantization error is increased.
- Due to the adjustable step size, the slope overload problem is solved. Hence ADM system has a low bit rate than the PCM system.
- Therefore the BW required is also less than a comparable PCM system.

2.16.3 Adaptive Delta Modulation (Alternate Method) :

- In the ADM system, the step size is not constant. Rather when the slope overload occurs the step size becomes progressive larger and therefore $x'(t)$ will catch up with $x(t)$ more rapidly.
- The ADM transmitter is as shown in Fig. 2.16.3.



(D-495) **Fig. 2.16.3 : ADM transmitter**

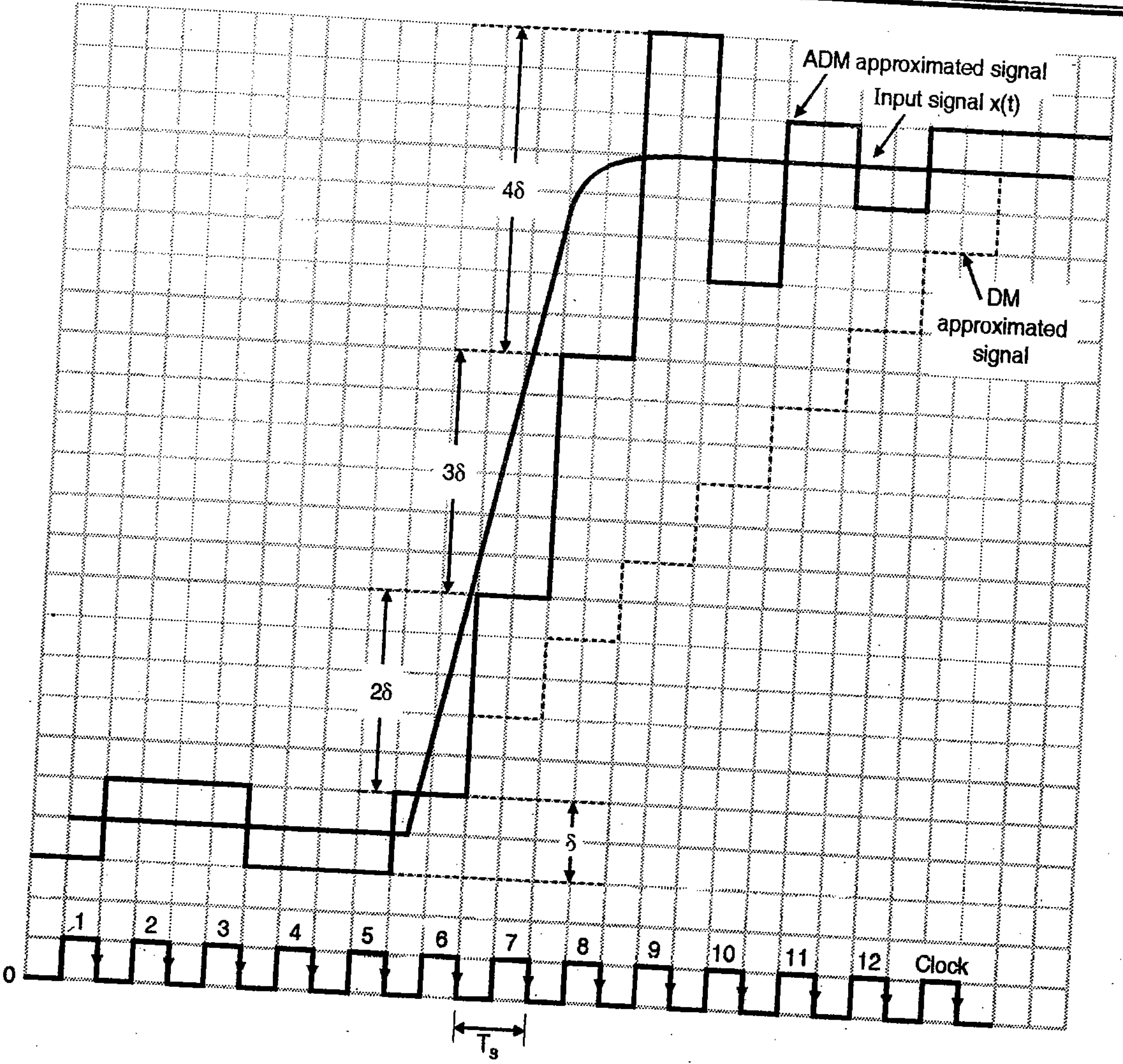
- If you compare this block diagram with that of the linear delta modulator, then you will find that except for the counter being replaced by the digital processor, the remaining blocks are identical. Let us understand the operation of the digital processor. For that carefully see the waveforms of Fig. 2.16.4.

Operation :

- In response to the kth clock pulse trailing edge, the processor generates a step which is equal in magnitude to the step generated in response to the previous i.e. (k – 1)th clock edge.
- If the direction of both the steps is same, then the processor will increase the magnitude of the present step by "δ". If the directions are opposite then the processor will decrease the magnitude of the present step by "δ".
- $S_o(t)$ in the Fig. 2.16.3, i.e. the output of the ADM system is given as,

$S_o(t) = +1$ if $x(t) > x'(t)$ just before the kth clock edge.

and $S_o(t) = -1$ if $x(t) < x'(t)$ just before the kth clock edge



(E-711) **Fig. 2.16.4 : Waveforms of ADM**

- Then the step size at the sampling instant k is given by,

$$\underset{\substack{\downarrow\\ \text{Step size}\\ \text{at } k^{th} \text{ clock}\\ \text{edge}}}{\delta(k)} = \underset{\substack{\downarrow\\ \text{Output at}\\ k^{th} \text{ edge}}}{S_o(k)} + \underset{\substack{\downarrow\\ \text{Basic}\\ \text{step}\\ \text{size}}}{\delta} \quad \underset{\substack{\downarrow\\ \text{Output at } (k-1)^{th}\\ \text{clock edge}}}{S_o(k-1)} \qquad ...(2.16.8)$$

- Let us take an example :

Refer to the waveforms of Fig. 2.16.4. Let us assume k = 6, i.e. consider the 6th clock edge.

$$\therefore \quad k-1 = 5$$

$$\therefore \quad \delta(k-1) = \delta(5) = \delta$$

$$S_o(k) = S_o(6) = +1$$

$$S_o(k-1) = S_o(5) = +1$$

- Substitute in Equation (2.16.8) to get,

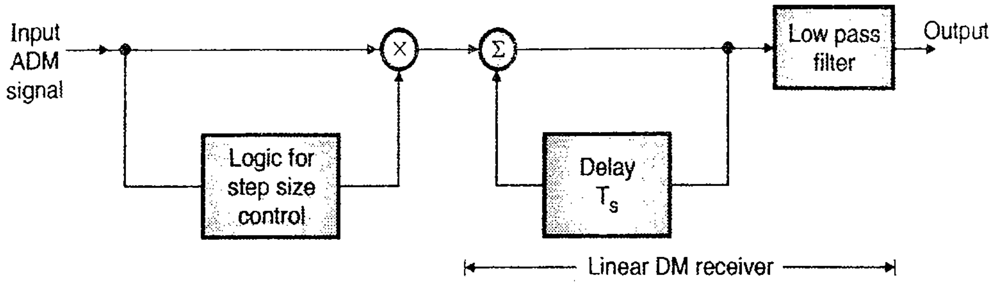
$$\delta(6) = \delta + \delta = 2\,\delta \qquad ...(2.16.9)$$

Look at the Fig. 2.16.4, the step size at the 6th clock edge is 2 δ.

- As shown in Fig. 2.16.4, due to variable step size, the slope overload error is reduced. But quantization error is increased. Due to the adjustable step size, the slope overload problem is solved. Hence ADM system has a low bit rate than the PCM system. Therefore the BW required is also less than a comparable PCM system.

2.16.4 ADM Receiver :

The block diagram of ADM receiver is shown in Fig. 2.16.5.



(D-497) Fig. 2.16.5 : ADM receiver

- The ADM signal is first converted into a D.M. signal with the help of the step size control logic and then applied to a D.M. receiver.
- At the output of low pass filter we get the original signal back.

2.16.5 Advantages of Adaptive Delta Modulation :

SPPU : Dec. 16

University Questions

Q. 1 Draw block diagram of adaptive delta modulator and explain the same. What are the advantages of adaptive delta modulator over delta modulator ? **(Dec. 16, 8 Marks)**

The advantages of ADM over DM are as follows :

1. Reduction in slope overload distortion and granular noise.
2. Improvement in signal to noise ratio.
3. Wide dynamic range due to variable step size.
4. Better utilization of bandwidth as compared to delta modulation.
5. Low signaling rate.
6. Simplicity of implementation.

2.16.6 Disadvantages of ADM :

For a relatively constant magnitude input signal x (t), the ADM will produce a high granular noise.

2.17 Delta-Sigma Modulation :

SPPU : Dec. 07, May 09, Dec. 10, May 16

University Questions

Q. 1 What are the limitations of Delta modulation ? How are they overcomed in Delta sigma modulation and Adaptive Delta modulation ? Explain with necessary diagrams. **(Dec. 07, 8 Marks)**

Q. 2 What is delta sigma modulation ? Explain the transmitter and receiver schemes of a delta sigma system. **(May 09, 6 Marks)**

Q. 3 Draw the block diagram of DM transmitter and explain its working. Comment on the drawbacks of DM. Explain how the drawback of accumulation of noise is eliminated by Delta-Sigma modulator ? **(Dec. 10, 10 Marks)**

Q. 4 What are the limitations of delta modulation ? How are they overcome in delta sigma modulation and adaptive delta modulation ? Explain with necessary diagrams. **(May 16, 7 Marks)**

Drawback of the conventional delta modulator :

- The quantizer input in the conventional delta modulator can be considered as an approximation to the derivative of the input message signal.
- Therefore due to the noise results there will be an accumulated error in the demodulated signal.
- This is the drawback of the conventional delta modulator.
- This drawback can be overcome by using the delta sigma modulator.
- In the delta sigma modulator, the input signal is passed through an integrator before applying it to the delta modulator circuit.

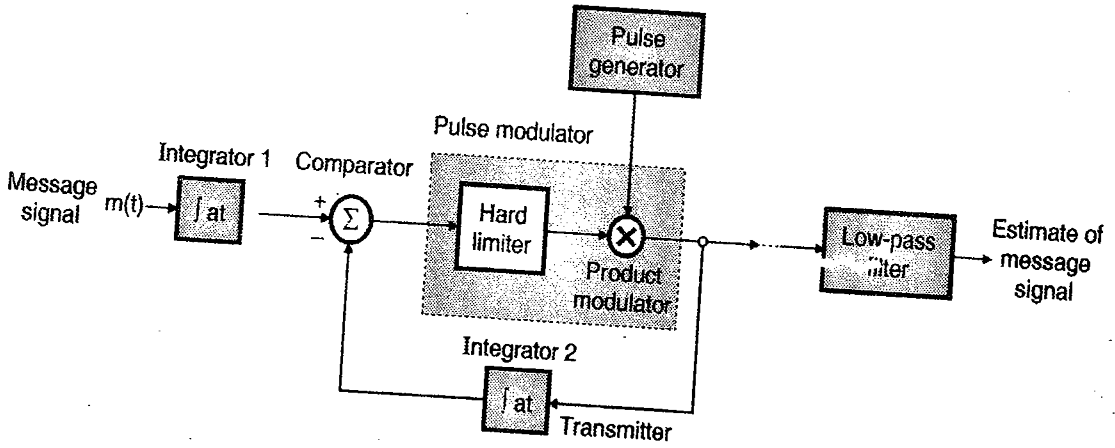
Advantages of the integrator :

The use of integrator has the following advantages :

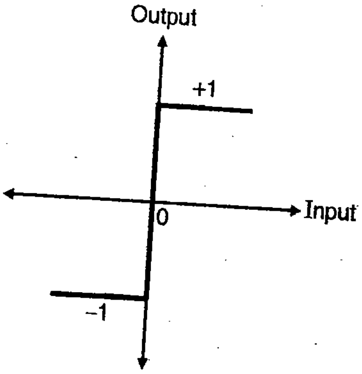
1. The low frequency components in the input signal are boosted which is similar to the process of pre-emphasis.
2. The correlation between the adjacent samples of delta modulator is increased. This improves the overall system performance, because the variance of the error signal at the quantizer input gets reduced.
3. It simplifies the receiver design.

Block diagram :

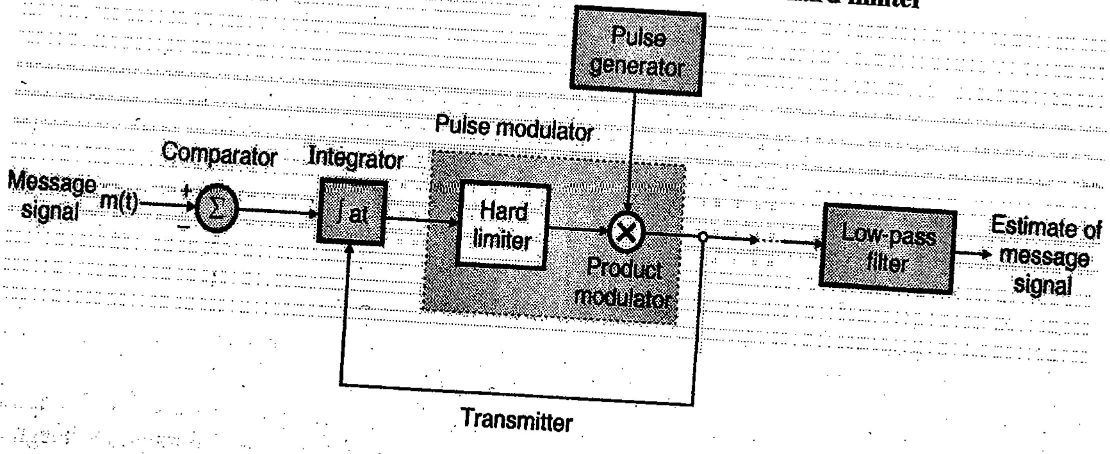
- Fig. 2.17.1(a) shows the block diagram of a delta sigma modulator.
- m(t) is the message signal which is a continuous time signal. It is first passed through an integrator and a comparator.
- The comparator output is applied to the pulse modulator block. It consists of a hard limiter, and a product modulator.
- The input output characteristics of the hard limiter is shown in Fig. 2.17.1(b). That means if the input to this block is negative it will produce a – 1 output whereas for all the positive inputs it produces a + 1 output.
- The hard limiter output (± 1) is applied to a multiplier (product modulator). The other input to the multiplier is the clock pulses produced by the external pulse generator.
- The frequency of clock pulses should be higher than the Nyquist rate.



(E-703) **Fig. 2.17.1(a) : Delta-sigma modulator**



(E-704) **Fig. 2.17.1(b) : Input output characteristics of hard limiter**



(E-705) **Fig. 2.17.2 : Simplified sigma-delta system**

- At the output of the product modulator we get the sampled version of limiter output. It is transmitted over the communication channel. Thus we transmit one bit per sample.
- The same output signal is applied to the second integrator.
- The output of integrator 2 is compared with the output of integrator 1 with the help of the comparator.

Receiver :

Fig. 2.17.2 shows that the receiver of sigma delta system is simply a low pass filter i.e. an integrator.

Simplified sigma-delta system :

- We know that the integration is a linear operation, therefore we can combine the integrators 1 and 2 into one integrator which is placed after the comparator.
- This will simplify the transmitter design to some extent.
- The simplified sigma-delta system is shown in Fig. 2.17.2.

2.18 LPC Speech Synthesis : SPPU : May 06

University Questions

Q. 1 Explain speech synthesis using LPC.
(May 06, 6 Marks)

- Linear Predictive Coding (LPC) is a new and completely different method of representing an analog signal digitally.
- The analog signal at the input is first synthesized. The parameters of the waveform synthesizer are then encoded and transmitted instead of transmitting the original signal. LPC is particularly well suited for speech synthesis and transmission.
- This method uses a transversal filter or its digital equivalent alongwith some auxiliary components to synthesize the required waveform.
- The parameters of the waveform synthesizer are then encoded for transmission (note that the actual signal is not transmitted).
- Fig. 2.18.1 shows the block diagram of a speech synthesizer. It consists of two input generators, a variable gain amplifier and a transversal filter connected in the feedback loop.
- The amplifier gain and the filter tap gains are adjusted to represent the acoustic properties of the vocal track.
- The white noise generator in Fig. 2.18.1 is connected in order to produce the unvoiced speech such as the hissing sounds.
- Voiced speech is generated by connecting the impulse train generator set at an appropriate pitch frequency.
- With a filter of about 10 tap gains and all the parameters values updated every 10 ms the synthesized speech is very much recognizable. But it sounds artificial like a robot.
- Some recorded message systems and talking toys, generate speech sounds by the synthesis method. They draw the parameter values stored in a digital memory. Such systems are called **voice coders** or **vocoders.**
- Block diagrams of LPC transmitter and receiver are as shown in Figs. 2.18.1(a) and (b).

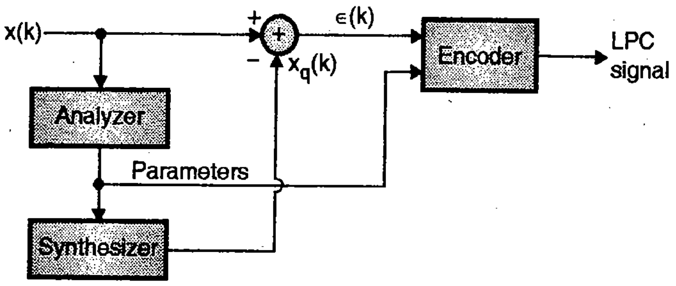
2.18.1 LPC Transmitter (or Encoder) :

SPPU : May 11

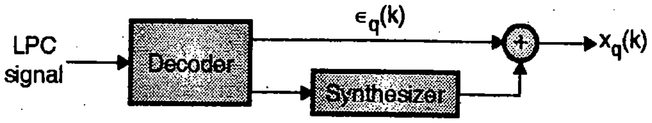
University Questions

Q. 1 Explain LPC encoder and decoder in detail with the help of block diagram. **(May 11, 8 Marks)**

- Fig. 2.18.1(a) shows the block diagram of a LPC transmitter.
- The voice input signal is first sampled to obtain the signal x (k) at the input of LPC transmitter. These sampled values are applied to an analyser which analyzes them to determine the parameters for synthesizer.
- The synthesizer reconstructs the approximated speech signal x_q (k).
- The reconstructed speech signal x_q (k) and the sampled signal x (k) are compared to produce the error signal "ϵ (k)".
- The error signal and parameter values are encoded by the encoder to form a digital signal which is called as the LPC signal.



(a) Transmitter



(b) Receiver

(E-720) **Fig. 2.18.1 : LPC transmission system**

2.18.2 LPC Receiver (or Decoder) :

SPPU : May 11

University Questions

Q. 1 Explain LPC encoder and decoder in detail with the help of block diagram. **(May 11, 8 Marks)**

- Fig. 2.18.1(b) shows the LPC receiver. From the received LPC signal the decoder or demultiplexer separates out the error signal "ϵ_q (k)" and the parameters.
- The parameters are applied to a synthesizer. The synthesizer output is then added to the error signal to produce speech signal x_q (k).
- A complete LPC code word consists of about 80 bits out of which 1 bit is used for the voiced/unvoiced switch, 6 bits for pitch frequency of the voice, and a few bits are used to represent error. LPC has a very low bit rate of about 3k to 8 kbits/sec.

Ex. 2.18.1 : Using predictability theory, prove that transmission of encoded error signal (rather than encoded signal itself) is sufficient for reasonable reconstruction of signal. With the help of block schematic suggest any one technique to transmit and receive encoded errors. What are the limitations and advantages of such techniques with reference to linear or uniform PCM ?

Soln. :

Predictive coding theory :

Here we are going to show that the transmission of encoded error signal (rather than the encoded signal itself) is sufficient for reconstruction of signal with reasonable accuracy.

1. Let an analog message signal x (t) be passed through a LPF and then sampled after every T_s seconds. Then we can denote the sampled values of x (t) as x (k T_s) where k is an integer. Let us represent these sampled values as only x (k) for simplification.

$$\therefore \quad x(k) = \text{Samples of } x(t)$$

2. When the sampling frequency f_s is much higher than the Nyquist rate, the samples are very close to each other. Then we can expect that x (k) is approximately equal to x (k – 1) i.e. the previous sample value.

$$\therefore \quad x(k) \approx x(k-1)$$

3. Therefore if the quantized value of previous sample x_q (k – 1) is known, then we can guess or predict the next value i.e. $\tilde{x}_q$ (k) as,

$$\tilde{x}_q(k) \approx x_q(k-1) \quad \text{...(1)}$$

where $\tilde{x}_q$ (k) denotes our prediction of x_q (k).

4. Equation (1) shows that a time delay circuit with a delay of T_s seconds will serve as a "predictor". The error between the predicted and actual values of x_q (k) is then given by,

$$\epsilon_q(k) = x_q(k) - \tilde{x}_q(k) \quad \text{..(2)}$$

where $\epsilon_q(k)$ = The quantized prediction error.

5. Now if we transmit this error signal rather than the signal itself then we can use the system shown in Fig. P. 2.18.1(a) to generate x_q (k) because

$$x_q(k) = \epsilon_q(k) + \tilde{x}_q(k) \quad \text{...(3)}$$

$$\text{But} \quad \tilde{x}_q(k) = x_q(k-1)$$

$$\tilde{x}_q(k) = \epsilon_q(k) + x_q(k-1) \quad \text{...(4)}$$

The block diagram of Fig. P. 2.18.1(a) will implement Equations (3) and (4) and hence acts as an accumulator. The effect of accumulation can be illustrated by writing.

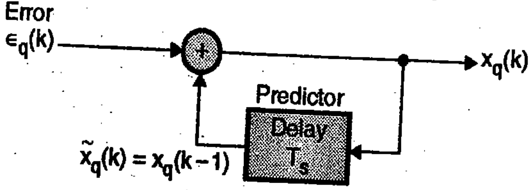
$$x_q(k) = \epsilon_q(k) + x_q(k-1)$$

$$\text{But} \quad x_q(k-1) = \epsilon_q(k-1) + x_q(k-2)$$

and so on.

$$\therefore \quad x_q(k) = \epsilon_q(k) + \epsilon_q(k-1) + x_q(k-2)$$

$$\therefore \quad x_q(k) = \epsilon_q(k) + \epsilon_q(k-1) + \epsilon_q(k-2) + \epsilon_q(k-3) \quad \text{...(5)}$$



(E-721) **Fig. P. 2.18.1(a)**

- This equation shows that we can reconstruct the original quantized signal x_q (k) by adding the prediction errors at various sampling instants.
- Thus we have proved that transmission of encoded error signals is sufficient for reconstruction of signal with a reasonable accuracy.
- Such prediction errors can be easily generated by a simple delta modulator system.

2.19 Comparison of Digital Pulse Modulation Systems :

SPPU : May 06, Dec. 13

University Questions

Q. 1 Compare DM and ADM systems. **(May 06, 6 Marks)**

Q. 2 Compare PCM, DPCM, Delta modulation and Adaptive Delta modulation on the basis of Sampling Frequency, Bit rate and bandwidth requirement. **(Dec. 13, 8 Marks)**

The PCM, DM, ADM and DPCM all are digital pulse modulation systems. Table 2.19.1 shows the comparison of these systems.

Table 2.19.1 : Comparison of PCM, DM, ADM and DPCM

Sr. No.	Parameter	PCM	DM	ADM	DPCM
1.	Number of bits per sample	N can be 4, 8, 16, 32, 64 etc.	N = 1	N = 1	N is more than 1 but less than that for PCM
2.	Step size	Depends on the number of Q levels.	Step size is fixed	Step size is variable	Step size is fixed
3.	Distortions / errors	Quantization error	Slope overload and granular noise	Granular noise	Slope overload and granular noise
4.	Signaling rate and bandwidth	Highest	Low, if the input is slow varying	Lowest	Lower than PCM
5.	System complexity	Complex	Simple	Simple	Simple
6.	Feedback from output	No feedback	Feedback is present	Feedback is present	Feedback is present
7.	Noise immunity	Very good	Very good	Very good	Very good
8.	Use of repeaters	Possible	Possible	Possible	Possible

Review Questions

Short Answer Questions :

Q. 1 Give the elements for pulse communication system.

Q. 2 Define pulse modulation. Give the types of pulse modulation.

Q. 3 Define Nyquist and Nyquist interval.

Q. 4 What is quantizing noise ?

Q. 5 Define companding, encoding.

Q. 6 State the applications of PCM signals.

Q. 7 Explain why is companding done in PCM ?

Q. 8 Explain Nyquist rate.

Q. 9 What are the advantages of ADM over DM ?

Q. 10 State the types of distortions observed in a DM system.

Q. 11 Explain the slope overload distortion. How can it be minimized ?

Q. 12 What is granular noise ?

Q. 13 What is quantization noise in PCM ? How to reduce it ?

Q. 14 How is the "information" transmitted in a PCM system ?

Q. 15 What is quantization ?

Q. 16 What is quantization error ? What is its maximum value ?

Q. 17 How to reduce the quantization error ?

Q. 18 What are the advantages of PCM ?

Q. 19 What are the disadvantages of PCM ?

Q. 20 Why is companding used ?

Q. 21 What information do you transmit in DM system ?

Q. 22 What is the cause of slope overload error in DM ?

Q. 23 How to reduce the slope overload error ?

Q. 24 What is the signaling rate of a DM system ?

Q. 25 What is granular noise ?

Q. 26 How to reduce granular noise ?

Q. 27 Why is quantization necessary ?

Q. 28 What is the difference between DM and ADM ?

Q. 29 State the advantages of ADM over DM.

Q. 30 Why is the signaling rate of DPCM is less than that of a PCM ?

Q. 31 What is the role of a predictor in DPCM system ?

Long Answer Questions :

Q. 1 Write a note on sample and hold circuit.

Q. 2 Explain the generation of flat topped samples using sample and hold circuit.

Q. 3 State and explain sampling theorem.

Q. 4 Draw and explain the block diagram for generation of PCM signal.

Q. 5 Write a note on quantization process.

Q. 6 Write notes on :

1. Delta Modulation (DM).
2. Adaptive delta modulation.

Q. 7 Explain different techniques of practical sampling.

Q. 8 Explain the delta modulation system. What is slope overload ?

Q. 9 Write a note on DPCM.

Q. 10 Compare PCM, DM, ADM and DPCM.

Q. 11 Explain the ADM transmitter and receiver.

Q. 12 How is slope overload reduced in ADM ?

Q. 13 Is the signal and noise separable in PCM ?

❑❑❑

CHAPTER 3

Baseband Digital Transmission

Unit II

Syllabus :

Digital multiplexing : Multiplexers and hierarchies, Data multiplexers, Data formats and their spectra, Synchronization : Bit synchronization, Scramblers, Frame synchronization, Inter-symbol interference, Equalization.

3.1 Introduction :

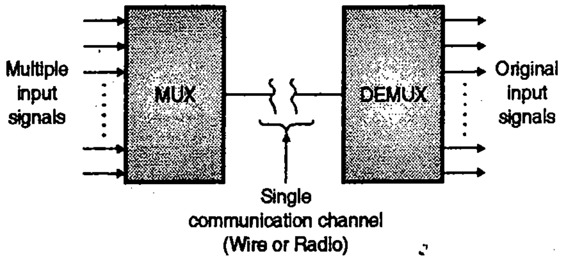
- In case of the baseband signaling, the waveforms at the receiver input are in the form of pulses. However the shapes of these pulses are not the ideal shapes.
- The filters used at the transmitter and the channel itself (which acts as another filter) are responsible for this degradation in the shape of received pulses. This is called as intersymbol interference (ISI).
- Therefore a detector should perform the following duties :
  1. Recover the baseband pulse with minimum amount of degradation.
  2. The SNR at the output of the receiver should be as high as possible.
  3. The ISI in the received pulses should be equal to zero.
- These goals can be achieved by means of a technique called equalization.
- In this chapter various topics related to baseband detection techniques have been discussed. In addition to that some other topics such as data formats, multiplexing, synchronization etc. also have been discussed.

3.2 Introduction to Multiplexing :

- Multiplexing is the process of simultaneously transmitting two or more individual signals over a single communication channel.
- Due to multiplexing it is possible to increase the number of communication channels so that more information can be transmitted.
- The typical applications of multiplexing are in telemetry and telephony or in the satellite communication.

3.2.1 Concept of Multiplexing :

- The concept of a simple multiplexer is illustrated in Fig. 3.2.1.
- The multiplexer receives a large number of different input signals.
- Multiplexer has only one output which is connected to the single communication channel.
- The multiplexer combines all input signals into a single composite signal and transmits it over the communication medium.
- Sometimes the composite signal is used for modulating a carrier before transmission.
- At the receiving end, of communication link, a demultiplexer is used to separate out the signals into their original form.



(L-105) **Fig. 3.2.1 : Concept of multiplexing**

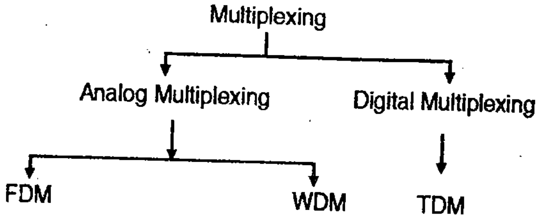
- The operation of demultiplexer is exactly opposite to that of a multiplexer. Demultiplexing is the process which is exactly opposite to that of multiplexing.

3.2.2 Types of Multiplexing :

- There are three basic types of multiplexing. They are :
  1. Frequency division multiplexing (FDM)
  2. Time division multiplexing (TDM).
  3. Wavelength division multiplexing (WDM).

3.2.3 Classification of Multiplexing :

- The multiplexing techniques can be broadly classified into two categories namely analog and digital.
- Analog multiplexing can be either FDM or WDM and digital multiplexing is TDM.
- Fig. 3.2.2 shows the classification of multiplexing techniques.



(L-106) Fig. 3.2.2 : Classification of multiplexing techniques

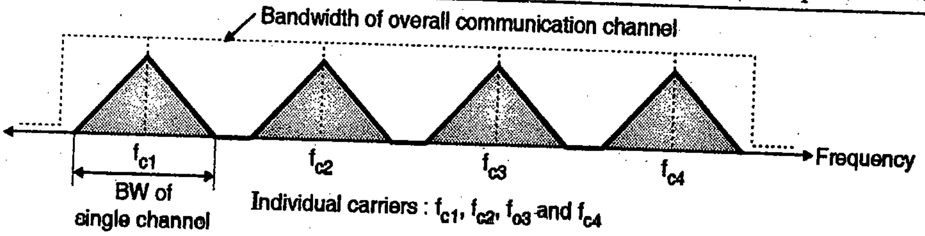
- Generally the FDM and WDM systems are used to deal with the analog information whereas the TDM systems are used to handle the digital information.
- In FDM many signals are transmitted simultaneously where each signal occupies a different frequency slot within a common bandwidth.
- In TDM the signals are not transmitted at a time, instead they are transmitted in different time slots.

3.3 Frequency Division Multiplexing (FDM) :

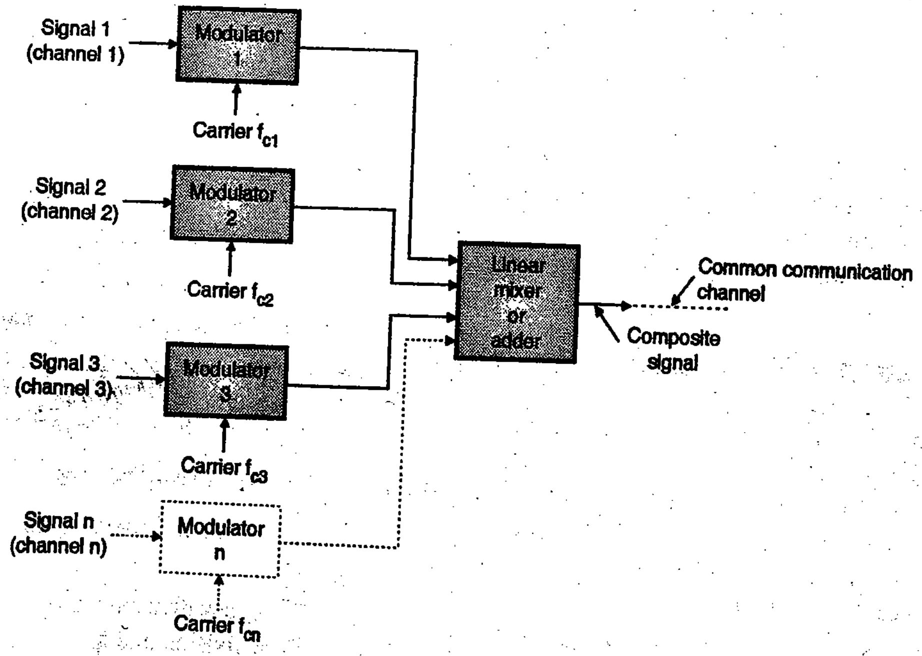
- The operation of FDM is based on sharing the available bandwidth of a communication channel among the signals to be transmitted.
- That means many signals are transmitted simultaneously with each signal occupying a different frequency slot within the total available bandwidth.
- Each signal to be transmitted modulates a different carrier. The modulation can be AM, SSB, FM or PM.
- The modulated signals are then added together to form a composite signal which is transmitted over a single channel.
- The spectrum of composite FDM signal is shown in Fig. 3.3.1(a).
- Generally the FDM systems are used for multiplexing the analog signals.

3.3.1 FDM Transmitter (Multiplexing Process) :

- Fig. 3.3.1(b) shows the block diagram of an FDM transmitter. The signals which are to be multiplexed will each modulate a separate carrier.



(L-107) Fig. 3.3.1(a) : Spectrum of FDM signal



(L-108) Fig. 3.3.1(b) : The FDM transmitter

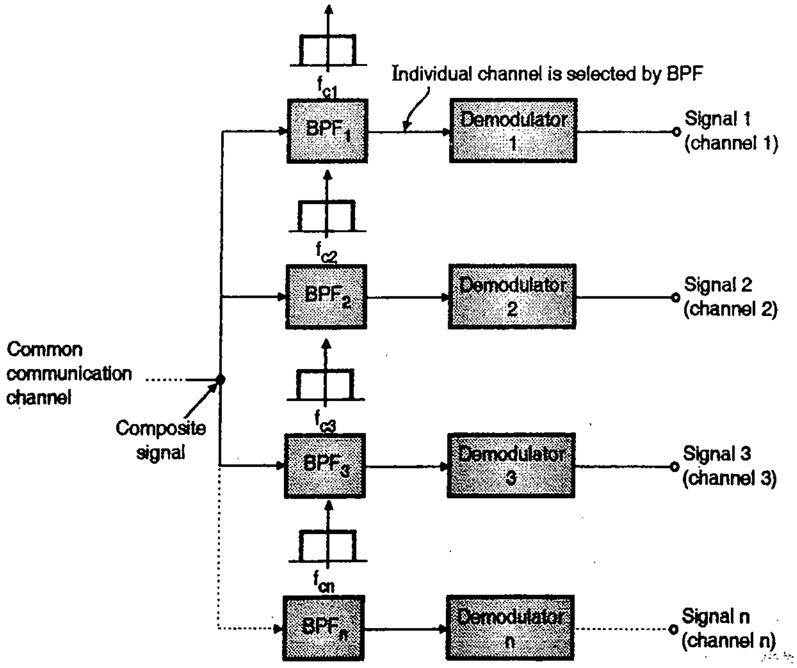
- The type of modulation can be AM, SSB, FM or PM.
- The modulated signals are then added together to form a complex signal which is transmitted over a single channel.

Operation of the FDM transmitter :

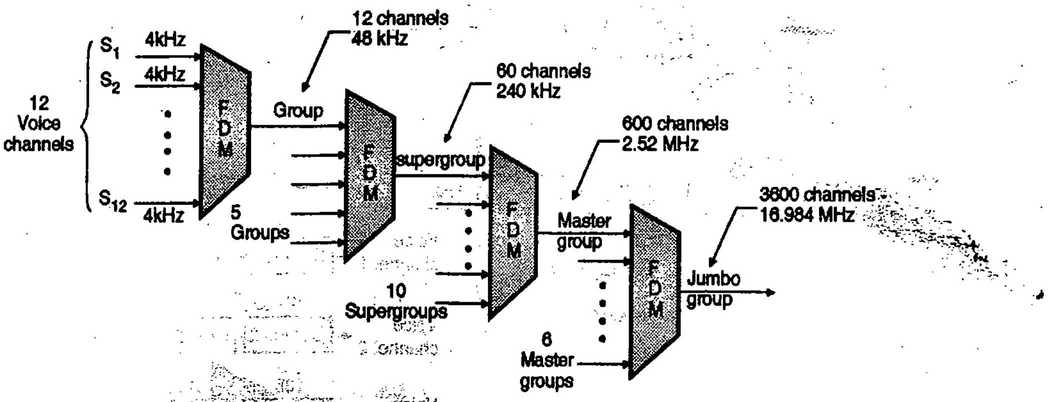
- Each signal modulates a separate carrier. The modulator outputs will contain the sidebands of the corresponding signals.
- The modulator outputs are added together in a linear mixer or adder. The linear mixer is different from the normal mixers. Here the sum and difference frequency components are not produced. But only the algebraic addition of the modulated outputs will take place.
- Different signals are thus added together in the time domain but they have their own separate identity in the frequency domain. This is as shown in the Fig. 3.3.1(a).
- The composite signal at the output of mixer is transmitted over the single communication channel as shown in Fig. 3.3.1(b). This signal can be used to modulate a radio transmitter if the FDM signal is to be transmitted through air.

3.3.2 FDM Receiver (Demultiplexing Process) :

- The block diagram of an FDM receiver is as shown in Fig. 3.3.1(c). The composite signal is applied to a group of band pass filters (BPF).
- Each BPF has a center frequency corresponding to one of the carriers used in the transmitter i.e. f_{c1}, f_{c2}, ... f_{cn} etc.
- The BPFs have an adequate bandwidth to pass all the channel information without any distortion.
- Each filter will pass through only its channel and reject all the other channels. Thus all the multiplexed channels are separated out.
- The channel demodulator then removes the carrier and recovers the original signal back.



(L-109) Fig. 3.3.1(c) : FDM receiver



(L-110) Fig. 3.3.2 : FDM hierarchy

3.3.3 The Analog Carrier System :

- To maximize the efficiency of their infrastructure, the telephone companies have used multiplexing technique for lower bandwidth lines.
- In this way it is possible to combine many switched or leased lines into fewer but bigger channels.
- One of such hierarchical system is used by AT and T. It is as shown in Fig. 3.3.2 and is made up of groups, super groups, master groups and jumbo groups.

The levels of multiplexing is also called as multiplexing hierarchy.

- The different levels of multiplexing which is also called multiplexing hierarchy is as follows :

Level (1) : Basic Group. [12 voice channels multiplexed together].

↓

Level (2) : Super Group. [Upto 5 basic groups multiplexed together i.e. upto 60 voice channels].

↓

Level (3) : Master Group. [Upto 10 super groups multiplexed together i.e. upto 600 voice channels].

↓

Level (4) : Jumbo Group. [Upto 6 master groups multiplexed together i.e. upto 3600 voice channels].

(G-1975)

- This hierarchy is used by AT and T and shown in Fig. 3.3.2.

3.3.4 Advantages of FDM :

1. A large number of signals (channels) can be transmitted simultaneously.
2. FDM does not need synchronization between its transmitter and receiver for proper operation.
3. Demodulation of FDM is easy.
4. Due to slow narrow band fading only a single channel gets affected.

3.3.5 Disadvantages of FDM :

1. The communication channel must have a very large bandwidth.
2. Intermodulation distortion takes place.
3. Large number of modulators and filters are required.
4. FDM suffers from the problem of crosstalk.
5. All the FDM channels get affected due to wideband fading.

3.3.6 Applications of FDM :

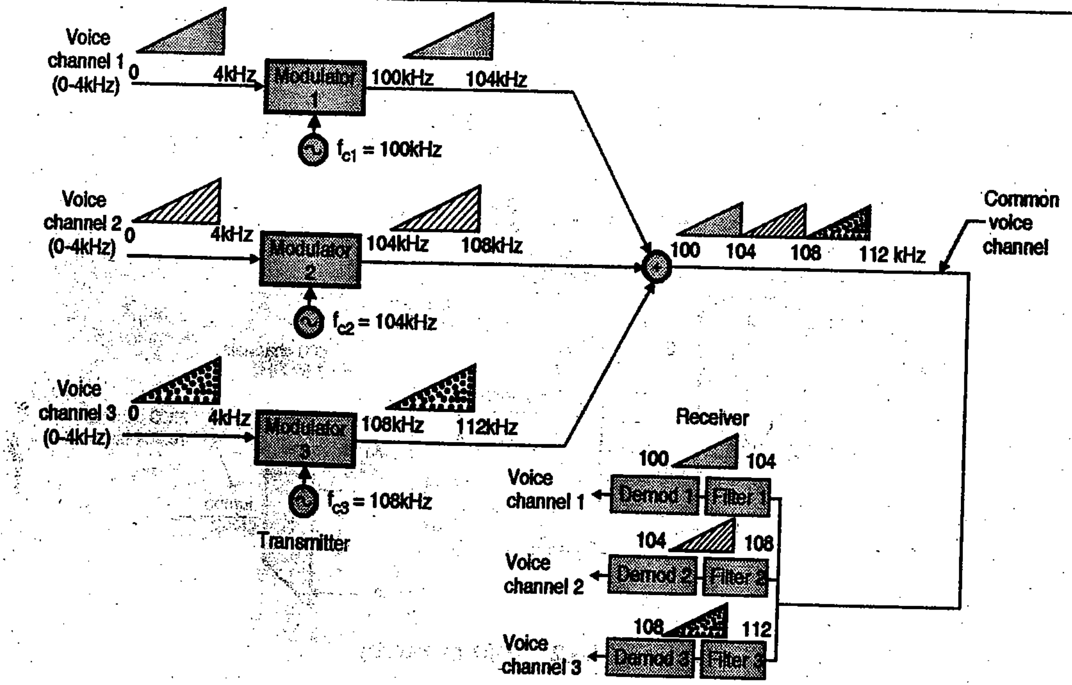
Some of the important applications of FDM are :

1. Telephone systems.
2. AM (amplitude modulation) and FM (frequency modulation) radio broadcasting.
3. TV broadcasting
4. First generation of cellular phones used FDM.

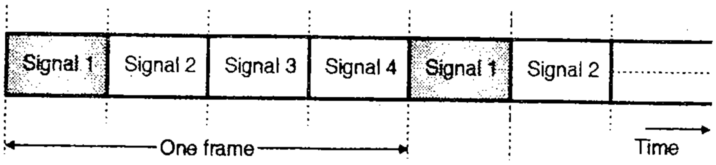
Ex. 3.3.1 : Draw the FDM system to combine three voice channels. Each voice channel occupies a bandwidth of 4 kHz. The common voice channel has a bandwidth of 12 kHz from 100 kHz to 112 kHz.

Soln. :

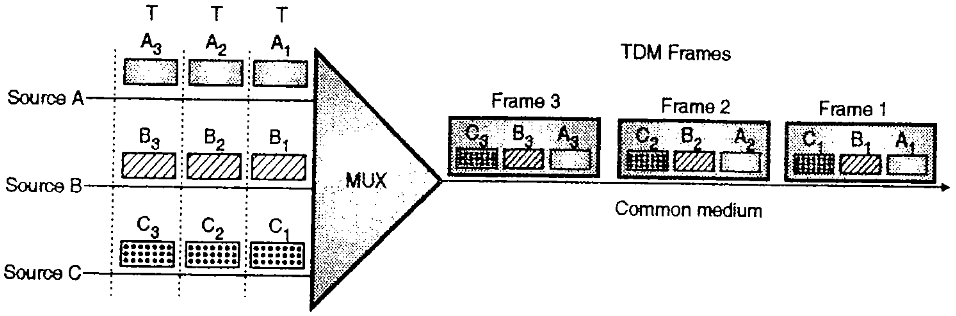
Fig. P. 3.3.1 shows the required FDM system.



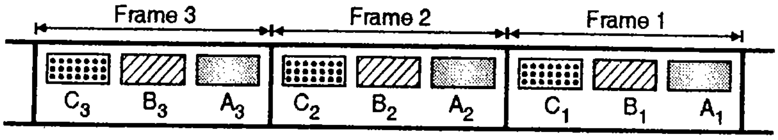
(L-117) Fig. P. 3.3.1 : Required FDM system



(L-122) **Fig. 3.4.1 : Principle of TDM**



(L-123) **Fig. 3.4.2 : TDM system**



(L-124) **Fig. 3.4.3 : TDM frames**

3.4 Synchronous Time Division Multiplexing (TDM) :

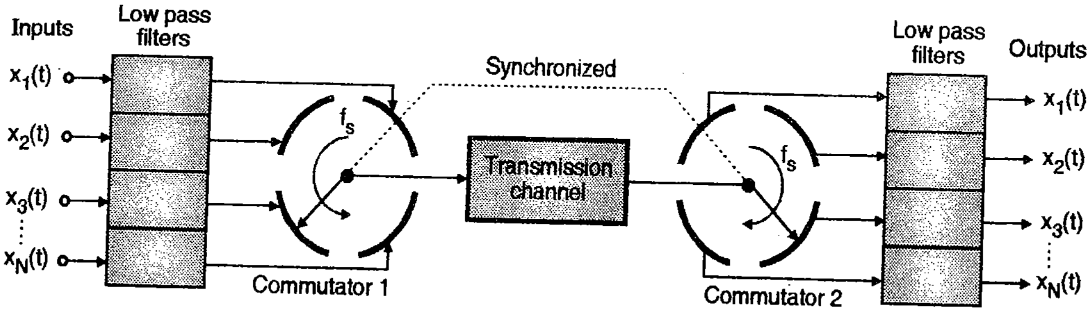
- The process called multiplexing is used in order to utilize common transmission channel or medium to transmit more than one signals simultaneously.
- TDM is a digital multiplexing process.
- In TDM all the signals to be transmitted are not transmitted simultaneously. Instead, they are transmitted one-by-one.
- Thus each signal will be transmitted for a very short time. One cycle or frame is said to be complete when all the signals are transmitted once on the transmission channel. The TDM principle is illustrated in Fig. 3.4.1.
- As shown in the Fig. 3.4.1 one transmission of each channel completes one cycle of operation called as a "Frame".
- The TDM system can be used to multiplex analog or digital signals, however it is more suitable for the digital signal multiplexing.
- The concept of TDM will be more clear if you refer to Fig. 3.4.2.
- The data flow of each source (A, B or C) is divided into units (say A_1, A_2 or B_1, C_1 etc.)
- Then one unit from each source is taken and combined to form one frame. The size of each unit such as A_1, B_1 etc. can be 1 bit or several bits.
- Fig. 3.4.3 shows the frames of TDM signal. For 3 inputs being multiplexed, a frame of TDM will consist of 3 units i.e. one unit from each source.
- Similarly for n number of inputs, each TDM frame will consist of n units.
- The TDM signal in the form of frames is transmitted on the common communication medium.

Data rate :

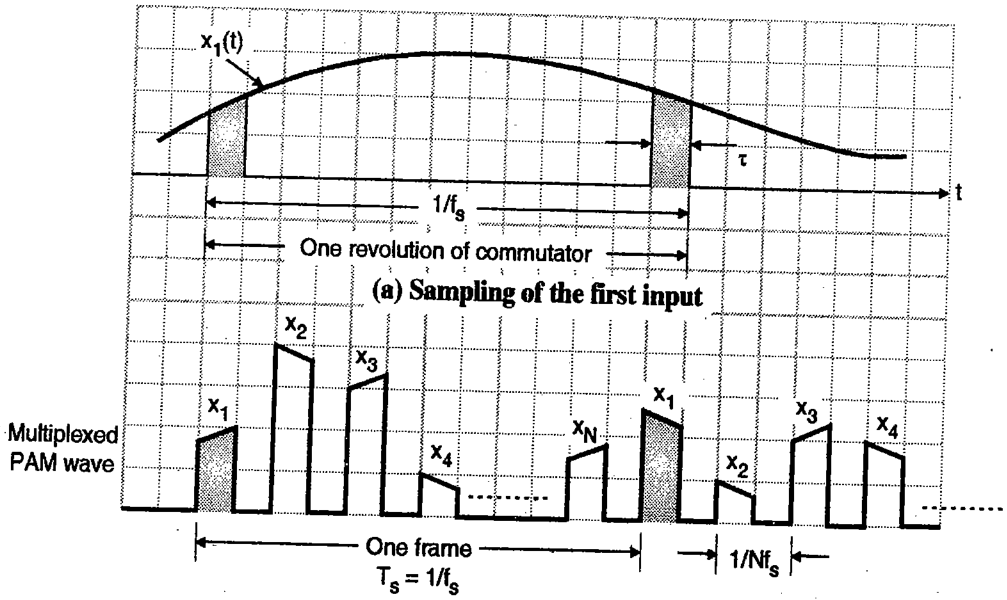
- For a TDM, the data rate of the multiplexed signal is always n times the data rate of individual sources, where n is the number of sources.
- So if three sources are being multiplexed, then the data rate of the TDM signal is three times higher than the individual data rate.
- Naturally the duration of every unit (A_1 or B_1 etc.) in TDM signal is n times shorter than the unit duration before multiplexing.

3.4.1 PAM - TDM System :

- The TDM system which is going to be discussed now, combines the concepts of PAM and TDM both. The TDM system is as shown in Fig. 3.4.4.



(L-125) Fig. 3.4.4 : PAM/TDM system



(b) Multiplexed PAM signal, transmitted on the transmission channel

(L-126) Fig. 3.4.5

The operation of the system is as follows :

- The multiplexer here is a single pole rotating switch or commutator. It can be a mechanical switch or an electronic switch. It rotates at f_s rotations per second.
- As the switch arm rotates, it is going to make contact with the position 1, 2, 3 or N for a short time. To these contacts are connected the N analog signals which are to be multiplexed.
- Thus the switch arm will connect these N input signals one by one to the communication channel.
- The waveform of a TDM signal which is being transmitted is as shown in Fig. 3.4.5. It shows that the rotary switch samples each channel during each of its rotations. Each rotation corresponds to one frame. Hence 1 frame is completed in T_s seconds where $T_s = 1/f_s$.
- At the receiver, there is one more rotating switch or commutator used for demultiplexing.
- It is important to note that this switch must rotate at the same speed as that of the commutator 1 at the transmitter and its position must be synchronized with commutator 1 in order to ensure proper demultiplexing.
- The same principle of multiplexing can be used for multiplexing more number of signals.

Interleaving :

- On the multiplexer side the commutator-1 opens in front of a connection, that connection has the opportunity to send its bit on to the channel.
- This process is called as interleaving.

Ex. 3.4.1 : 3 signals having a data rate of 2 kbps are grouped together by means of time division multiplexing. Each unit consists of 1 bit. Calculate :

1. The bit duration before multiplexing.
2. The transmission rate of TDM.
3. The duration of each time slot in TDM.
4. The duration of one TDM frame.

Soln. :

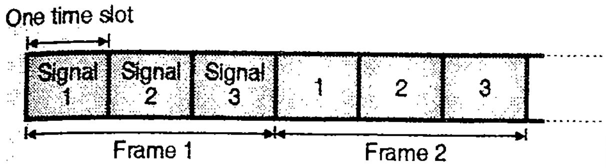
Step 1 : Duration of a bit before multiplexing :

- Each signal has a data rate of 2 kbps. That means 2000 bits per second.
- Hence the duration of each bit is,

$$T_b = \frac{1}{2000} = 0.5 \text{ mS} \quad \text{...Ans.}$$

Step 2 : Transmission rate of TDM :

- The TDM frame is shown in Fig. P. 3.4.1.



(L-127) **Fig. P. 3.4.1 : TDM frames**

- As discussed earlier the transmission rate of TDM is n times higher than the bit rate of each source.

$$\text{Transmission rate of TDM} = n \times 2000$$

$$= 3 \times 2000$$

$$= 6000 \text{ bps or } 6 \text{ kbps} \quad \text{...Ans.}$$

Step 3 : Duration of time slot in TDM :

$$\text{Duration of each time slot in TDM} = \frac{1}{6000}$$

$$= 166.67 \quad \text{...Ans.}$$

Step 4 : Frame duration :

$$\text{Duration of 1 frame} = n \times \text{duration of one slot}$$

$$= 3 \times 166.67\ \mu\text{S}$$

$$= 0.5 \text{ mS} \quad \text{...Ans.}$$

Note : The duration of a TDM frame is always equal to the duration of one unit before multiplexing.

Ex. 3.4.2 : Three channels are to be multiplexed using TDM technique. The rate of each channel is 150 bytes per second. In TDM, one byte per channel is to be multiplexed.

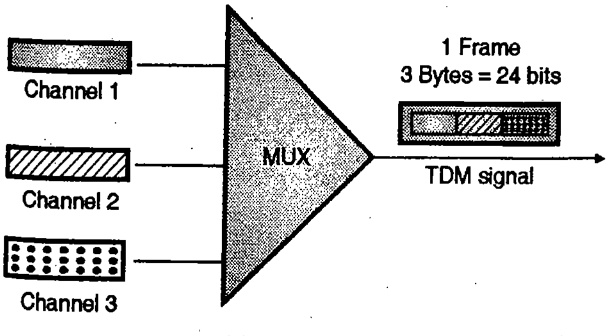
Calculate :
1. Frame size
2. Frame duration
3. Frame rate and
4. Bit rate of the TDM signal.

Soln. :

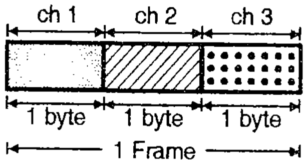
Given : Bit rate of each channel 150 bytes per sec, n = 3.
1 byte per channel to be multiplexed.

Step 1 : Draw the system block diagram :

Fig. P. 3.4.2(a) shows the block diagram of the TDM system and Fig. P. 3.4.2(b) shows one TDM frame.



(a) TDM system
Fig. P. 3.4.2 (Contd...)



(b) One frame of TDM
(L-128) **Fig. P. 3.4.2**

Step 2 : Frame size :

Each frame consists of one byte from each channel. So frame size is 3 bytes or 24 bits.

Step 3 : Frame duration and frame rate :

- The duration of a TDM frame is always equal to the duration of one unit before multiplexing.
- Here one unit before multiplexing is 1 byte i.e. 8 bits.

$$\therefore \quad \text{Frame duration} = 1 \text{ byte duration}$$

- But each channel transmits at 150 bytes/sec.

$$\therefore \quad \text{Frame duration} = \frac{1}{150} = 6.666 \text{ mS} \quad \text{...Ans.}$$

$$\text{Frame duration} = \frac{1}{\text{Frame duration}}$$

$$= \frac{1}{6.666 \times 10^{-3}}$$

$$= 150 \text{ frames/sec.} \quad \text{...Ans.}$$

Step 4 : Bit rate of TDM signal :

$$\text{Bit rate of TDM} = \text{Number of bits per frame} \times \text{Number of frames per second.}$$

$$= 24 \times 150 = 3600 \text{ bps} \quad \text{...Ans.}$$

3.4.2 Signaling Rate (r) :

The signaling rate of a TDM system is defined as the number of pulses transmitted per second. It is denoted by "r". Let us now derive the expression for the signaling rate of the PAM-TDM system.

- Let W = Maximum frequency of all the input signals x_1 to x_N.
- Therefore as per Nyquist criteria, the sampling frequency $f_s \geq 2$ W. Therefore the speed of rotation of the commutators is f_s rotations per second with $f_s \geq 2W$.
- As shown in Fig. 3.4.6, one revolution of commutators corresponding to one frame contains one sample from each input signal.

$$\therefore \quad 1 \text{ Revolution} \Rightarrow 1 \text{ frame} \Rightarrow N \text{ pulses} \quad \text{...(3.4.1)}$$

- 1 frame period is $(1/f_s)$ i.e. T_s seconds. Therefore in "T_s" seconds "N" number of pulses are transmitted. Hence the pulse to pulse spacing within the frame is given by,

$$\text{Pulse to pulse spacing} = \frac{T_s}{N} = \frac{1}{Nf_s} \quad \text{...(3.4.2)}$$

- As the period of one pulse (ON + OFF) is $(1/Nf_s)$ seconds, the number of pulses per second is given by, Number of pulses per second = Nf_s

- This is nothing but the signaling rate.

$\therefore$ Signaling rate of a TDM system $= r$

$= Nf_s$ pulses/second.

But as $f_s \geq 2W$.

Signaling rate of a TDM system $= r \geq 2NW$ pulses/second ...(3.4.3)

- A TDM system is supposed to have its signaling rate as high as possible. It is evident from the expressions above that the signaling rate can be increased by increasing the sampling rate f_s and/or the number of input signals N.

3.4.3 Transmission Bandwidth of a TDM Channel :

- The minimum transmission bandwidth of a PAM-TDM channel is given by,

$$B_T = \frac{1}{2} \text{signaling rate}$$

$\therefore$ Minimum transmission bandwidth

$$B_T \geq \frac{1}{2} \times 2\,NW$$

$\therefore$ Minimum transmission bandwidth

$$B_T = NW \qquad \text{...(3.4.4)}$$

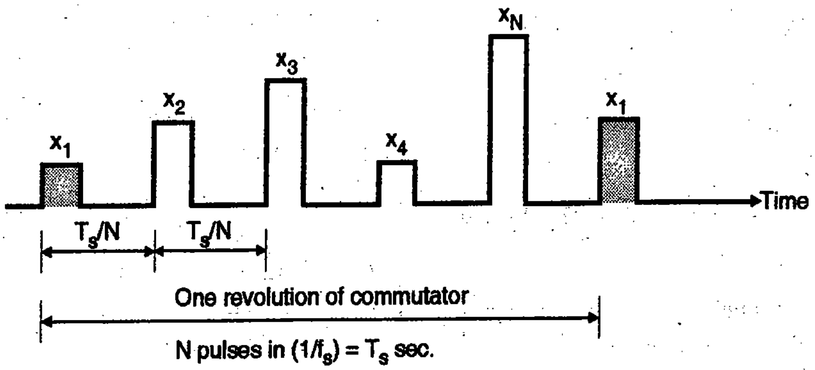
3.4.4 Synchronization in TDM System :

Synchronization in TDM PAM system :

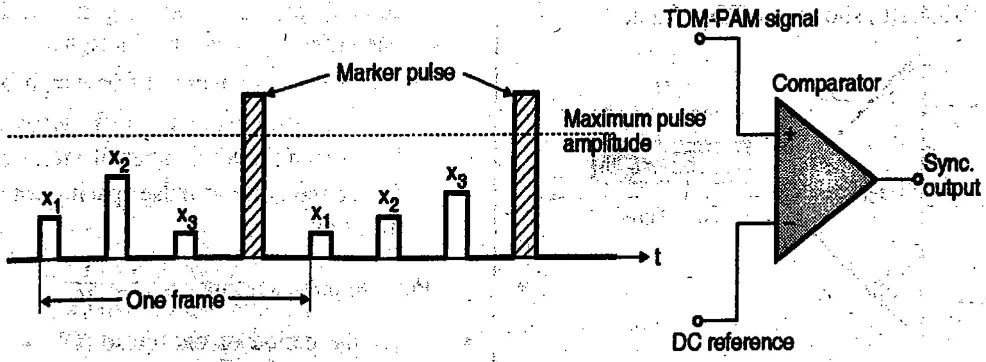
- The multiplexed PAM signals can be received properly if and only if the transmitter and receiver commutators are synchronized to each other in terms of the speed and the position.
- In order to ensure synchronization, a marker pulse is introduced at the end of each frame in the transmitted signal as shown in Fig. 3.4.7.
- The amplitude of this pulse is higher than the maximum permissible amplitude of the multiplexed channels.
- At the receiver the received signal is compared with a DC reference level. The comparator responds to only the marker pulse to produce output.
- Thus the marker pulse is separated from the remaining multiplexed channels.
- Due to the introduction of synchronizing pulse, only three signals instead of four can now be transmitted.

Synchronization in digital TDM system :

- In digital TDM, the inputs are digital bit streams. All the digital pulses are of same amplitudes. So the synchronizing techniques for TDM-PAM system cannot be used here.
- If the synchronization is lost then a bit belonging to one channel may be received by a wrong channel.



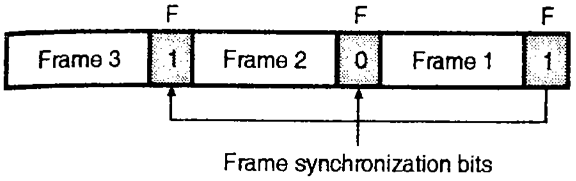
(L-129) Fig. 3.4.6 : Calculation of number of pulses per second for PAM-TDM system



(D-516) Fig. 3.4.7 : Frame synchronization and detection

Synchronization bit :

- In order to establish synchronization between the transmitter and receiver, one synchronization bit is added at the beginning of each TDM frame as shown in Fig. 3.4.8.



(D-524) **Fig. 3.4.8 : Frame synchronization in TDM**

- These bits are called frame synchronizing bits or simply framing bits.
- The framing bits will follow a pattern frame to frame. For example the pattern shown in Fig. 3.4.8 is 101.
- The framing bit pattern will allow the demux to synchronize itself to the mux.

3.4.5 Advantages of TDM :

1. Full available channel bandwidth can be utilized for each channel.
2. Intermodulation distortion is absent.
3. TDM circuitry is not very complex.
4. The problem of crosstalk is not severe.

3.4.6 Disadvantages of TDM :

1. Synchronization is essential for proper operation.
2. Due to slow narrowband fading, all the TDM channels may get wiped out.

3.4.7 Applications of TDM :

1. Multiplexing of digital signals.
2. Digital telephony
3. Satellite communications.
4. Fiber optic communication
5. Wireless communication applications.

3.4.8 Bit Padding :

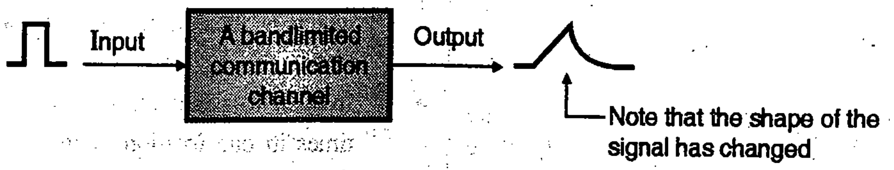
- Till now we have assumed that the data rate of all the channels is the same.
- But practically it won't be so. We will have to multiplex the channels having different data rates.
- These data rates are not integer multiples of each other. But in order to multiplex them using TDM, they need to be integer multiples of each other.
- This can be achieved by a technique called **bit padding.**
- In the bit padding technique the multiplexer adds extra bits to the bit stream of a source so that the bit rates of all the sources to be multiplexed become integer multiples of each other.
- For example if the bit rate of one source is 3.5 times the bit rate of the other source then by using the bit padding, we can make it 4 times the bit rate of the other.
- These extra bits however do not contain any information. So they are discarded by the demultiplexer.

3.4.9 Crosstalk and Guard Times :

- Crosstalk basically means interference between the adjacent TDM channels. It is the unwanted coupling of information from one channel to the other.
- The guard time (T_g) is the time spacing introduced between the adjacent TDM channels.

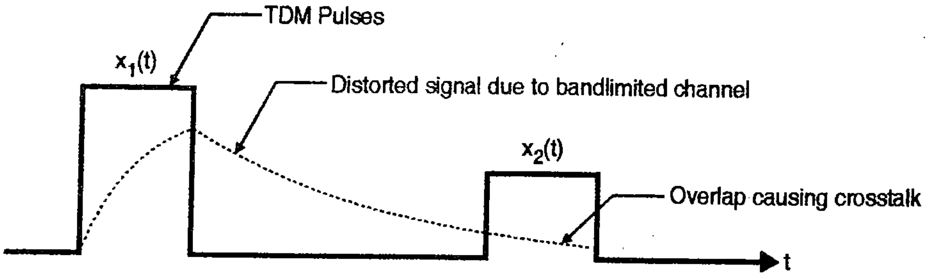
Cause of crosstalk :

- The communication channel over which the TDM signal is travelling should ideally have an infinite bandwidth in order to avoid the signal distortion.
- But practically all the communication channels have a finite bandwidth. Such channels are called as the bandlimited channels.
- Whenever we pass a signal over such bandlimited channel, the shape of the signal will change as shown in Fig. 3.4.9(a).
- When a PAM-TDM signal is transmitted over a bandlimited channel, the signal corresponding to x_1 (t) will get mixed with x_2 (t) as shown in Fig. 3.4.9(b) and the overlap will result into crosstalk.



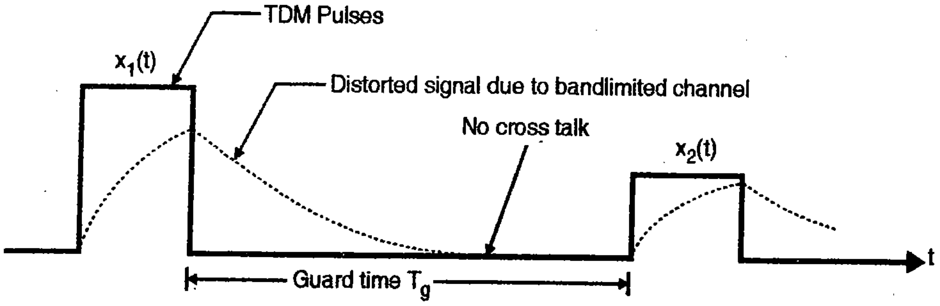
(a) Transmission of signal over a bandlimited channel

Fig. 3.4.9 (Contd...)



(b) Crosstalk in TDM

(E-781) **Fig. 3.4.9**



(E-782) **Fig. 3.4.10 : Elimination of crosstalk due to guard time**

- One more cause for the crosstalk between the adjacent TDM signals is the use of bandlimiting filters. Due to these filters the shapes of the TDM pulses are distorted, they get overlapped and crosstalk will take place.

Guard time (T_g) :

- The crosstalk resulting from the pulse overlap shown in Fig. 3.4.10 can be reduced by introducing guard time (T_g) of sufficient duration between the adjacent TDM pulses as shown in Fig. 3.4.10.
- If the crosstalk is to be kept below – 30 dB then the value of guard time (T_g) that should be introduced between the adjacent TDM signals is given by,

$$\text{Guard time } T_g > 0.55/B_T \qquad ...(3.4.5)$$

where B_T = Bandwidth of the channel.

- This equation shows that the guard time required to avoid the crosstalk will decrease with increase in the channel bandwidth B_T. As T_g reduces we can increase the signalling rate of the PAM-TDM system.

3.4.10 Solved Examples on TDM / PAM :

Ex. 3.4.3 : Two analog signals m_1 (t) and m_2 (t) are to be transmitted over a common channel by means of time division multiplexing. The highest frequency of m_1 (t) is 4 kHz and that of m_2 (t) is 4.5 kHz. What is the minimum value of permissible sampling rate ?

Soln. :

The highest frequency component of the composite signal consisting of m_1 (t) and m_2 (t) is 4.5 kHz. Therefore the minimum value of permissible sampling rate is,

$$f_{s\,(min)} = 2 \times 4.5 \text{ kHz} = 9 \text{ kHz} \quad \text{...Ans.}$$

Ex. 3.4.4 : A signal x_1 (t) is bandlimited to 3 kHz. There are three more signals x_2 (t), x_3 (t) and x_4 (t) which are bandlimited to 1 kHz each. These signals are to be transmitted by a TDM system.

1. Design a TDM scheme where each signal is sampled at its Nyquist rate.
2. What must be the speed of the commutator ?
3. Calculate the minimum transmission bandwidth of the channel.

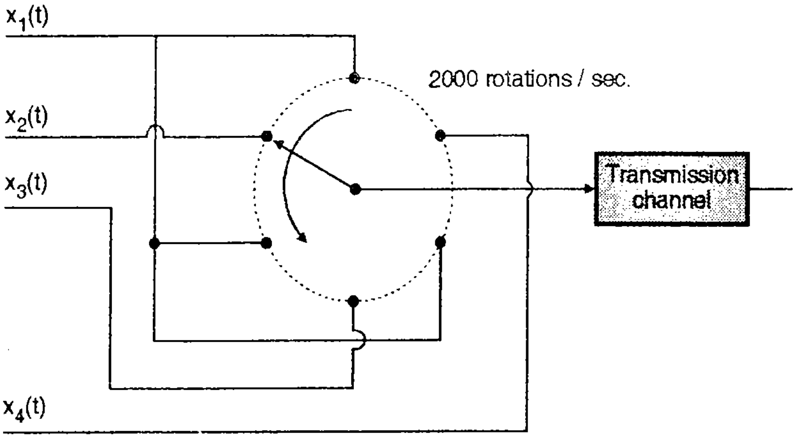
Soln. :

(a) Table P. 3.4.4 shows different message signal with corresponding Nyquist rates.

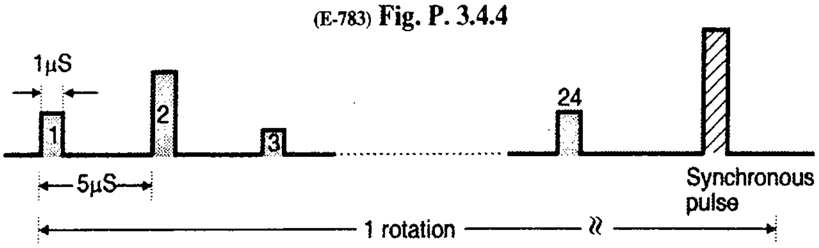
Table P. 3.4.4

Message signal	Bandwidth	Nyquist rate
x_1 (t)	3 kHz	6 kHz
x_2 (t)	1 kHz	2 kHz
x_3 (t)	1 kHz	2 kHz
x_4 (t)	1 kHz	2 kHz

- If the sampling commutator rotates at the rate of 2000 rotations per second then the signals x_2 (t), x_3 (t) and x_4 (t) will be sampled at their Nyquist rate. But we have to sample x_1 (t) also at its Nyquist rate which is three times higher than that of the other three.
- In order to achieve this we should sample x_1 (t) three times in one rotation of the commutator. Therefore the commutator must have atleast 6 poles connected to the signals as shown in Fig. P. 3.4.4.



(E-783) **Fig. P. 3.4.4**



(E-784) **Fig. P. 3.4.5**

(b) The speed of rotation of the commutator is 2000 rotations/sec.

(c) Number of samples produced per second is calculated as follows :

x_1 (t) produces $3 \times 2000 = 6000$ samples/sec.

x_2 (t), x_3 (t) and x_4 (t) produce 2000 samples/sec. each.

$$\therefore \text{ Number of samples per second } = 6000 + (3 \times 2000)$$

$$= 12000 \text{ samples/sec.}$$

$$\therefore \text{ Signaling rate } = 12000 \text{ samples/sec.}$$

(d) The minimum channel bandwidth is,

$$B_T = \frac{1}{2} \text{ signaling rate}$$

$$= 12000/2$$

$$\therefore B_T = 6000 \text{ Hz} \quad \text{...Ans.}$$

Ex. 3.4.5 : Twenty four voice signals are sampled uniformly and then time division multiplexed. The sampling operation uses flat top samples with 1 μS duration. The multiplexing operation includes provision for synchronization by adding an extra pulse of appropriate amplitude and 1 μS duration. The highest frequency component of each voice signal is 3.4 kHz.

(a) Assuming a sampling rate of 8 kHz, calculate the spacing between successive pulses of the multiplexed signal.

(b) Repeat (a), assuming the use of Nyquist rate sampling.

Soln. :

(a) It has been given that :

1. Sampling rate = 8 kHz = 8000 samples/sec.
2. There are 24 voice signals + 1 synchronizing pulse.
3. Pulse width of each voice channel and synchronizing pulse is 1 μS.

Time taken by the commutator for 1 rotation

$$= \frac{1}{8000} = 125 \ \mu\text{sec.}$$

Number of pulses produced in 1 rotation

$$= 24 + 1 = 25$$

∴ The leading edges of the pulses are at $\frac{125}{25} = 5$ μsec. distance as shown in Fig. P. 3.4.5.

Therefore spacing between successive pulses

$$= 5 - 1 = 4 \ \mu\text{S} \quad \text{...Ans.}$$

(b) Nyquist rate of sampling = 2 × 3.4 kHz = 6.8 kHz.

That means 6800 samples are produced per second. One rotation of commutator takes

$1/6800 = 147\ \mu S$ time.

$\therefore$ 147 μsec corresponds to 25 pulses

$\therefore$ 1 pulse corresponds to 5.88 μsec.

As the pulse width of each pulse is 1 μ sec, the spacing between adjacent pulses will be 4.88 μsec. and if we assume $\tau = 0$ then the spacing between the adjacent pulses will be 5.88 μ sec.

Ex. 3.4.6 : Six message signals each of bandwidth 5 kHz are time division multiplexed and transmitted. Calculate the signalling rate and the minimum channel bandwidth of the PAM/TDM channel.

Soln. :

The number of channels N = 6

Bandwidth of each channel, W = 5 kHz

$$\text{Minimum sampling rate} = 2 \times 5\ \text{kHz} = 10\ \text{kHz}$$

$$\text{Signaling rate} = \text{Number of bits per second} = 6 \times 10\ \text{kHz} = 60\ \text{Kbits/sec.}$$ **...Ans.**

Minimum channel bandwidth to avoid cross talk in PAM/TDM is,

$$B_T = NW = 6 \times 5\ \text{kHz} = 30\ \text{kHz}$$ **...Ans.**

Ex. 3.4.7 : Sketch a channel interleaving scheme for time division multiplexing the following PAM signals : Find 4 kHz telephone channels and one 20 kHz music channel. Find the pulse repetition rate of the multiplexed signal and estimate the minimum system bandwidth required.

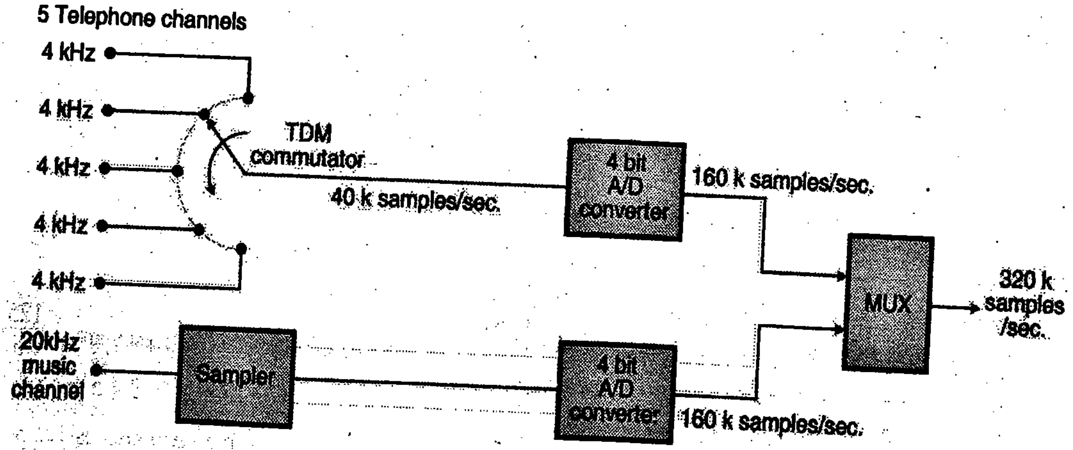
Soln. :

Each telephone channel of bandwidth 4 kHz must be sampled at Nyquist rate i.e. 2×4 kHz = 8 kHz. using a TDM commutator. The 20 kHz music channel must be sampled at 40 kHz (Nyquist rate) hence a separate sampler is required. The sampled signals are applied to two 4-bit A-D converters to obtain the equivalent digital signals. These signals are finally multiplexed using a multiplexer as shown in Fig. P. 3.4.7.

The TDM commutator output has a pulse repetition rate of 40K samples/sec as there are 5 channel and sampling rate is 8 kHz. Similarly the output of the separate sampler has a pulse repetition rate of 40 K samples/sec. The outputs of A-D converters have pulse repetition rates of $40 \times 4 = 160$ K samples/sec. Therefore pulse repetition rate at the output of a multiplexer is 160 + 160 = 320 K samples/sec.

$\therefore$ Pulse repetition rate of the system = 320 kHz.

$\therefore$ Bandwidth required = Bit rate = 320 kHz **....Ans.**



(E-785) **Fig. P. 3.4.7 : PAM-TDM system for Ex. 3.4.7**

3.4.11 Comparison of FDM and TDM Systems :

Sr. No.	FDM	TDM
1.	The signals which are to be multiplexed are added in the time domain. But they occupy different slots in the frequency domain.	The signals which are to be multiplexed can occupy the entire bandwidth but they are isolated in the time domain.
2.	FDM is usually preferred for the analog signals.	TDM is preferred for the digital signals.
3.	Synchronization is not required.	Synchronization is required.
4.	The FDM requires a complex circuitry at the transmitter and receiver.	TDM circuitry is not very complex.
5.	FDM suffers from the problem of crosstalk due to imperfect band pass filters.	In TDM the problem of crosstalk is not severe.
6.	Due to wideband fading in the transmission medium, all the FDM channels are affected.	Due to fading only a few TDM channels will be affected.
7.	Due to slow narrowband fading taking place in the transmission channel only a single channel may be affected in FDM.	Due to slow narrowband fading all the TDM channels may get wiped out.

3.5 Digital Multiplexers : SPPU : Dec. 07

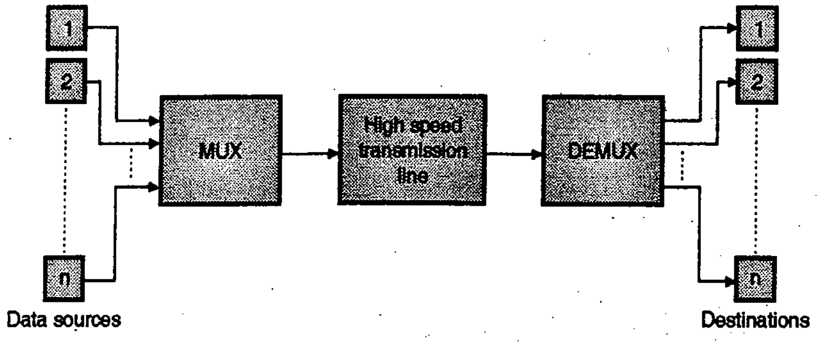
University Questions

Q. 1 What is digital multiplexing ? Draw AT and T hierarchy and compare the output rate with CCIT hierarchy. **(Dec. 07, 8 Marks)**

- We have already introduced the idea of TDM.
- In this section the multiplexing of digital signals at different bit rates has been discussed.
- With the help of digital multiplexing we can combine many digital signals such as computer outputs, digital voice, digitized facsimile and TV signals.
- Fig. 3.5.1 illustrates the concept of digital multiplexing and demultiplexing.
- The digital data can be multiplexed by using a bit-by-bit interleaving procedure.
- This can be achieved by using a selector switch which selects one bit from each input in a sequential manner and places it over the high speed transmission line (which acts as the transmission channel).
- At the receiving end the bits received on the common line are separated out and delivered to their respective destinations.

3.5.1 Principle of Digital Multiplexing :

- Digital multiplexing operates on the principle of "interleaving symbols" from two or more digital signals.
- This is similar to TDM but it does not have to undergo the periodic sampling and there is no need to preserve the shape of the waveform as in case of TDM.
- The signals which are to be multiplexed may come from digital data sources or from analog sources that have been converted into digital signals.



(E-793) Fig. 3.5.1 : Concept of multiplexing-demultiplexing

3.5.2 Functional Operations of a Digital Multiplexer :

SPPU : May 06, May 08, Dec. 08, May 09

University Questions

Q. 1 Write the functions performed by a multiplexer. What are three main categories of multiplexers ? **(May 06, 6 Marks)**

Q. 2 Write the functions performed by a multiplexer. Explain the three main categories of multiplexers. **(May 08, 8 Marks)**

Q. 3 Write the functions performed by a multiplexer. What are three main categories of multiplexers ? **(Dec. 08, May 09, 8 Marks)**

- A digital multiplexer is used to combine together the input bits from different sources to form one signal for transmission via a digital communication system.
- The multiplexed signal consists of source digits interleaved bit by bit or word by word.
- The important functions that must be performed by a multiplexer are as follows :
  1. To form a frame. A frame consists of atleast one bit from every input.
  2. A number of unique bits slots within the frame should be assigned to each input.
  3. To insert control bits which can be used for frame identification and synchronization.
  4. To make arrangement to allow any variations of the input bit rates.

3.5.3 Classification of Digital Multiplexers :

SPPU : May 06, Dec. 06, May 08, Dec. 08, May 09, Dec. 11, May 13

University Questions

Q. 1 Write the functions performed by a multiplexer. What are three main categories of multiplexers ? **(May 06, 6 Marks)**

Q. 2 Explain synchronous and quasi-synchronous multiplexers. **(Dec. 06, 8 Marks)**

Q. 3 Write the functions performed by a multiplexer. Explain the three main categories of multiplexers. **(May 08, 8 Marks)**

Q. 4 Write the functions performed by a multiplexer. What are three main categories of multiplexers ? **(Dec. 08, May 09, 8 Marks)**

Q. 5 What are the different types of multiplexers used in digital communication system ? Explain quasi-synchronous multiplexer in detail with a neat sketch. **(Dec. 11, 8 Marks, May 13, 10 Marks)**

- Various digital sources that are to be multiplexed, will have different bit rates.
- In practice, the bit rate variation poses the most serious design problem. There are three categories of digital multiplexers, as follows :
  1. Synchronous multiplexers
  2. Asynchronous multiplexers
  3. Quasi-synchronous multiplexers.

1. Synchronous multiplexers :

- In the synchronous digital multiplexers, a master clock governs all the sources. Therefore all the sources will operate at the same bit rate. So there is absolutely no variation in the bit rates of various sources.
- As the bit rate variations are completely eliminated, the synchronous multiplexing systems will have a very high throughput efficiency.
- But then these multiplexer need to make elaborate provisions to distribute the master clock signal, to all the sources.

2. Asynchronous multiplexers :

- The asynchronous digital multiplexers are used for the digital data sources which operate in the startstop mode.
- These sources produce data in the form of "bursts" of characters and the spacing between the bursts is not fixed. It keeps varying.
- We can use techniques such as "buffering" and "character interleaving" in order to combine these sources into a synchronous multiplexed bit stream.

3. Quasi-synchronous multiplexers :

- These multiplexers are used when the input bit rates have the same nominal value but vary within specific bounds.
- These multiplexers are arranged in a hierarchy of increasing bit rates to constitute the basic building blocks of an interconnected digital telecommunication system.

3.5.4 Multiplexing Hierarchy for Digital Communication :

SPPU : Dec 07, May 08, May 09, Dec. 10, May 11, Dec. 13, May 15, Dec. 15, May 16, Dec. 16

University Questions

Q. 1 What is digital multiplexing ? Draw AT and T hierarchy and compare the output rate with CCIT hierarchy. **(Dec. 07, 8 Marks)**

Q. 2 Diagram a digital multiplexing system that accommodates both analog and digital signals in a standard multiplexing hierarchy including the North American and CCIT. **(May 08, 8 Marks, May 09, 10 Marks)**

Q. 3 Explain T_1 carrier system and AT & T multiplexing hierarchy. **(Dec. 10, 8 Marks)**

Q. 4 Explain T_1 carrier system and hence compare AT-T and CCIT hierarchy of multiplexing. **(May 11, 7 Marks)**

Q. 5 What is digital hierarchy used in digital communication system ? Explain anyone with a neat sketch. **(Dec. 13, May 15, 6 Marks)**

Q. 6 Explain digital signal hierarchy using T1 carrier system. **(Dec. 15, 6 Marks)**

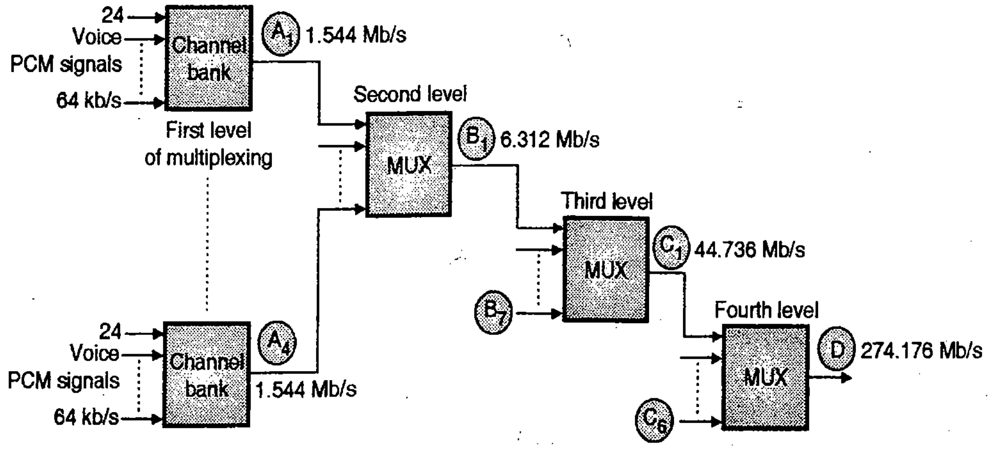
Q. 7 Draw and explain CCIT hierarchy of multiplexing. **(May 16, 7 Marks, Dec. 16, 6 Marks)**

- There are two slightly different multiplexing patterns used for digital communication namely the AT & T hierarchy and CCIT hierarchy as shown in Table 3.5.1.

Table 3.5.1 : Multiplexing hierarchies

	AT & T		CCIT	
	Number of inputs	**Output rate Mb/s**	**Number of inputs**	**Output rate Mb/s**
First level	24	1.544	30	2.048
Second level	4	6.312	4	8.448
Third level	7	44.736	4	34.368
Fourth level	6	274.176	4	139.264

- In both the hierarchies there are four levels of multiplexing. For both the types of systems, a 64 voice PCM unit is used as basic input and the structural layout is same as shown in Fig. 3.5.2.
- Fig. 3.5.2 shows the multiplexing hierarchy for digital communication.
- At the first level of multiplexing, a number of 64 PCM voice channels are multiplexed together.
- The number of such voice inputs is 24 for each channel bank (for AT & T system). The bit rate at point A_1 in Fig. 3.5.2 is 1.544 Mb/s.
- Four such channel banks produce outputs A_1, A_2, A_3 and A_4. These outputs are further multiplexed by using a second level multiplexer.
- The bit rate at the second level multiplexer output i.e. at point B_1 is 6.312 Mb/s.
- Seven such outputs i.e. B_1, B_2....., B_7 are multiplexed further by using a third level multiplexer.
- The bit rate at the output of the third level multiplexer i.e. at point C_1 is 44.736 Mb/s.
- Six such outputs i.e. C_1, C_2...., C_6 are then multiplexed in the fourth level multiplexer. The bit rate at the output of the fourth level multiplexer i.e. at point D is 274.176 Mb/s.



(E-794) **Fig. 3.5.2 : Multiplexing hierarchy for digital communication**

Calculation of bit rate at each level :

For an AT & T system the bit rates at various levels of multiplexing are as follows :

1. Bit rate at the first level of multiplexing = 64 kb/sec × 24 Channels

   = 1.536 Mb/sec ...(3.5.1)

   Bit rate assigned at the output of first level = 1.544 Mb/sec ...(3.5.2)

2. Bit rate at the output of second level = 1.544 Mb/sec × 4 Inputs

   = 6.176 Mb/sec ...(3.5.3)

   Bit rate assigned at the output of second level = 6.312 Mb/sec ...(3.5.4)

3. Bit rate at the output of third level = 6.312Mb/sec × 7 Inputs

   = 44.184 Mb/sec ...(3.5.5)

   Bit rate assigned at the output of third level = 44.736 Mb/sec ...(3.5.6)

4. Bit rate at the output of fourth level = 44.736 Mb/sec × 6 Inputs

   = 268.416 Mb/sec ...(3.5.7)

   Bit rate assigned at the output of fourth level = 274.176 Mb/sec ...(3.5.8)

5. Total number of voice PCM channels = 24 × 4 × 7 × 6 = 4032 ...(3.5.9)

Note : Observe that at every level of multiplexing the output bit rate is lower than the bit rate assigned at that level. This higher value allows us to add control bits and additional bits called "stuff bits" required to be added to yield a steady output rate.

Bit rate (r) and transmission channel bandwidth B_T :

The bit rate at the output of fourth level of AT & T system is given by,

$$\text{Bit rate, } r = 274.176 \text{ Mb/sec.}$$

$$\therefore \text{ Transmission channel bandwith, } B_T \geq \frac{r}{2}$$

$$\therefore B_T \geq \frac{274.176}{2} \qquad \therefore B_T \geq 137 \text{ Mb/sec.}$$

Thus to transmit 4032 PCM voice signals the required bandwidth of the multiplexing system is greater than 137 Mb/sec.

Bandwidth efficiency :

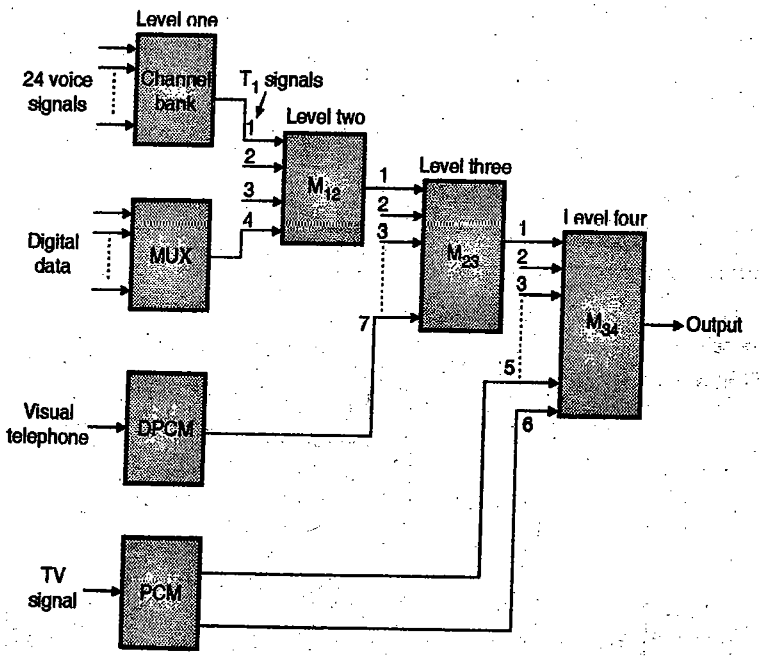
- The bandwidth efficiency of a multiplexing system is defined as,

  Bandwidth Efficiency

  $$= \frac{\text{Number of voice signals} \times \text{Bandwidth of each signals}}{\text{Bandwidth of the multiplexing system}}$$

  $$= \frac{4032 \times 4 \text{ kHz}}{137 \text{ MHz}} = 11.77\% \qquad \text{...(3.5.10)}$$

- Thus bandwidth efficiency of a digital multiplexing system is only 11.77% which is extremely poor as compared to a much higher bandwidth efficiency of an analog multiplexing system (typically above 85%).
- However this disadvantage of poor bandwidth efficiency is outweighed by the other advantages of digital transmission.



(E-795) Fig. 3.5.3 : A digital multiplexer used to multiplex different types of signals

3.5.5 Advantages of Digital Multiplexing :

SPPU : May 08, May 09

University Questions

Q. 1 Diagram a digital multiplexing system that accommodates both analog and digital signals in a standard multiplexing hierarchy including the North American and CCIT. **(May 08, 8 Marks, May 09, 10 Marks)**

1. Hardware cost is reduced due to the use of digital ICs.
2. Power cost also gets reduced due to the use of regenerative repeaters.
3. More flexibility as compared to the analog multiplexers. This is illustrated in Fig. 3.5.3 which shows a digital multiplexer which can multiplex voice signals, digital data, TV as well as videophone together.

3.5.6 Quasi-synchronous Multiplexing :

SPPU : Dec. 06, Dec. 11, May 13

University Questions

Q. 1 Explain synchronous and quasi-synchronous multiplexers. **(Dec. 06, 8 Marks)**

Q. 2 What are the different types of multiplexers used in digital communication system ? Explain quasi-synchronous multiplexer in detail with a neat sketch. **(Dec. 11, 8 Marks, May 13, 10 Marks)**

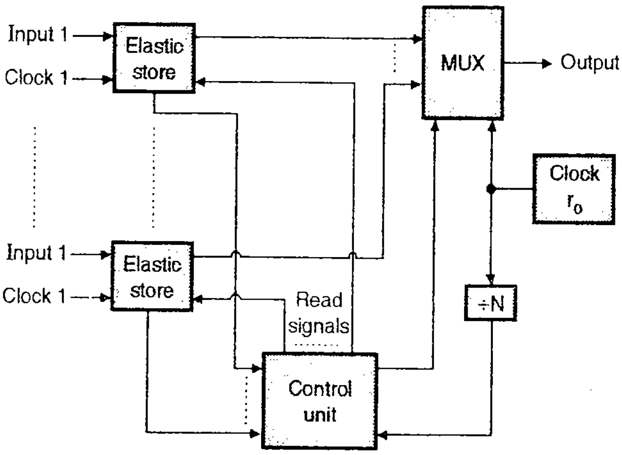
- As mentioned in section 3.5.3, the quasi-synchronous multiplexing becomes necessary when the variations in the bit rate of various input sources are small.
- This can happen with the M_{12} multiplexer shown in Fig. 3.5.3, because the input signals T_1 arriving at the input of this multiplexer are generated by channel banks operating at stable but unsynchronized local clock signals.

Features of a quasi-synchronous multiplexer :

- The quasi-square multiplexers has the three essential features as mentioned below :

1. It should have a sufficiently high output bit rate so that it can accommodate the maximum input bit rate.
2. It should have special buffers called "elastic stores" in which input bits can be stored temporarily.
3. It should "stuff" bits to pad the output stream.

- The block diagram of a quasi-square multiplexer is shown in Fig. 3.5.4. It is a combination of two processes namely, bit stuffing and synchronous multiplexing.



(E-796) **Fig. 3.5.4 : Quasi-synchronous multiplexer**

Operation :

- There are N input signals whose bit rates vary from r_{min} to r_{max} around a nominal value $\bar{r}$.
- The output bit rate of the MUX will be constant equal to $r_0 > N\ r_{max}$. This constant output rate is achieved by using the technique explained below.
- The control unit receives signals from all the elastic stores and returns read signals back to them. The elastic store which receives a read signal will put its output bits to MUX input.
- The control unit also supplies control and stuff bits to the MUX. Stuff bits are the extra bits added to the output bit stream of the MUX in order to make the bit rate of each signal same.
- The number of bits to be stuffed for each channel will be different from other channel. Thus with the help of the "bit stuffing" technique we can make the bit rate of all the inputs (1 to N) equal.

Analytical description :

- For an analytical description of quasi-synchronous multiplexer assume that :

T_f = Duration of each frame

r_0 = Rate of MUX output

D = Number of message and stuff bits

X = Number of control bits.

Then the total number of output bits per frame is given by :

$$T_f r_0 = D + X \quad \text{... (3.5.11)}$$

- Note that D is a fixed parameter, even then the proportion of message and stuff bits depends on the input rates. If all the inputs have the same rate $\bar{r}$, then

$$D = T_f N \bar{r} + s N \quad \text{... (3.5.12)}$$

where s = Average number of stuff bits per channel per frame.

Thus if s = 1/2 then each channel gets one stuff bit every other frame under the nominal operating condition.

- From Equation (3.5.11) we can write that,

$$r_o = \frac{D+X}{T_f} \quad \text{...(3.5.13)}$$

- From Equation (3.5.12) we get,

$$T_r = \frac{D - sN}{N \bar{r}}$$

- Substituting this into Equation (3.5.13) we get,

$$r_o = \frac{(D+X) N \bar{r}}{D - sN}$$

$$\text{or} \quad r_o = N \bar{r} \left[\frac{D+X}{D-sN}\right] \quad \text{...(3.5.14)}$$

- This expression relates r_o and s when the other parameters have been specified. If r_o and s are given, then we can find how much variations in the input rate can be allowed.

Expression for the maximum rate (r_{max}) :

- When the input rate r is maximum i.e. equal to r_{max}, there is no need of bit stuffing, hence s = 0. Substituting this into Equation (3.5.12).

We get,

$$D \geq T_f N r_{max} \quad \text{...(3.5.15)}$$

$$\therefore \quad r_{max} \leq \frac{D}{T_f N}$$

- Substituting Equation (3.5.12) for D we get,

$$r_{max} \leq \frac{T_f N \bar{r} + sN}{T_f N}$$

$$\therefore \quad r_{max} \leq \bar{r} + \frac{s}{T_f} \quad \text{...(3.5.16)}$$

- Now substitute $T_f = \frac{D+X}{r_o}$ from Equation (3.5.13) to get,

$$r_{max} \leq \bar{r} + \frac{r_o}{(D+X)} \cdot s \quad \text{...(3.5.17)}$$

- This is the required expression for the maximum input rate r_{max}.

Expression for the minimum rate (r_{min}) :

- When the input rate is minimum, we have to stuff additional bits in order to increase it. But there is a limit on the number of bits being added.
- We usually want at the most one stuff bit per channel per frame in order to simplify destuffing. So substitute s = 1 and r = r_{min} into Equation (3.5.12) to get,

$$D = T_f N r_{min} + N$$

$$\therefore (D - N) \leq T_f N r_{min} \quad \text{...(3.5.18)}$$

$$\therefore \quad r_{min} \geq \frac{D-N}{T_f N}$$

- Substitute the values of D from Equation (3.5.12) to write,

$$r_{min} \geq \frac{T_f N \bar{r} + sN - N}{T_f N}$$

$$\therefore \quad r_{min} \geq \frac{T_f N \bar{r} + (s-1) N}{T_f N}$$

$$\therefore \quad r_{min} \geq \bar{r} + \frac{(s-1)}{T_f} \quad \text{...(3.5.19)}$$

- Substitute the value of T_f as $\frac{D+X}{f_o}$ from Equation (3.5.13) we get,

$$r_{min} \geq \bar{r} + \frac{(s-1) r_o}{D+X}$$

$$\therefore \quad r_{min} \geq \bar{r} - \frac{(1-s) r_o}{(D+X)} \quad \text{...(3.5.20)}$$

- This is the required expression for minimum bit rate.

Ex. 3.5.1 : An AT and T, M_{12} multiplexer combines $N = 4T_1$ signals with $\bar{r}$ = 1.544 Mb/sec. A complete frame consists of 24 control bits and 1152 message and stuff bits. If the average number of stuff bits per channel per frame is 1/3, calculate the output bit rate.

Soln. :

Given :

$N = 4$, $\bar{r} = 1.544 \times 10^6$ bits/sec, $s = 1/3$ $D = 1152$ bits, $X = 24$ bits/frame

From Equation (3.5.14) we have,

$$\text{Output bit rate} \quad r_o = N \bar{r} \left[\frac{D+X}{D-sN}\right]$$

$$= 4 \times 1.544 \times 10^6 \left[\frac{1152+24}{1152-(4/3)}\right]$$

$$\therefore \quad r_o = 6.312 \text{ Mb/sec.} \quad \text{...Ans.}$$

Ex. 3.5.2 : Suppose that 24 voice signals arrive at a channel bank already encoded as PCM with $\bar{r}$ = 64 k bits/sec. The channel bank is a quasi-synchronous multiplexer whose output frame is divided into 24 subframes, each subframe containing 3 control bits and 8 message and stuff bits. Calculate r_o if s = 1/3 and determine the throughput efficiency.

Soln. :

Given :

$N = 24, X = 3 \times 24 = 72$ bits,

$D = 8 \times 24 = 192$ message and stuff bits.

$\bar{r} = 64$ kb/sec, $s = 1/3$

1. **To calculate output bit rate :**

From Equation (3.5.14) we have,

$$\text{Output bit rate, } \quad r_o = N\bar{r}\left[\frac{D+X}{D-sN}\right]$$

$$= 24 \times 64 \times 10^3 \cdot \left[\frac{192+72}{192-(24/3)}\right]$$

$$\therefore \quad r_o = 2.2 \times 10^6 = 2.2 \text{ Mb/sec.} \quad \text{...Ans.}$$

2. **To determine the throughput efficiency :**

The throughput efficiency is defined as :

$$\eta = \frac{\text{Number of input signals} \times \text{Average bit rate per signal}}{\text{Output bit rate}}$$

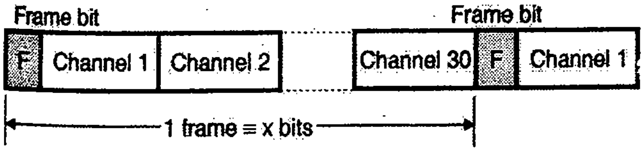
$$\therefore \eta = \frac{N \cdot \bar{r}}{r_o} = \frac{24 \times 64 \times 10^3}{2.2 \times 10^6} \times 100\%$$

$$\therefore \eta = 69.81\% \quad \text{...Ans.}$$

Ex. 3.5.3 : For CCIT hierarchy, assume that the first level multiplexer is a synchronous voice PCM bank with 30 input signals. The output bit rate of this multiplexer is 2.048Mb/sec. If bit robbing has not been implemented, calculate the number of framing plus signaling bits per frame.

Soln. :

- One frame of transmitted signal is as shown in Fig. P. 3.5.3.
- It consists of signal bits for all the 30 input signals plus one frame bit. Let us assume that there are "x" bits per frame.



(E-790) Fig. P. 3.5.3 : Frame format for a CCIT digital multiplexer

1. Channels 1 to 30 each are 64kb/s PCM encoded voice channels. Therefore the minimum sampling rate required to multiplex them is,

$$f_{s(min)} = 64 \times 10^3 \times 2 = 128 \text{ kHz} \quad \text{... (1)}$$

2. Number of frames transmitted per second

$$= f_{s(min)} = 128 \times 10^3$$

3. Number of bits per frame = x

4. Bit rate = Number of frames/sec × Number of bits/frame

$$= 128 \times 10^3 x$$

$$\text{But it is given that bit rate} = 2.048 \text{ Mb/ sec}$$

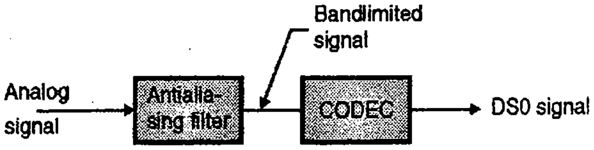
$$\therefore 128 \times 10^3 x = 2.048 \times 10^6$$

$$x = 16 \quad \text{.... Ans.}$$

Thus number of bits per frame = 16.

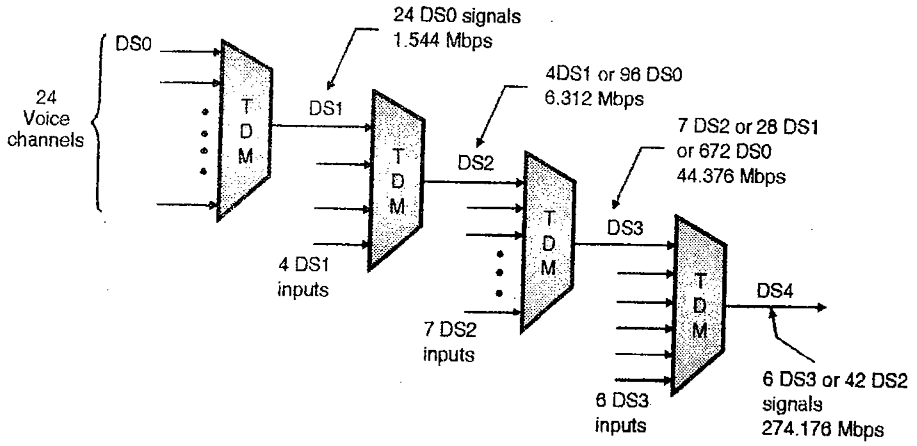
3.6 North American Hierarchy :

- The first digital signal in true sense is the PCM voice signal.
- A PCM voice signal uses 64 kbits/sec. i.e. 8000 samples per second × 8 bits to represent each voice sample.
- Such a signal is called as **digital signal at level zero (DS0).** It is also called as a T_1 signal.
- Note that due to 8000 samples/sec, sampling rate, the time duration between adjacent samples will be (1/8000) i.e. 125 μS.
- This is (DS0) the fundamental building block of all the digital communication systems.
- But practically the DS0 signal is never transmitted because most of the telephone lines are analog.
- Hence in a telephone central office, the subscribers analog line is passed through an antialiasing filter. The bandlimited signal at the output of this filter is applied to a codec, which converts it into DS0 signal.
- The generation of DS0 signal is shown in Fig. 3.6.1.



(G-1303) Fig. 3.6.1 : Generation of DS0 signal

- 24 such DS0 lines are multiplexed into a DS1 (digital signal at level 1). Commonly this signal is also called as T_1 signal.
- The telephone companies implement TDM (time division multiplexing) through the hierarchy of digital signals. This is called as digital signal (DS) service.
- Fig. 3.6.2 shows the DS hierarchy and the bit rates supported at various levels of this hierarchy.
- The multiplexed signal is converted into a frame at the DS1 or T_1 level.



(G-1304) **Fig. 3.6.2 : D.S. hierarchy**

3.6.1 T Lines :

SPPU : Dec. 15

University Questions

Q. 1 Explain digital signal hierarchy using T1 carrier system. **(Dec. 15, 6 Marks)**

- DS0, DS1, DS2, ... etc. are the names of the services. The telephone companies use the T lines (T0, T1, T2, ... etc.) to implement these services.
- The T lines have capacities which precisely match with the bit rates of the corresponding services as shown in Table 3.6.1.

Table 3.6.1 : Relation between DS and T lines

Service	Line	Rate (Mbps)	Number of voice channels
DS – 1	T – 1	1.544	24
DS – 2	T – 2	6.312	96
DS – 3	T – 3	44.736	672
DS – 4	T – 4	274.176	4032

- Thus T – 1 line implements DS – 1 service, T – 2 implements DS – 2 service and so on.
- DS0 is defined as the basic service.

Note : T lines are digital lines which are designed to carrying digital data, audio or video.

- But the T lines can also be used for analog communication. For example T1 line can be used for the telephone applications.

3.6.2 The T_1 System (PCM-TDM System) :

SPPU : Dec. 05, Dec. 10, May 11

University Questions

Q. 1 What are T_1 multiplexing standards for 1. rate, 2. number of voice channels, 3. medium, 4. line code, 5. repeater spacing, 6. max system length, 7. system BER rate. **(Dec. 05, 8 Marks)**

Q. 2 Explain T_1 carrier system and AT & T multiplexing hierarchy. **(Dec. 10, 8 Marks)**

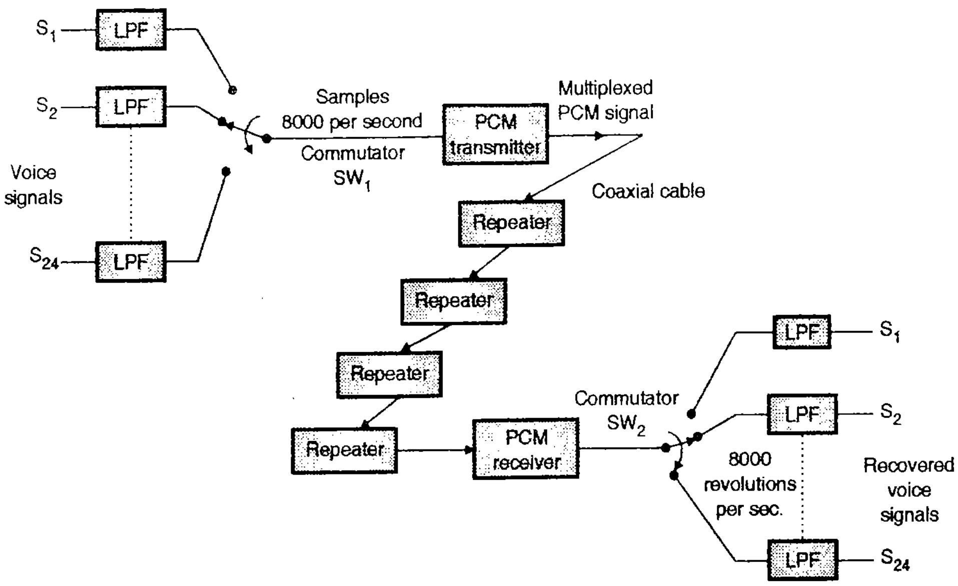
Q. 3 Explain T_1 carrier system and hence compare AT-T and CCIT hierarchy of multiplexing. **(May 11, 7 Marks)**

- When a large number of PCM signals are to be transmitted over a common channel, multiplexing of these PCM signals is required.
- Fig. 3.6.3 shows the basic time division multiplexing scheme for PCM voice channels called as the T_1 **digital system.**
- This system is used to convey a number of voice signals over telephone lines using wideband coaxial cable. Thus the communication medium used is a coaxial cable.

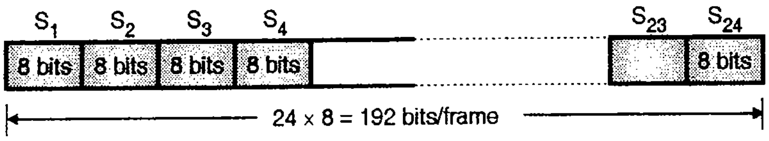
Operation of the T_1 system :

The operation of the PCM-TDM system shown in Fig. 3.6.3 is as follows :

- This system has been designed to multiplex 24 voice channels marked as S_1 to S_{24}. Each signal is bandlimited to 3.3 kHz, and the sampling is done at a standard rate of 8 kHz. This sampling rate is higher than the Nyquist rate. The sampling is done by the commutator switch SW_1.
- These voice signals are selected one by one and connected to a PCM transmitter by the commutator switch SW_1, as it completes its rotation. The commutator switch remains in contact with each voice channel for a short time. Thus it samples each of the 24 channels.
- Each sampled signal is then applied to the PCM transmitter which converts it into a digital signal by the process of A to D conversion and companding. Each sampled voice signal is converted into an 8-bit PCM word.
- The resulting digital waveform is transmitted over a co-axial cable. This waveform is called as the PCM-TDM signal.
- Periodically, after every 6000 ft., the PCM-TDM signal is regenerated by amplifiers called "Repeaters". They eliminate the distortion introduced by the channel and remove the superimposed noise and regenerate a clean noise free PCM-TDM signal at their output. This ensures that the received signal is free from the distortions and noise.



(G-1312) **Fig. 3.6.3 : Block diagram of a basic PCM-TDM system**



(G-1313) **Fig. 3.6.4 : One frame and bits per frame**

- At the destination the signal is companded, decoded and demultiplexed, using a PCM receiver. The PCM receiver output is connected to different low pass filters via the commutator switch SW_2. The LPF outputs are applied to the destination receivers (subscribers).
- Synchronization between the transmitter and receiver commutators SW_1 and SW_2 is essential in order to ensure proper communication.

Bits/Frame :

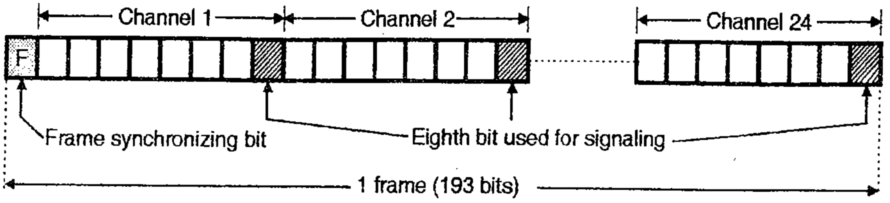
- The commutators sweep continuously from S_1 to S_{24} and back to S_1 at the rate of 8000 revolutions per second (Sampling rate = 8000 samples/sec.).
- This will generate 8000 samples per second of each signal (S_1 to S_{24}). Each sample is then encoded (converted) into an eight bit digital word. One complete revolution of commutator switches corresponds to generation of one frame which consists of all 24 voice channels.
- Thus the digital signal generated during one complete sweep (revolution) of the commutator is given by :

$$\begin{aligned} 1 \text{ Frame} &\equiv 1 \text{ revolution} \\ &= 24 \text{ channels} \\ &= 24 \times 8 \text{ bits} \\ &= 192 \text{ bits} \end{aligned}$$

- One frame of PAM-TDM is shown in Fig. 3.6.4. Each voice signal from S_1 to S_{24} is encoded into eight bits.
- One frame corresponds to one revolution which is the time taken to transmit each signal once. Hence 1-frame corresponds to one-revolution of the commutator.

Frame synchronization :

- As we have already seen, the synchronization between the transmitter and receiver commutators is essential.
- Without such synchronization the receiver cannot know which received bits were generated by whom at the transmitter and are meant for which subscriber on the receiving side.
- To provide such synchronization, an extra bit is transmitted preceding the 192 data bits carrying the information in each frame, as shown in Fig. 3.6.5.
- This bit is called as the frame synchronizing bit "F". Thus one frame synchronizing bit is transmitted per frame.
- This makes the total number of bits per frame to be 193. The time slots for the 24 signals and the extra frame synchronizing bit is as shown in Fig. 3.6.5.



(G-1314) Fig. 3.6.5 : The PCM T_1 frame using frame synchronization and channel associated signaling

- Twelve successive F slots are used to transmit a 12 bit code. The code is 1101 1100 1000.
- This code is transmitted repeatedly once every 12 frames and it is used at the receiver to achieve synchronization between the transmitter and receiver commutators.

Bit rate :

- Bit rate means number of bits transmitted by a system per second. In the T1 system; as each signal is sampled 8000 times per second :

  1 frame (1 revolution of commutator)

  $\equiv 1/8000 = 125\ \mu sec.$
- But 1 frame consists of 193 bits.

$$\therefore \text{ 193 bits are transmitted in } 125\mu sec.$$

$$\therefore \text{ Number of bits in 1 sec. } = \frac{193}{125 \times 10^{-6}}$$

$$= 1.554 \times 10^{6}$$

$$\therefore \text{ Bit rate of } T_1 \text{ system } = 1.544 \text{ Mbits/sec.}$$

Bandwidth of T_1 system :

$$\text{Minimum bandwidth } B_T = \frac{1}{2} \text{ bit rate}$$

$$= \frac{1}{2} \times 1.544 \times 10^{6}$$

$$= 772 \text{ kHz}$$

Duration of each bit :

$$193 \text{ bits} = 125\ \mu S$$

$$1 \text{ bit} = (125 / 193)\ \mu S$$

$$= 0.6476\ \mu S$$

Channel associated signaling :

- When the PCM-TDM system is being used for the telephony, it is expected to transmit certain control signals along with the voice information. The control information is of two types : signalling and supervisory.
- The signaling information consists of the signals such as a call is being initiated or a call is being terminated, or the address of calling party etc.
- In analog system such a signaling information is transmitted over a separate channel other than the voice channel. But in the T_1 system which is a digital system, a separate channel is not used.
- In T_1 system the signaling information is sent using the same data bit slots which are used to send the voice information. The technique used is "bit slot sharing".
- In the "bit slot sharing" method, for the first five frames, all the 24 channels are encoded into an 8 bit digital code. That means all the 8-bits in each PCM word will carry the voice information.
- However in the sixth frame, all the channels are coded into a 7 bit code and the LSB (least significant bit) of each channel is used to transmit the signaling information. This is as shown in Fig. 3.6.5. That means MSB 7-bits carry voice and the LSB bit carries the signalling information.
- This is called as "channel associated signalling". This pattern is repeated after every six frames.

3.6.3 E Lines :

- E-line is actually the Europian version of T lines. The T lines and E lines are conceptually identical but their capacities and number of voice channels which they can carry will be different.
- Table 3.6.2 shows the E lines, their capacities and the number of voice channels which they can carry.

Table 3.6.2 : E lines and their capacity

Line	Rate (Mbps)	Number of voice channels
E – 1	2.048	30
E – 2	8.448	120
E – 3	34.368	480
E – 4	139.264	1920

Ex. 3.6.1 : Five telemetry signals each of BW 1 kHz are to be transmitted by binary PCM-TDM. The maximum tolerable error in the sampling amplitude is 0.5% of peak signal amplitude. The signals are sampled atleast 20% above the Nyquist rate. Framing and synchronization require additional 0.5% extra bits. Determine the minimum transmission data rate and minimum required BW.

Soln. :

1. **Number of quantization levels Q :**

Let the peak signal amplitude be A. The maximum tolerable error is

$$\epsilon_{max} = 0.5\% \text{ of A.}$$

$$= 0.005 \text{ A} \qquad \text{...(1)}$$

But in a PCM system $\epsilon_{max} \pm S/2$ where S is the step size.

$$\therefore \frac{S}{2} = 0.005 \text{ A}$$

$$\therefore S = 0.01 \text{ A} \quad \ldots(2)$$

But S = 2A/Q where Q is number of quantization levels.

$$\therefore \frac{2A}{Q} = 0.01 \text{ A}$$

$$\therefore Q = 200 \quad \ldots(3)$$

2. **Number of digits/word N :**

$$\text{As } Q = 2N$$

$$N = \log 2\, Q = \log_2 200 = \frac{\log_{10} 200}{\log_{10} 2}$$

$$\therefore N = 7.64$$

Let us round off N to 8.

$$\therefore N = 8 \quad \ldots(4)$$

3. **Sampling frequency :**

It has been stated that sampling rate should be atleast 20% higher than the Nyquist rate.

$$\therefore f_s = 1.2 \times \text{Nyquist rate}$$

$$= 1.2 \times 2 \times 1 \text{ kHz} = 2.4 \text{ kHz} \quad \ldots(5)$$

4. **Minimum data rate (r) :**

Fig. P. 3.6.1 shows one frame of the PCM-TDM signal.

$\therefore$ 1 revolution ≡ 1 frame ≡ Transmission of 5 channels + Framing and sync signals

$$\therefore 1 \text{ frame} \equiv [5 \times 8 \text{ bits}] + [0.005 \times 40 \text{ bits}]$$

$$= 40 + 0.2 = 40.2 \text{ bits.}$$

$$\text{But 1 frame timing } T_s = 1/f_s = 1/(2.4 \times 10^3)$$

$$= 0.5 \text{ msec.}$$

$$\therefore 0.5 \text{ msec} \equiv 40.2 \text{ bits}$$

$$\therefore \text{Number of bits per second (r)} = \frac{40.2}{0.5 \times 10^{-3}} = 80400$$

$$\therefore \text{Minimum data rate (r)} = 80.4 \text{ k bits/s} \quad \textbf{...Ans.}$$

5. **Minimum bandwidth :**

$$\text{Minimum BW} = \frac{1}{2} \text{ data rate}$$

$$= \frac{1}{2} \times 80400$$

$$= 40200 \text{ Hz or } 40.2 \text{ kHz} \quad \textbf{...Ans.}$$

Ex. 3.6.2 : 24 Voice channels of 4 kHz bandwidth, each sampled at Nyquist rate and encoded into 8 bit PCM are multiplexed with 1 bit/frame as synchronization bit. What is resultant bit rate at the output of multiplexer ? Sketch the frame configuration.

Soln. :

Given : $n = 24$, $W = 24$ kHz, $f_s = 2\,W = 8$ kHz

N = 8 bit/channel, 1 frame synchronizing bit frame.

Bit rate ?

1. The frame configuration is as shown in Fig. P. 3.6.2.
2. Number of bits per frame = 1 + (24 × 8) = 193 bits.
3. As the Nyquist rate = 8 kHz, there are 8000 revolutions of he commutator per second.

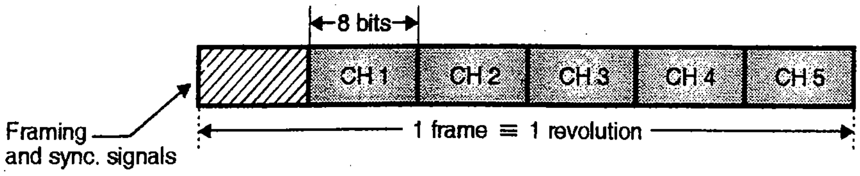
$$\therefore 1 \text{ Revolution} = \frac{1}{8000} = 125\ \mu\text{S}$$

4. Each revolution corresponds to 1 frame and each frame contains 193 bits.

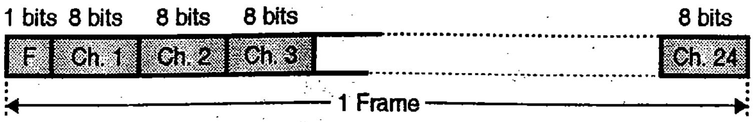
$$\therefore 125\ \mu\text{S} = 193 \text{ bits}$$

$$\therefore \text{Number of bits per second} = \frac{193}{125 \times 10^{-6}}$$

$$\therefore \text{Bit rate} = 1.54 \text{ Mbits/sec} \quad \textbf{...Ans.}$$



(E-791) **Fig. P. 3.6.1**



(E-792) **Fig. P. 3.6.2**

Ex. 3.6.3 : A signal m(t) is band limited to 3 kHz is sampled at a rate of 33 1/3 % higher than the Nyquist rate. The maximum acceptable error in sample amplitude (quantization error) is 0.5% of peak amplitude m_p. The quantized signals are binary encoded. Find minimum bandwidth of a channel required to transmit the encoded binary signal, if such 24 channels are time division multiplexed, determine minimum transmission bandwidth.

May 07, 8 Marks

Soln. :

Given : $f_m = 3$ kHz, $\in_{max} = 0.5\%$ of maximum value m_p, Number of channels n = 24.

To find : $BW_{(min)}$.

Step 1 : Number of quantization levels Q :

$$\text{Given that } \in_{max} = 0.5\% \text{ of } m_p$$

$$= 0.005\, m_p$$

$$\text{But in PCM system } \in_{max} = \pm S/2 \text{ where S is the step size.}$$

$$\therefore \quad \frac{S}{2} = 0.005\, m_p$$

$$\therefore \quad S = 0.01\, m_p.$$

$$\text{But} \quad s = \frac{2m_p}{Q} \text{ where}$$

Q = Number of quantization levels.

$$\therefore \quad \frac{2\,m_p}{Q} = 0.01\, m_p$$

$$\therefore \quad Q = \frac{2}{0.01} = 200$$

Step 2 : Number of bits/word N :

$$\text{As} \quad Q = 2^N$$

$$\therefore \quad N = \log_2 Q = \log_2 200 = \frac{\log_{10} 200}{\log_{10} 2}$$

$$\therefore \quad N = 7.64$$

Round it off to 8

$$\therefore \quad N = 8$$

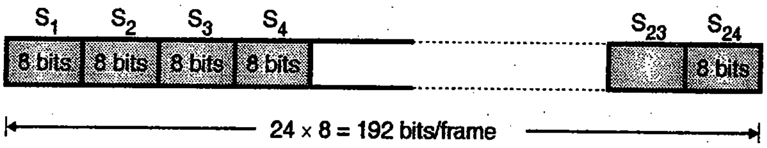
Step 3 : Sampling frequency :

Sampling frequency should be 33 1/3 % higher than the Nyquist rate.

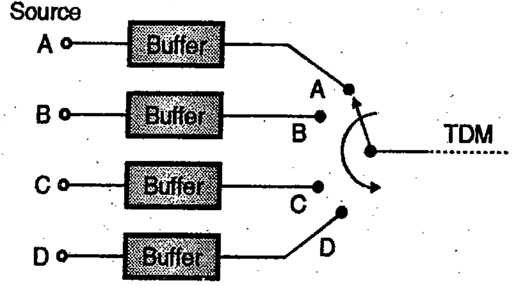
$$\therefore \quad f_s = 1.333 \times 2 \times f_m = 1.333 \times 2 \times 3 \text{ kHz} = 8 \text{ kHz.}$$

Step 4 : Minimum data rate (r) :

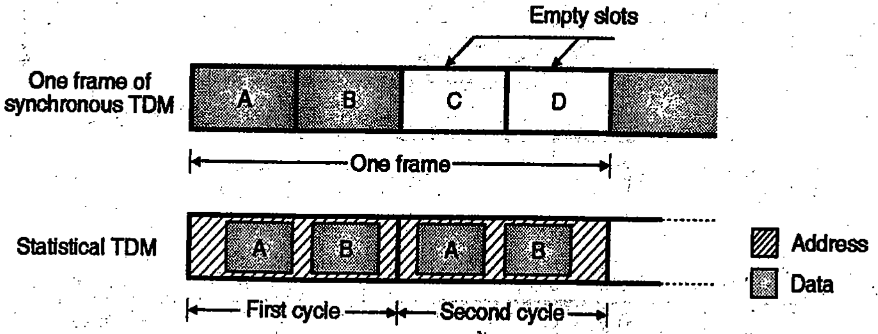
Fig. P. 3.6.3 shows one frame of PCM-TDM signal.



(E-815) **Fig. P. 3.6.3**



(L-138)**(a) Block diagram**



(b) Frame format

(L-139) **Fig. 3.7.1 : Statistical TDM**

$$\therefore \quad 1 \text{ revolution} \equiv 1 \text{ frame}$$

$$\equiv \text{Transmission of 24 channel.}$$

$$\therefore \quad 1 \text{ frame} = 24 \times 8 \text{ bits} = 192 \text{ bits}$$

$$1 \text{ frame time} = \frac{1}{f_s} = \frac{1}{8 \times 10^3} = 125\ \mu S$$

$$\therefore \quad 125\ \mu S \equiv 192 \text{ bits}$$

$$\therefore \quad \text{Number of bits/sec.} = \frac{192}{125 \times 10^{-6}} = 1.536 \text{ Mbps.}$$

Step 5 : Minimum BW :

$$\text{Minimum BW} = \frac{1}{2} \text{data rate} = \frac{1}{2} \times 1.536 \times 10^6$$

$$= 768 \text{ kHz} \qquad \text{...Ans.}$$

3.6.4 Applications of Synchronous TDM :

1. For analog telephone system (T – 1 system).
2. Some second generation cellular telephone companies use synchronous TDM.

3.7 Statistical TDM :

- The TDM system that we have discussed earlier is known as the synchronous TDM. This system has a major drawback. In synchronous TDM, many of the time slots in a frame are wasted due to absence of data on some of the time slots.
- Therefore an alternative system called as statistical TDM or asynchronous TDM or intelligent TDM is used.
- The block diagram of the statistical TDM system is as shown in Fig. 3.7.1(a) and its frame format is as shown in Fig. 3.7.1(b).

Operating principle :

- In statistical TDM, the time slots are not permanently assigned to all the available users (like synchronous TDM). Instead, the time slots are allocated dynamically on demand only to those channels holding data for transfer.
- Each TDM channel is called as an I/O line. Thus the statistical TDM has many I/O lines and one high speed multiplexed line.
- Each I/O line has a buffer associated with it. As shown in Fig. 3.7.1, there are N number of I/O lines. Out of these only K channels are transmitted which hold data for transfer. The remaining (N – K) channels are not considered for transmission.
- In statistical TDM, the multiplexer will "scan" the input buffers of all the channels, sequentially. During the scan time, it collects the data until a frame is filled. As soon as a frame is filled, it is transmitted.
- The data is transferred on the transmission medium. The received frame is then distributed among the output buffers by the output multiplexer.

3.7.1 Data Rate of Statistical TDM :

- In statistical TDM system, all the channels are not transmitted in every frame. Hence the data rate on the multiplexed line will be less than the sum of the data rates of all the sources.
- Thus a statistical multiplexer can use a transmission medium of lower data rate to support the same number of sources as the synchronous multiplexer.
- That means if we have a synchronous and statistical TDM with equal data rates, then the statistical TDM will support more number of sources.

3.7.2 Slot Size :

- The slot carries both data and address, the ratio of the data size to address size should be reasonable to ensure high efficiency.
- In statistical TDM, the data block contains many bits while address bits are very few.

3.7.3 No Synchronization Bit :

- The statistical TDM frames need not be synchronized. So it is not necessary to use the synchronizing bit.

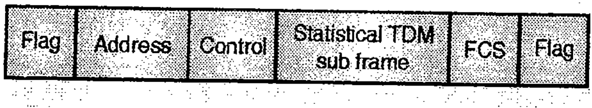
3.7.4 Bandwidth :

- In statistical TDM, the capacity of multiplexed link is generally less than the sum of capacities of individual channels.
- Therefore the bandwidth requirement of the multiplexed link is less than that for the synchronous TDM.

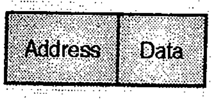
3.7.5 Frame Format for Statistical TDM :

- Fig. 3.7.1(b) shows the comparison of synchronous and statistical TDM frames. It shows that there are four data sources and they are to be transmitted in four time slots t_0 to t_3.
- In the synchronous TDM, the data sources A and B only have data present whereas the data sources C and D do not have any data. Hence time slots $(t_1 - t_2)$ and $(t_2 - t_3)$ corresponding to these sources are empty.
- On the other hand, the statistical TDM system does not send the empty slots. Thus during the first cycle, data from the sources A and B only is transmitted.
- As the data from A and B only is being sent and the time slots of C and D are utilized for this purpose, the positional identity of slots is lost in statistical TDM. We cannot predict ahead of time about data from which source will be in which slot.

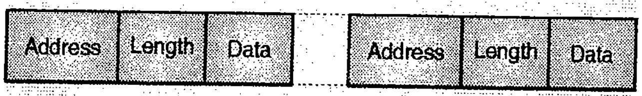
- Due to this unpredictability, the address information is required to be sent alongwith the data information. This address is then used for proper delivery of data.
- Thus each slot of statistical TDM carries an address as well as data. The detailed frame structure is as shown in Fig. 3.7.2.



(a) Overall frame



(b) One source per frame



(c) Multiple sources per frame

(L-140) Fig. 3.7.2 : Frame formats of statistical TDM

Now refer to Fig. 3.7.2.

- The frame structure for the statistical TDM should be such that it should minimize the overhead bits. This is important for improving the throughout efficiency.
- The statistical TDM system uses a synchronous protocol such as HDLC. Therefore within the HDLC frame, the data frame contains the control bits for the sake of multiplexing operation.
- Fig. 3.7.2 shows two such frame formats. Fig. 3.7.2(b) shows only one source of data per frame. This source is identified by the associated address. The length of the data field is variable and its end is identified by the end of overall frame.

But the one source per frame scheme will work properly only for light loads. It is quite inefficient under the heavy load condition.

Improvement in efficiency :

1. The throughput efficiency can be improved by allowing the multiple data sources to be packaged in a single frame as shown in Fig. 3.7.2(c).
2. When many sources are packaged in a single frame, it is necessary to specify the length of data for each source. Therefore the statistical TDM subframe consists of a sequence of data fields. Each data field is labelled with an address and a length.

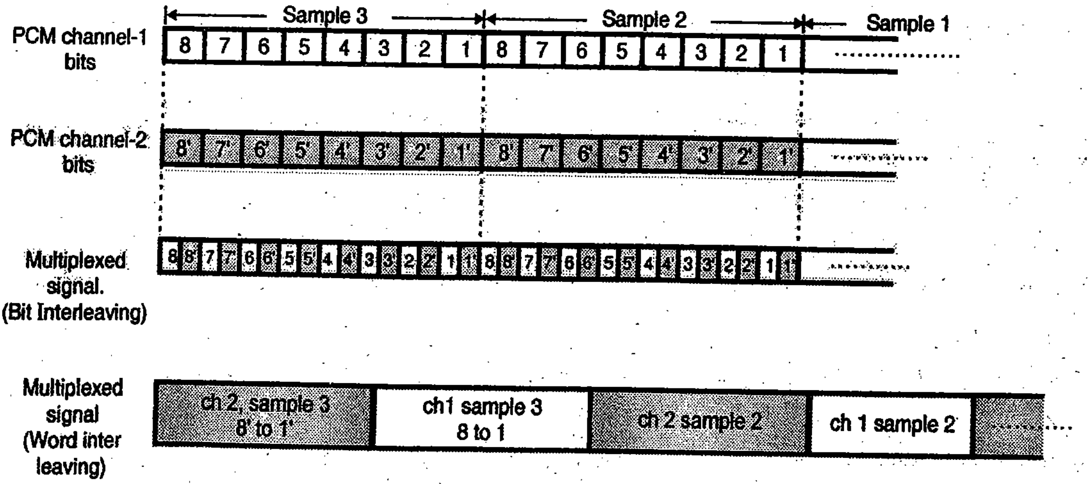
3.7.6 Comparison of FDM, Synchronous TDM and Statistical TDM :

Table 3.7.1 : Comparison of data multiplexer techniques

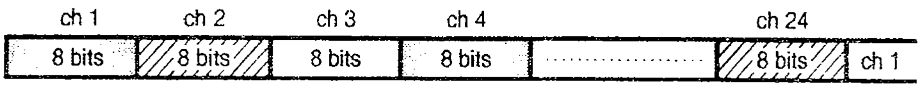
Sr. No.	Parameter	FDM	Synchronous TDM	Statistical TDM
1.	Line utilization efficiency	Poor	Good	Very good
2.	Flexibility	Poor	Good	Very good
3.	Channel capacity	Poor	Good	Excellent
4.	Error control	Not possible	Not possible	Possible
5.	Multidrop capacity	Very good	Difficult to achieve	Possible
6.	Transmission delay	Does not exist	Low	Random
7.	Cost	High	Low	Moderate

3.8 Time Interleaving versus Word Interleaving :

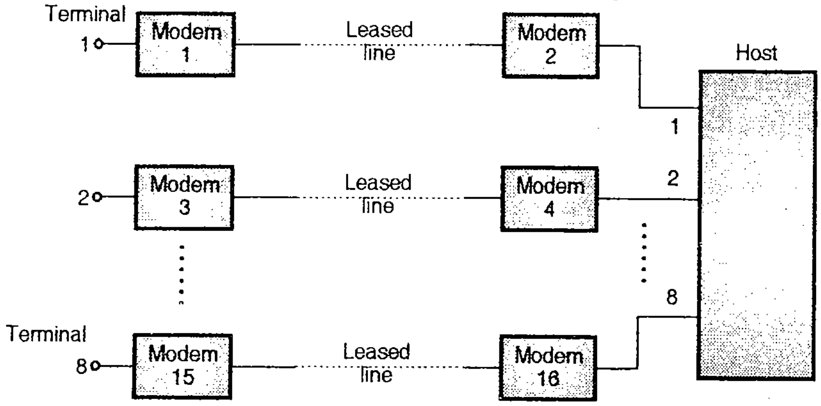
- When time division multiplexing of two or more PCM systems is to be done, it is necessary to interleave the signals obtained from various terminals in time domain as shown in Fig. 3.8.1(a).



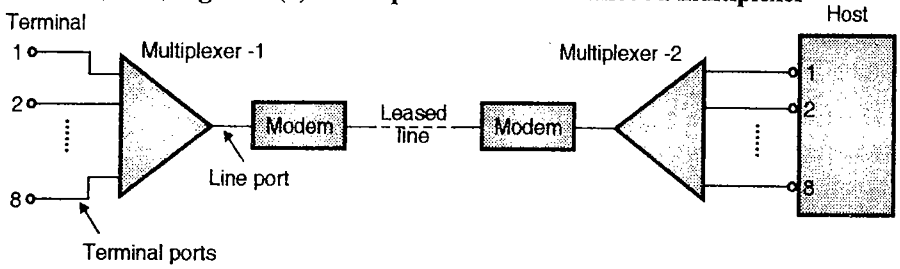
(E-1352) Fig. 3.8.1(a) : Bit and word interleaving



(E-1353) **Fig. 3.8.1(b) : Word interleaving in T_1 system**



(G-1081) **Fig. 3.9.1(a) : Multiple connections without a multiplexer**



(G-1082) **Fig. 3.9.1(b) : Multiple connections using data multiplexers**

- Two methods of interleaving PCM transmission are :
  1. Bit interleaving
  2. Word interleaving
- In the process of bit interleaving, the bits from two PCM channels are interleaved whereas in the word interleaving, the entire word from the channels are interleaved.
- T_1 carrier system uses word interleaving. 8 bit samples from each channel are interleaved to form a single 24 channel TDM frame as shown in Fig. 3.8.1(b).
- High speed TDM and delta modulation systems use bit interleaving.

3.9 Data Multiplexers :

- We know that a MODEM is used for the interconnecting terminals and computers when the distances involved are large. The **data multiplexer** is another data transmission intermediary device which can be used to allow sharing of the transmission media.

  Multiplexing is used for reducing the cost of the transmission media and modem.
- The application of the data multiplexer is as shown in Fig. 3.9.1(a). Fig. 3.9.1(a) shows a system without the data multiplexers. For transmitting and receiving eight signals we need 8-leased lines and 16-Modems.

Effect of Multiplexer :

- Now refer to Fig. 3.9.1(b) in which the terminals and the host are connected to each other through two data multiplexers.
- Due to the data multiplexers, the number of modems used is reduced to two from sixteen, and the leased line requirement has reduced to only one.
- The multiplexer inputs connected to the eight input terminals are called as the multiplexer ports, and its output connected to the leased line is called as the line port. (See multiplexer-1 in Fig. 3.9.1(b)).
- The multiplexer has a built-in de-multiplexer. This is used for the signals coming from the far end. (See multiplexer-2 in Fig. 3.9.1(b)).

Advantages of the data multiplexers :

1. The system becomes economical due to reduction in the number of modems and leased line.
2. Centralized monitoring of all the channels is possible.
3. The data multiplexers can have built in diagnostic hardware/software in order to monitor the performance of individual data channels.
4. If the data multiplexer is designed to have some excess capacity then addition of a few more number of users is easy and inexpensive.

Disadvantage :

If any one of the multiplexer or the lease line fails, then all the terminals will be cutoff from the host.

Types of data multiplexers :

The data multiplexers are of two types :

1. Data multiplexers using FDM
2. Data multiplexers using TDM.

The data multiplexers using the TDM, are further classified into two types. They are, multiplexers using the synchronous TDM and the multiplexers using the statistical TDM.

3.10 Introduction to Discrete PAM Signals (Data Formats) :

SPPU : May 12, Dec. 13, May 15

University Questions

Q. 1 What are line codes and its characteristics ? Compare the power spectral density of unipolar NRZ and RZ formats by deriving suitable expressions. **(May 12, 8 Marks)**

Q. 2 Explain need of line coding. State its properties. Draw and give mathematical expression of Power Spectral density for unipolar NRZ, Polar RZ, AMI, and Manchester. **(Dec. 13, 8 Marks)**

Q. 3 Explain various data formats. **(May 15, 6 Marks)**

- There are various techniques used to convert the analog signal to digital signal.
- But the other way of obtaining digital data is from the source such as computers.
- The information from such a source is inherently discrete in nature.
- If such a discrete signal is transmitted over a band-limited channel, then the signal gets dispersed. That means the pulses spread out and overlap each other to cause distortion.
- Such a distortion is called as intersymbol interference (ISI). In order to avoid this we should not transmit the discrete signal as it is on the transmission medium.
- This data is first converted into a PAM suitable format and then transmitted over a communication channel. The various formats used as required are also called as **line codes.** Fig. 3.10.1 shows various formats to represent a binary data sequence.
- Various data formats or line codes are :
  1. Non-return to zero (NRZ) and return to zero (RZ) unipolar format.
  2. NRZ and RZ polar format.
  3. Non-return to zero bipolar format.
  4. Manchester format.
  5. Polar quaternary NRZ format.
- A digital message representation at baseband commonly takes the form of an amplitude modulated pulse train. Such a signal is expressed as,

$$x(t) = \sum_{k} a_k \, p(t - kT) \quad ...(3.10.1)$$

Here a_k=Amplitude of k^{th} symbol in the message sequence.

- p (t) is the pulsed carrier whose pulses are modulated by a_k and T is the time period (maximum duration) of the carrier pulses. If the carrier pulse p (t) is unmodulated, then it will be a rectangular pulse with a variable duty cycle. We can represent this mathematically as,

$$p(t) = 1, \quad \text{for } t = 0$$
$$= 0, \quad \text{for } t = \pm T, \pm 2T \quad ...(3.10.2)$$

- x (t) in Equation (3.10.1) represents a baseband signal which is continuous in time. In order to obtain various formats mentioned above, we have to sample x (t) at some regular intervals. This is necessary to know about which symbol is being transmitted (0 or 1). The ideal time instants to sample x (t) are at k = 0, ± 1, ± 2...etc.
- Thus x (t) is first sampled at k = 0, ± 1, ± 2 ..... and then represented in the form of a train of modulated pulsed carrier.
- Different types of line coding formats are also called as digital PAM signals. These signals have the following important properties.

3.10.1 Properties of Discrete PAM Signals :

SPPU : May 07, May 08, May 09, Dec. 13

University Questions

Q. 1 Explain properties of line codes and draw the power spectral for various codes. **(May 07, 8 Marks)**

Q. 2 Consider a binary sequence with a long sequence of 1's followed by a single 0 and then a long sequence of 1's . Draw the waveform for this sequence, using the following signaling formats.

1. Unipolar NRZ signaling.
2. Bipolar RZ signaling.
3. AMI RZ signaling.
4. Split-phase (manchester) signaling.

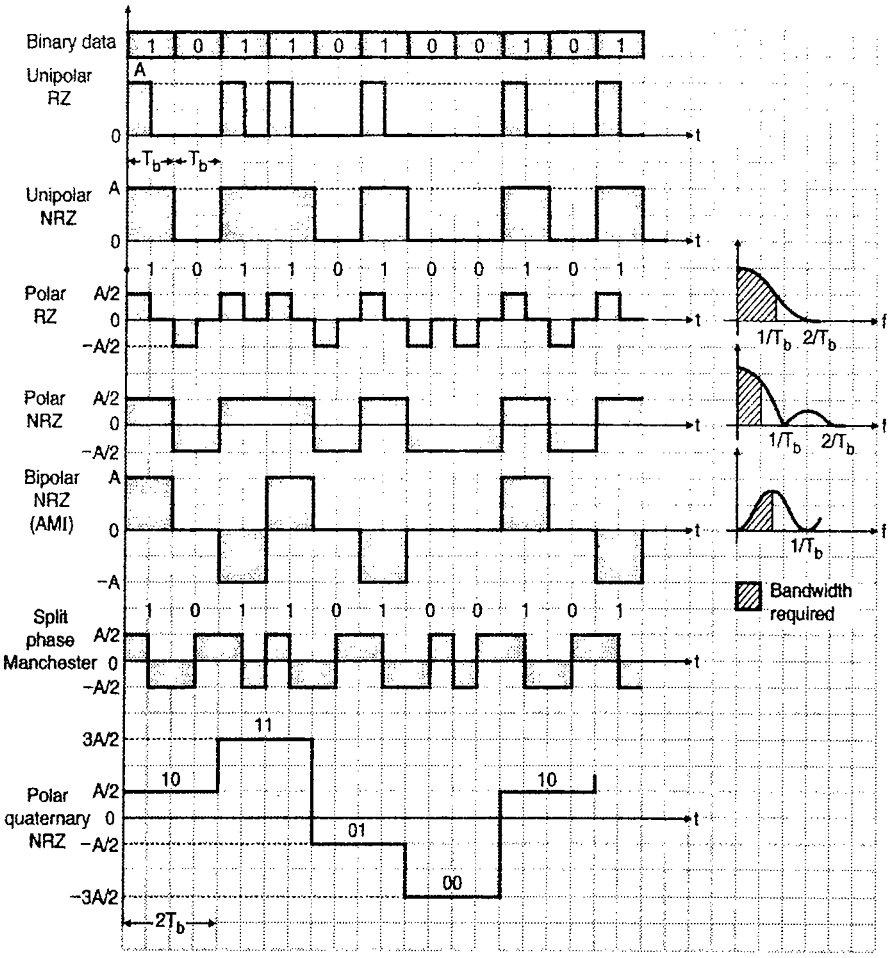
(May 08, 8 Marks)

Q. 3 What are desirable properties of line codes ? Compare RZ and NRZ line coding formats on the basis of above properties along with their merits and demerits. **(May 09, 6 Marks)**

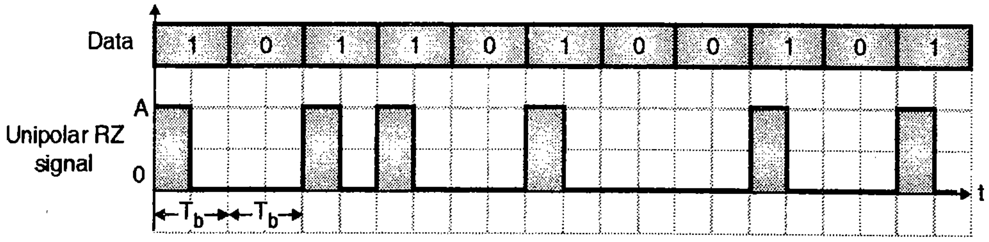
Q. 4 Explain need of line coding. State its properties. Draw and give mathematical expression of Power Spectral density for unipolar NRZ, Polar RZ, AMI, and Manchester. **(Dec. 13, 8 Marks)**

Following are some of the important properties of PAM signals :

1. All the cable systems and other communication systems, do not allow transmission of a dc signal. Therefore, the line signal must have a zero average (dc) value. NRZ bipolar formats usually satisfy this requirement. For this reason, long strings of element sequences having same polarity should not be transmitted.
2. As the code adds redundancy, the code efficiency should be as high as possible.
3. To ensure synchronization at the receiver, the line signal should undergo a sufficient number of zero crossings that means, the transmitted signal should always undergo transitions.



(E-729) **Fig. 3.10.1 : Various discrete PAM signals**



(L-262) **Fig. 3.10.2 : Unipolar RZ format**

4. The crosstalk between channels should be minimized. To do so the amount of energy in the signal at low frequencies should be small.

3.10.2 Unipolar RZ Format : SPPU : May 15

University Questions

Q. 1 Explain various data formats. **(May 15, 6 Marks)**

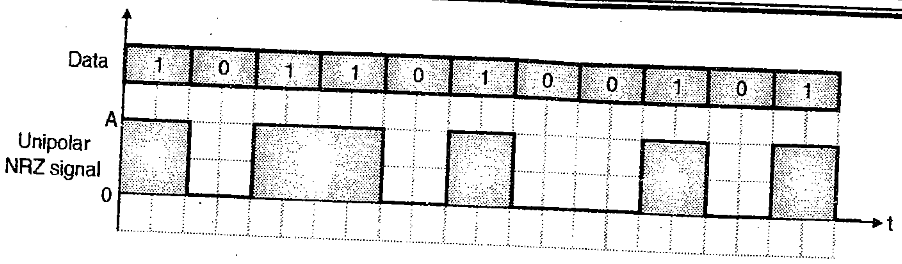
- The return to zero (RZ) unipolar format is as shown in Fig. 3.10.2.
- In this format each "0" is represented by an off pulse (0) and each "1" by an on pulse with amplitude A and a duration of $T_b/2$, followed by a return to zero level.
- Therefore this is called as return to zero (RZ) format. As the voltage level is either + A or zero, this is a unipolar format. (Unipolar means only one polarity).
- Due to the unipolar nature, the unipolar RZ format has a nonzero dc value. The dc value does not contain any information.

3.10.3 Unipolar NRZ Format : SPPU : May 15

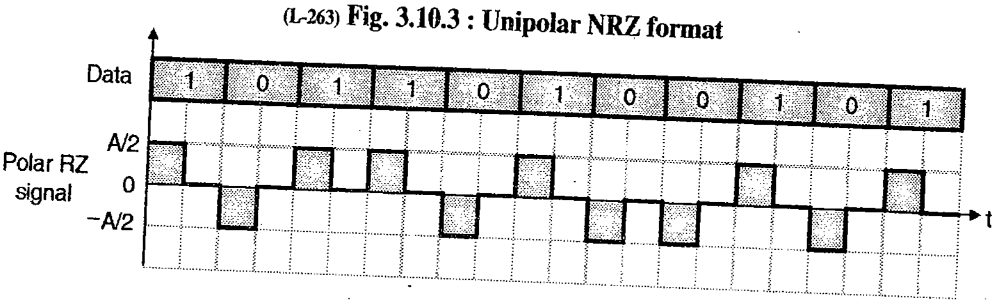
University Questions

Q. 1 Explain various data formats. **(May 15, 6 Marks)**

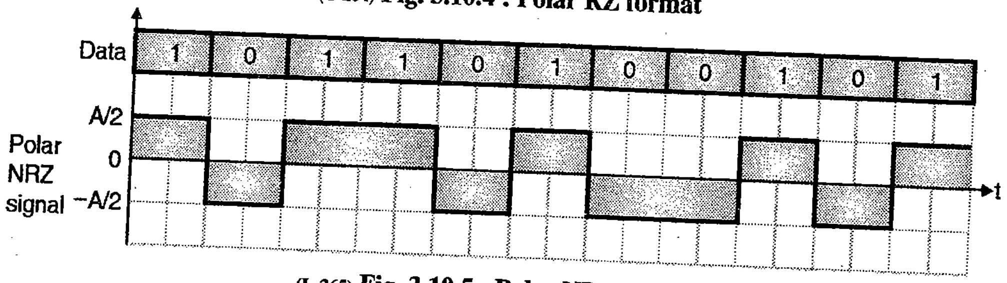
- A non-return to zero (NRZ) format is as shown in Fig. 3.10.3.
- In this format a logic "1" is represented by a pulse of full bit duration T_b and amplitude + A while a logic "0" is represented by an off pulse or zero amplitude.
- During the on time, the pulse does not return to zero after half bit period. Therefore the name NRZ format.
- As the pulses have either + A or 0 amplitude it is called as a unipolar format.



(L-263) **Fig. 3.10.3 : Unipolar NRZ format**



(L-264) **Fig. 3.10.4 : Polar RZ format**



(L-265) **Fig. 3.10.5 : Polar NRZ format**

- Internal computer waveforms are usually of this type. Due to the unipolar nature, the unipolar NRZ format also will have a nonzero average (dc) value which does not carry any information.
- Due to longer pulse duration, the NRZ pulses carry more "energy" than the RZ pulses. But they need synchronization at the receiver as there is no separation between the adjacent pulses.

3.10.4 Polar RZ Format :

SPPU : May 15

University Questions

Q. 1 Explain various data formats. **(May 15, 6 Marks)**

- The disadvantage of the two unipolar formats discussed earlier is that they result in a dc component that does not carry any information and wastes power.
- The polar RZ format is as shown in Fig. 3.10.4. It shows that opposite polarity pulses of amplitude ± "A/2" are used to represent logic "1" and "0".
- Therefore it is called as a "polar" format. As the pulses return to zero after half the bit duration "$T_b/2$" this format is a RZ format.

3.10.5 Polar NRZ Format :

SPPU : May 15

University Questions

Q. 1 Explain various data formats. **(May 15, 6 Marks)**

- In the polar NRZ format, as shown in Fig. 3.10.5 a pulse of amplitude "+ A/2" of duration T_b is used to represent a logic "1" and a pulse of amplitude "– A/2" of the same duration represents a logic "0".
- Unlike the unipolar waveform, a polar waveform has no dc component if the 0s and 1s in the input data occur in equal proportion.

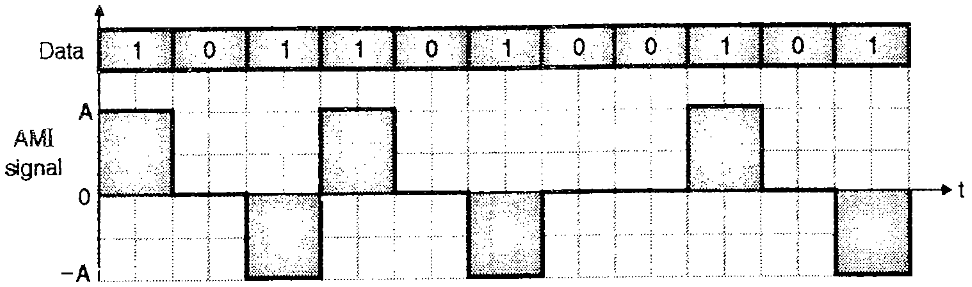
3.10.6 Bipolar NRZ Format (AMI) :

SPPU : May 15

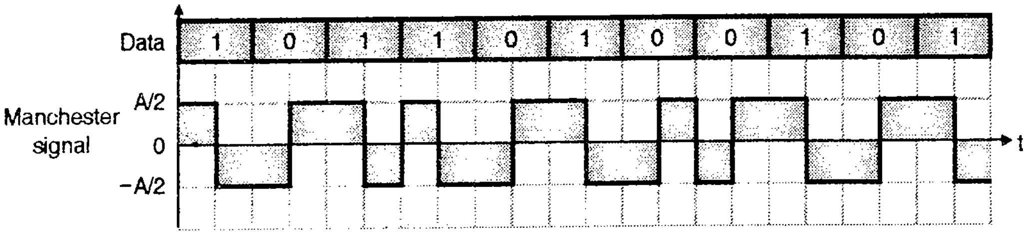
University Questions

Q. 1 Explain various data formats. **(May 15, 6 Marks)**

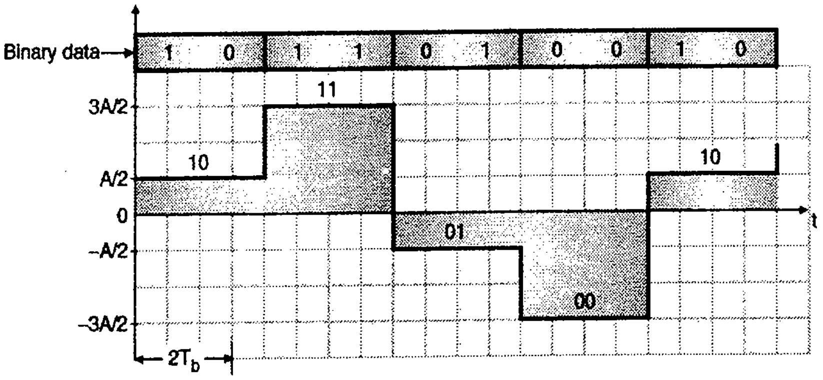
- The bipolar NRZ format is as shown in Fig. 3.10.6. Here the successive "1s" are represented by pulses with alternating polarity, and no pulse is transmitted for a logic "0".
- Note that in this representation there are three levels : + A, 0 and – A.
- Therefore this is also known as "pseudoternary or alternative mark inversion (AMI)" format.



(L-268) Fig. 3.10.6 : Bipolar NRZ format (AMI)



(L-266) Fig. 3.10.7 : Split phase manchester format



(L-275) Fig. 3.10.8 : Polar quaternary NRZ format

3.10.7 Split Phase Manchester Format :

SPPU : May 15

University Questions

Q. 1 Explain various data formats. (May 15, 6 Marks)

- The split phase Manchester format is as shown in Fig. 3.10.7.
- In this format, symbol "1" is represented by transmitting a positive pulse of "+ A/2" amplitude for one half of the symbol duration, followed by a negative pulse of amplitude "– A/2" for remaining half of the symbol duration.
- For symbol "0" these two pulses are transmitted in reverse order.
- This waveform does not have any dc component.
- The Manchester format has a built in synchronization capability as it crosses zero at regular intervals. But this capability is attained at the expense of a bandwidth requirement of twice that of the NRZ unipolar, polar and bipolar formats.
- Local area networks (LAN) such as Ethernet and Cheapernet are increasingly using the Manchester code for signal transmission over the network.

3.10.8 Polar Quaternary NRZ Format :

- Fig. 3.10.8 shows quaternary NRZ format derived by grouping the message bits in the blocks of two and using four amplitude levels to represent the four possible combinations 00, 01, 10 and 11.
- To these four combinations, four amplitude levels are assigned, as shown in Table 3.10.1.

Table 3.10.1

Message combination	$x(t) = a_k$
0 0	– 3 A/2
0 1	– A/2
1 0	A/2
1 1	3 A/2

- In the waveforms of Fig. 3.10.8, the first combination of two bits is "10" hence it is represented by a level "A/2".
- The second combination is "11" hence it is represented by a level of "3A/2". Thus here for a message of two bits only one pulse of duration $D = 2\,T_b$ is transmitted.

$$\therefore \quad D = 2\,T_b \qquad \text{...(3.10.3)}$$

$$\text{The signaling rate : } r = \frac{1}{2\,T_b} \text{ messages/sec} \qquad \text{...(3.10.4)}$$

- If there are "M" levels obtained from the combination of "k" bits (here M = 4 and k = 2) then :

$$M = 2^k \qquad \text{...(3.10.5)}$$

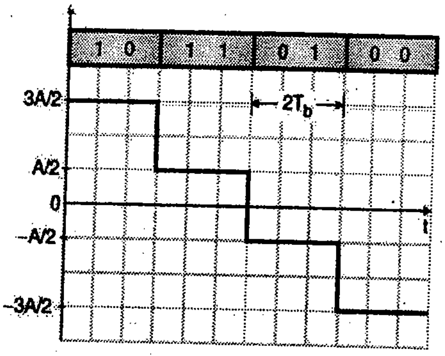
3.10.9 Gray Code :

- There is another scheme used for coding the quaternary format. It is called as the gray code.
- The gray coding scheme is illustrated in the following table. The adjacent bits are arranged in such a way that they differ by only one bit.

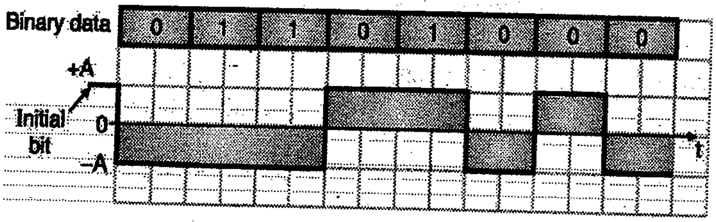
Table 3.10.2 : Gray encoding

Message combination	x (t)
0 0	– 3 A/2
0 1	– A /2
1 1	A/2
1 0	3 A/2

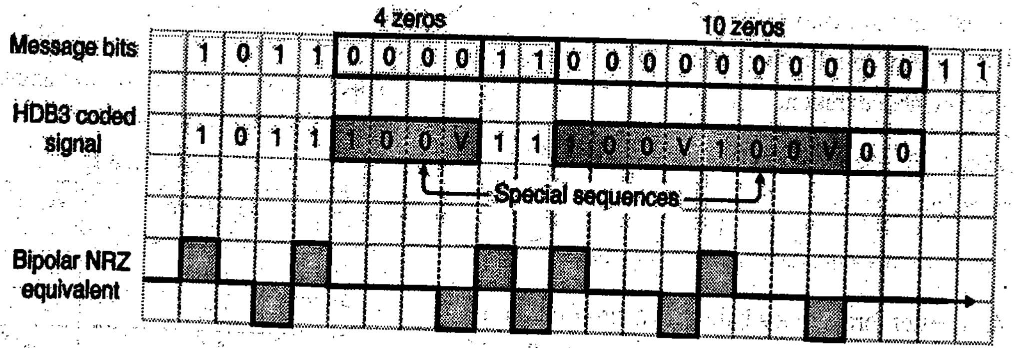
- The polar quaternary format with gray coding is shown in Fig. 3.10.9.



(L-276) Fig. 3.10.9 : Polar quaternary format with gray coding



(L-277) Fig. 3.10.10 : Differential encoding format



(L-284) Fig. 3.10.11

3.10.10 Differential Encoding :

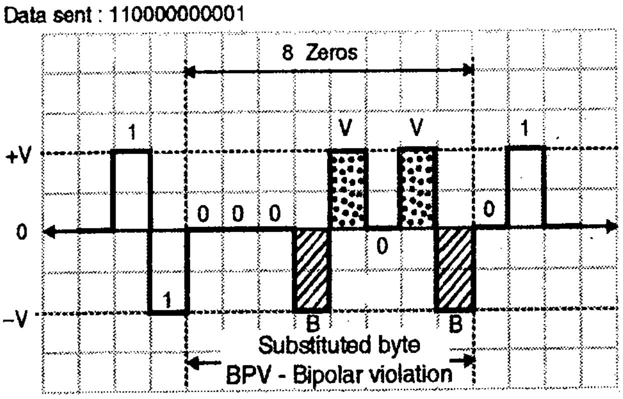
- The differential encoding format is shown in Fig. 3.10.10.
- The advantage of differential encoding is that it is immune from the polarity inversion-ambiguity problem.
- Differential encoding starts with an arbitrary initial bit.
- Corresponding to every 0 at the input, the differential encoding format make a transition from + A to – A or – A to + A whereas no transition takes place corresponding to a logic 1 at the input.
- The original binary information can be recovered by sampling the received wave and comparing the polarities of the adjacent samples.

3.10.11 High Density Bipolar (HDB) Signalling :

- In case of the bipolar NRZ or AMI signal, the transmitted signal is equal to zero when a binary "0" is to be transmitted. This is true even for the unipolar RZ and unipolar NRZ signals.
- The absence of transmitted signal can cause problems in synchronization at the receiver, if long sequence of binary "0"s are being transmitted.
- This problem can be solved by adding (transmitting) pulses when long strings of 0's exceeding a number n are being transmitted. This type of coding is called as High Density Bipolar coding. It is denoted by HDBN. Here N = 1, 2, 3 .... The most widely used HDB format is with N = 3 i.e. HDB3.
- In the string of message bits when (N + 1) or mn number of zeros occur, they are replaced by special binary sequences of (N + 1) length. As shown in Fig. 3.10.11, these sequences contain some binary 1's which are necessary for synchronization at the receiver.

3.10.12 B8ZS line code :

- The long form of B8ZS is bipolar with 8-zero substitution.
- We have discussed about the line codes in this chapter. We know that in order to have synchronization between the transmitter and receiver, the line code needs to cross the zero line frequently.
- As per U.S. T_1 standard, not more than 15 0's can be sent in succession to ensure proper synchronization.
- In order to solve the problems related to synchronization a new line code called B8ZS (Binary 8-zeros suppression) was developed.
- Whenever eight successive 0's are detected, the implementation of this line code will automatically insert a special 8 bit sequence containing a bipolar violation.
- This can be easily detected and corrected by the CSU/DSU (channel service unit/digital service unit).
- Refer Fig. 3.10.12 to get a clear idea about the B8ZS line code.
- The violations (BPV in Fig. 3.10.12), will distinguish a byte substituted for all 0's from a normal byte which contains 1's.
- The B8ZS does not allow more than 8 - consecutive 0's and the bipolar violation pattern uniquely identifies the eight 0's.
- Note that the voltage levels in the "violating byte", has a zero average (dc) value.



(L-285) Fig. 3.10.12

3.10.13 Selection of Line Codes :

SPPU : May 12

University Questions

Q. 1 What are line codes and its characteristics ? Compare the power spectral density of unipolar NRZ and RZ formats by deriving suitable expressions. **(May 12, 8 Marks)**

The selection of a line code for a particular application is based on the following parameters.

1. **DC component :**

   All the cable systems and other communication systems, do not allow transmission of a dc signal. Therefore the line code must have a zero average (dc) value. NRZ bipolar formats usually satisfy this requirement. For this reason, long strings of zeros or ones should not be transmitted.

2. **Self clocking capability :**

   Any digital communication system needs symbol or bit synchronization. To ensure synchronization at the receiver the line code waveform must undergo a sufficient number of zero crossings. That means the waveform must always undergo transitions after regular intervals. This is known as the inherent synchronizing or clocking feature. Some of the codes such as the Manchester code have this feature inherently.

3. **Error detection :**

   Some of the line codes such as "duobinary" are capable of detecting the data errors without introducing additional error detection bits into the data sequence.

4. **Bandwidth compression :**

   The bandwidth of a line code should be as small as possible. The multilevel codes need less bandwidth as compared to the other codes. This happens due to effective utilization of bandwidth.

5. **Differential encoding :**

   The differential encoding is useful for those communication systems where the transmitted waveform sometimes experiences an inversion. In differential encoding, the polarity of encoded waveform is inverted without affecting the data detection.

6. **Noise immunity :**

   The selected line code should have a very high noise immunity (ability to minimize effects of noise). This is necessary to have minimum number of errors introduced due to noise. The NRZ formats have better noise immunity than that of the unipolar RZ format.

7. **Minimum crosstalk :**

   The crosstalk between the adjacent channels that are being transmitted should be minimized. To achieve this the amount of energy in the signal at low frequencies should be small.

3.11 Power Spectra of Discrete PAM Signals :

- As mentioned in the preceding section, we can represent all the discrete PAM signals by a single expression as,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, p(t-kT) \qquad ...(3.11.1)$$

where a_k = Coefficient

$p(t)$ = The basic pulse shape

T = Symbol duration

- Assume that the basic pulse p (t) is centered at the origin (t = 0) and normalized such that p (0) = 1.
- Table 3.11.1 lists the values of coefficients a_k for different PAM formats i.e. line codes.

Table 3.11.1 : Values of a_k for various data formats

Sr. No.	NRZ format	Coefficient a_k		Basic pulse p (t)
		for symbol 0	for symbol 1	
1.	Unipolar NRZ	$a_k = 0$	$a_k = A$	Basic pulse p(t) is a rectangular pulse of unit amplitude and duration T_b.
2.	Polar NRZ	$a_k = -A$	$a_k = A$	
3.	Bipolar NRZ	$a_k = 0$	$a_k = A$ or $-A$ alternately for 1s.	
4.	Manchester	$a_k = -A$	$a_k = +A$	p(t) consists of double pulses of amplitude ± 1 and duration T_b.
5.	Polar Quaternary	$a_k = -3A/2$ for 00 $a_k = A/2$ for 10	$-A/2$ for 01 and $3A/2$ for 11	p(t) is a rectangular pulse of unit amplitude and duration $2\,T_b$.

Data signaling rate :

- It is defined as the number of bits of data transmitted per second. It is measured in bits/second.
- The data signaling rate is also called as data rate and it is defined as follows :

$$R_b = 1/T_b \qquad ...(3.11.2)$$

where T_b represents the bit duration.

Modulation rate :

It is defined as the rate at which the signal level is changed. The units of modulation rate are bauds or symbols per second.

Power spectra :

- We can represent line code format as follows :

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, p(t-kT)$$

- Each format x(t) can be considered as a random process and each coefficient a_k as a random variable.
- These coefficients are assumed to be generated by a discrete stationary random source which is characterized by the following relation between auto correlation and ensemble average.

$$R_\tau(n) = E[a_k\, a_{k-n}] \qquad ...(3.11.3)$$

where $R_\tau(n)$ = Auto-correlation

and E = Expectation operator

- The power spectral density (PSD) of the line code signal x(t) is given by,

$$S(f) = \frac{1}{T} |P(f)|^2 \sum_{n=-\infty}^{\infty} R_\tau(n)\, e^{-j2\pi n f T} \quad ...(3.11.4)$$

where P(f) = Fourier transform of the basic pulse p(t).

- Note that the values of P(f) and R_τ (n) depends on the type of PAM format.

3.11.1 Power Spectral Density of NRZ Unipolar Format :

SPPU : May 12. Dec. 12. Dec. 13

University Questions

Q. 1 What are line codes and its characteristics ? Compare the power spectral density of unipolar NRZ and RZ formats by deriving suitable expressions. **(May 12, 8 Marks)**

Q. 2 Derive and sketch the power spectral density of polar RZ and polar NRZ signaling. **(Dec. 12, 10 Marks)**

Q. 3 Explain need of line coding. State its properties. Draw and give mathematical expression of Power Spectral density for unipolar NRZ, Polar RZ, AMI, and Manchester. **(Dec. 13, 8 Marks)**

- Assume that the 0s and 1s have equal probability.

$P(a_k = 0) = p(a_k = A) = 1/2.$

- Hence for n = 0, Equation (3.11.3) gets modified to,

$$R_\tau(n) = E[a_k a_k] = E[a_k^2] \quad ...(3.11.5)$$

$$E[a_k]^2 = 0^2 P(a_k = 0) + (A)^2 P(a_k = A)$$

$$= A^2 \times \frac{1}{2} = A^2/2 \quad ...(3.11.6)$$

- Now let us obtain the product $a_k a_{k-n}$ for n ≠ 0. The four possible values of this product are 0, 0, 0 and A^2.
- If we assume that the successive symbols in a binary sequence are equiprobable (with a probability equal to 1/2), then the probability of the four product values 0, 0, 0 and A^2 will be equal to 1/4 each.
- Therefore for n ≠ 0 we can write,

$$E[a_k a_{k-n}] = (3 \times 0 \times 1/4) + (A^2 \times 1/4)$$

$$= A^2/4 \quad ...\ n \neq 0 \quad ...(3.11.7)$$

- Hence the autocorrelation function R_τ (n) can be expressed as :

$$R_\tau(n) = A^2/2 \quad ...\text{ for } n = 0 \quad ...(3.11.8)$$

$$= A^2/4 \quad ...\text{ for } n \neq 0 \quad ...(3.11.9)$$

- The basic pulse p(t) is a rectangular pulse of amplitude equal to 1 and duration T_b. Therefore its Fourier transform is given by,

$$P(f) = T_b \,\text{sinc}(f T_b) \quad ...(3.11.10)$$

where the sinc function is defined as,

$$\text{sinc}\, x = \frac{\sin(\pi x)}{\pi x}$$

- Substitute the expressions for P(f) and R_τ (n) into Equation (3.11.4) and substitute $T = T_b$ to get,

Power spectral density (PSD) as follows :

$$S(f) = \frac{1}{T_b} \times [T_b \,\text{sinc}(f T_b)]^2$$

$$\left[\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{A^2}{4} e^{-j2\pi n f T_b} + \sum_{n=0} \frac{A^2}{2} e^{-j2\pi n f T_b} \right] \quad ...(3.11.11)$$

$$\therefore \quad S(f) = T_b \,\text{sinc}^2(f T_b)$$

$$\left[\frac{A^2}{2} e^0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{A^2}{4} e^{-j2\pi n f T_b} \right]$$

$$= \frac{A^2 T_b}{2} \text{sinc}^2(f T_b) + \frac{A^2 T_b}{4} \text{sinc}^2(f T_b) \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} e^{-j2\pi n f T_b} \quad ...(3.11.12)$$

- According to Poisson's formulae we can write,

$$\sum_{n=-\infty}^{\infty} e^{-j2\pi m f T_b} = \frac{1}{T_b} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T_b}\right) \quad ...(3.11.13)$$

where δ (f) is the unit impulse at f = 0 in the frequency domain.

- Substitute Equation (3.11.13) into Equation (3.11.12) and note that the sinc function passes through zero at $f = \pm 1/T_b, \pm 2/T_b$ ... We can write Equation (3.11.12) in the simplified form as,

$$S(f) = \frac{A^2 T_b}{4} \text{sinc}^2(f T_b) + \frac{A^2}{4} \delta(f) \quad ...(3.11.14)$$

- This expression can be plotted as shown in Fig. 3.11.1. (curve "a").

3.11.2 PSD of NRZ Polar Format :

SPPU : Dec. 12. Dec. 13. Dec. 14

University Questions

Q. 1 Derive and sketch the power spectral density of polar RZ and polar NRZ signaling. **(Dec. 12, 10 Marks)**

Q. 2 Explain need of line coding. State its properties. Draw and give mathematical expression of Power Spectral density for unipolar NRZ, Polar RZ, AMI, and Manchester. **(Dec. 13, 8 Marks)**

Q. 3 Derive the expression for power spectral density of polar NRZ signal. **(Dec. 14, 6 Marks)**

- We will follow a similar procedure to the one followed in the previous section to obtain the PSD of NRZ polar format.

- For the NRZ polar format

$$\left.\begin{aligned} R_\tau(n) &= a^2 &&\dots\text{for } n = 0 \\ \text{and} \quad &= 0 &&\dots\text{. for } n \neq 0 \end{aligned}\right\} \quad \dots(3.11.15)$$

- The basic pulse p(t) is same for all the PAM formats. So its Fourier transform also will remain same.

$$\therefore \quad P(f) = T_b \operatorname{sinc}(f T_b)$$

- Hence the power spectral density is given by,

$$S(f) = A^2 T_b \operatorname{sinc}^2(f T_b) \qquad \dots(3.11.16)$$

- The normalized form of this expression has been plotted in Fig. 3.11.1 as curve "b".

Note : Curve "b" of Fig. 3.11.1 shows that most of the power of polar NRZ format lies inside the main lobe of the sinc shaped curve extending upto the bit rate $1/T_b$.

Similarly it is possible to obtain the expressions for the power spectral densities of other PAM formats.

3.11.3 PSD of NRZ Bipolar Format :

SPPU : Dec. 13

University Questions

Q. 1 Explain need of line coding. State its properties. Draw and give mathematical expression of Power Spectral density for unipolar NRZ, Polar RZ, AMI, and Manchester. **(Dec. 13, 8 Marks)**

- The bipolar NRZ format has three levels A, 0 and – A.
- If we assume that the 1s and 0s in the binary input data have equal probability of occurance then the probabilities of occurrence of these levels are as follows :

$$P(a_k = A) = 1/4$$

$$P(a_k = 0) = 1/2$$

$$P(a_k = -A) = 1/4$$

- Hence for n = 0 we can write that,

$$\begin{aligned} E[a_k^2] &= A^2 P(a_k = A) \\ &\quad + (0)^2 P(a_k = 0) + (-A)^2 P(a_k = -A) \\ &= A^2/2 \end{aligned}$$

- Similarly for n = 1 the input sequence can assume four possible formats as (0, 0), (0, 1), (1, 0) and (1, 1).
- The corresponding values of the product $a_k a_{k-1}$ are 0, 0, 0 and 1 respectively.
- Assuming that the 0s and 1s in the binary sequence have equal probability, we can write that

$$E[a_k a_{k-1}] = 3(0)(1/4) + (-A^2)(1/4) = -A^2/4$$

- Finally for n > 1 we find that $E[a_k a_{k-1}] = 0$
- Therefore for the bipolar NRZ format

$$R_\tau(n) = \begin{cases} A^2/2 & \dots\text{ for } n = 0 \\ -A^2/4 & \dots\text{ for } n = \pm 1 \\ 0 & \dots\text{ for } n > 1 \end{cases} \qquad \dots(3.11.17)$$

- The basic pulse p(t) has the Fourier transform given by,

$$P(f) = T_b \operatorname{sinc}(f T_b) \qquad \dots(3.11.18)$$

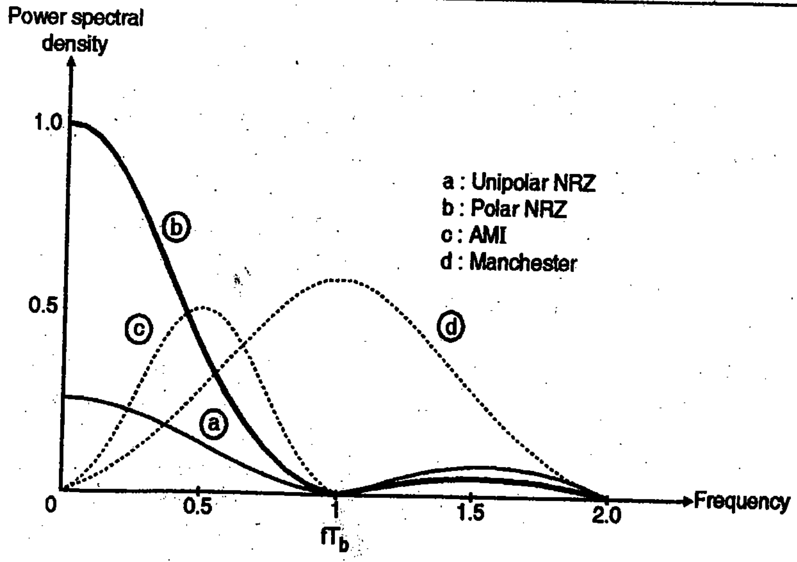
- Hence substitute Equation (3.11.17) and (3.11.18) into the Equation (3.11.11) that states that,

$$S(f) = \frac{1}{T_b} |P(f)|^2 \sum_{n=-\infty}^{\infty} R_\tau(n) e^{-j2\pi n f T_b}$$

To get the power spectral density of NRZ bipolar format as follows :

$$\begin{aligned} S(f) &= T_b \operatorname{sinc}^2(f T_b)\left[\frac{A^2}{2} - \frac{A^2}{4}\left(e^{j2\pi f T_b} + e^{-j2\pi f T_b}\right)\right] \\ &= \frac{A^2}{2} T_b \sin^2 c(f T_b)\left[1 - \frac{(e^{j2\pi f T_b} + e^{-j2\pi f T_b})}{2}\right] \\ &= \frac{A^2 T_b}{2} \sin^2 c(f T_b)[1 - \cos(2\pi f T_b)] \\ &= A^2 T_b \sin^2 c(f T_b) \sin^2(\pi f T_b) \qquad \dots(3.11.19) \end{aligned}$$

- The normalized form of this equation is shown plotted in "c" in Fig. 3.11.1.



(E-735) **Fig. 3.11.1 : Power spectra of different line codes**

3.11.4 PSD of the Manchester Format :

SPPU : Dec. 13

University Questions

Q. 1 Explain need of line coding. State its properties. Draw and give mathematical expression of Power Spectral density for unipolar NRZ, Polar RZ, AMI, and Manchester. **(Dec. 13, 6 Marks)**

- Assume that the input binary data consists of 0s and 1s of equal probability (1/2).
- Then the autocorrelation function of this format is same as that for the NRZ polar format i.e.

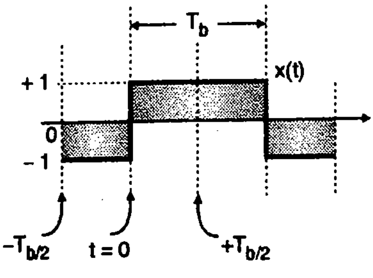
$$R_\tau(n) = \begin{cases} A^2 & \dots n = 0 \\ 0 & \dots n = 1 \end{cases} \qquad \dots(3.11.20)$$

- The basic pulse p (t) of the Manchester format is as follows :

Step 1 : Express the Manchester signal x(t) mathematically in time domain :

- The basic pulse p(t) of the Manchester signal is as shown in Fig. 3.11.2.
- Referring to Fig. 3.11.2 we can express the Manchester signal as

$$\left.\begin{aligned} p(t) &= \pm 1 && \dots -T_b/2 \le t \le 0 \\ &= \mp 1 && \dots 0 \le t \le +T_b/2 \end{aligned}\right\} \dots(3.11.21)$$



(E-736) **Fig. 3.11.2 : Manchester signal**

Step 2 : Obtain the Fourier transform of p(t) :

The Fourier transform of p(t) is given by,

$$P(f) = \int_{-T_b/2}^{0} \pm 1\, e^{-j2\pi ft}\, dt - \int_{0}^{T_b/2} \mp 1\, e^{-j2\pi ft}\, dt \quad \dots(3.11.22)$$

$$\therefore\ P(f) = \frac{\pm 1}{-j\,2\pi f}\left[e^{-j2\pi ft}\right]_{-T_{b/2}}^{0} \mp \frac{1}{-j2\pi f}\left[e^{-j2\pi ft}\right]_{0}^{T_{b/2}}$$

$$P(f) = \frac{\pm 1}{-j2\pi f}\left[e^{0} - e^{j\pi fT_b}\right] \mp \frac{1}{-j2\pi f}\left[e^{-j\pi fT_b} - e^{0}\right]$$

$$P(f) = \underbrace{\frac{\pm 1}{-j2\pi f}\left[1 - e^{j\pi fT_b}\right]}_{\text{Term 1}} \mp \underbrace{\frac{1}{-j2\pi f}\left[e^{-j\pi fT_b} - 1\right]}_{\text{Term 2}}$$

Solving separately for Term 1 and Term 2.

Term 1 :

$$\frac{\pm 1}{-j2\pi f}\left[e^{j\pi fT_b/2} \times e^{-j\pi fT_b/2} - e^{j\pi fT_b/2} \times e^{j\pi fT_b/2}\right]$$

$$= \frac{\pm 1}{-j2\pi f}\left[e^{j\pi fT_b/2} \times \left(e^{-j\pi fT_b/2} - e^{j\pi fT_b/2}\right)\right]$$

$$= \frac{\pm 1}{\pi f}\, e^{j\pi fT_b/2}\left(\frac{e^{j\pi fT_b/2} - e^{-j\pi fT_b/2}}{2j}\right)$$

$$= \pm \frac{1}{\pi f}\, e^{j\pi fT_b/2} \sin \pi f\, T_b / 2 \qquad \dots(3.11.23)$$

Term 2 :

$$\mp \frac{1}{-j2\pi f}\left[e^{-j\pi fT_{b-}1}\right]$$

$$= \mp \frac{1}{-j2\pi f}\left[e^{-j\pi fT_b/2} \cdot e^{-j\pi fT_b/2} - e^{j\pi fT_b/2} \times e^{-j\pi fT_b/2}\right]$$

$$= \mp \frac{1}{-j2\pi f}\left[e^{-j\pi fT_b/2}\left(e^{-j\pi fT_b/2} - e^{j\pi fT_b/2}\right)\right]$$

$$= \mp \frac{1}{\pi f}\, e^{-j\pi fT_b/2}\left[\frac{e^{j\pi fT_b/2} - e^{-j\pi fT_b/2}}{2j}\right]$$

$$= \mp \frac{1}{\pi f}\, e^{-j\pi fT_b/2} \sin(\pi f\, T_b / 2) \qquad \dots(3.11.24)$$

Adding both the terms we get,

$$P(f) = \pm \frac{1}{\pi f}\, e^{j\pi fT_b/2} \sin(\pi f\, T_b/2)$$

$$\mp \frac{1}{\pi f}\, e^{-j\pi fT_b/2} \sin(\pi f\, T_b/2)$$

$$= \pm \frac{1}{\pi f} \sin(\pi f\, T_b/2)\left[e^{j\pi fT_b/2} - e^{-j\pi fT_b/2}\right]$$

$$= \frac{\pm 2j\,1}{\pi f} \sin(\pi f\, T_b/2)\left[\frac{e^{j\pi fT_b/2} - e^{-j\pi fT_b/2}}{2j}\right]$$

$$= \pm \frac{2j\,1}{\pi f} \sin(\pi f\, T_b/2)\,(\pi f\, T_b/2)$$

$$= \pm \frac{2j\,1}{\pi f} \sin^2(\pi f\, T_b/2)$$

Multiply and divide by $T_b / 2$ to get,

$$P(f) = \pm \frac{2j\,(T_b/2)}{\pi f\, T_b/2} \sin^2 (\pi f\, T_b/2)$$

$$P(f) = \pm j\,(T_b) \left[\frac{\sin^2 (\pi + T_b/2)}{\pi f\, T_b/2} \right] \quad ...(3.11.25)$$

Rearranging, we get

$$P(f) = \pm j\, T_b \frac{\sin (\pi f\, T_b/2)}{\pi f\, T_b/2} \sin (\pi f\, T_b/2)$$

$$= \pm T_b \sin c \frac{(f\,T_b)}{2} \sin \frac{(\pi\, f\, T_b)}{2} \quad ...(3.11.26)$$

This is the required Fourier transform.

Step 3 : Obtain the expression for S (f) :

- The psd S (f) is given by,

$$S(f) = \frac{1}{T_b} |P(f)|^2 \sum_{n=-\infty}^{\infty} R_\tau (n) \cdot e^{-j2\pi n f T_b}$$

- Substituting the expressions for P (f) and R_τ (n) we get,

$$S(f) = T_b \,\text{sinc}^2 \frac{(f\,T_b)}{2} \sin^2 \frac{(\pi\, f\, T_b)}{2} \cdot A^2$$

$$\therefore\ S(f) = A^2\, T_b \,\text{sinc}^2 \frac{(f\,T_b)}{2} \sin^2 \frac{(\pi\, f\, T_b)}{2} \quad ...(3.11.27)$$

- This is the required expression for the PSD. The normalized form of this PSD is plotted as curve "d" in Fig. 3.11.1.

- Most of the power lies inside the bandwidth equal to $2 / T_b$ which is twice as large as that of unipolar, polar and bipolar formats of NRZ type.

Features of the PSD of a Manchester signal :

- The main lobes extend from 0 to $2f_b$ rather than 0 to f_b (as NRZ).
- At f = 0, S (f) = 0 as shown in Fig. 3.11.3.
- If the Manchester signal is transmitted through an ideal low pass filter with a cut off frequency of $2\, f_b$ then about 95% power will be transmitted but if the cutoff frequency is reduced to f_b then only about 70% power is passed.
- The bandwidth required is $2f_b$ which is double the bandwidth of the NRZ signal.

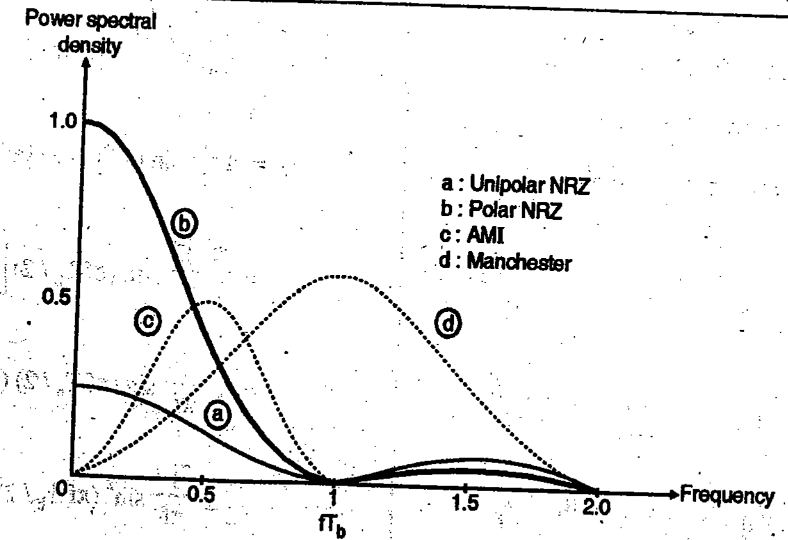
3.11.5 Comparison of Discrete PAM Formats on the Basis of Power Spectra :

SPPU : May 07

University Questions

Q. 1 Explain properties of line codes and draw the power spectral for various codes. **(May 07, 8 Marks)**

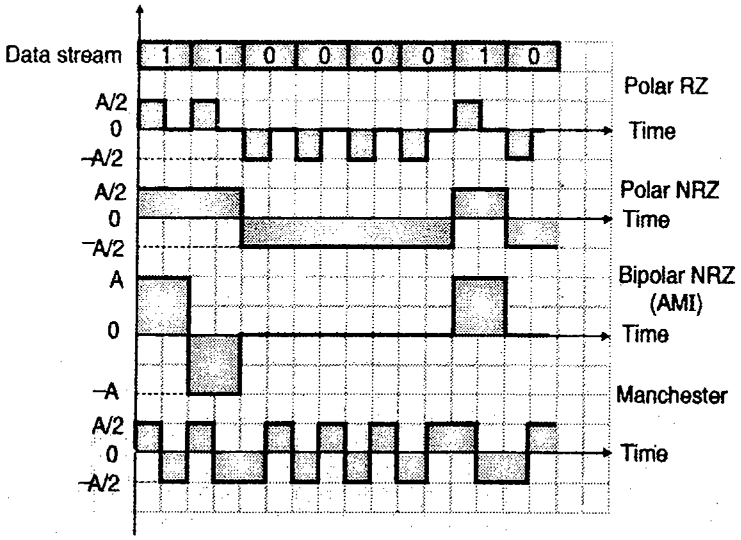
- Power spectra is a graph of power spectral density plotted against frequency. It gives the distribution of power of different frequency components.
- The power spectra for four important line codes are as shown in Fig. 3.11.3.



(E-348) **Fig. 3.11.3 : Power spectra of different line codes**

Table A : Comparison of line codes

Sr. No.	Parameter	Polar RZ	Polar NRZ	AMI	Manchester	Polar Quaternary NRZ
1.	Transmission of DC component	Yes	Yes	No	No	Possible
2.	Signaling rate	$1/T_b$	$1/T_b$	$1/T_b$	$1/T_b$	$1/2\ T_b$
3.	Noise immunity	Low	Low	High	High	High
4.	Synchronizing capability	Poor	Poor	Very good	Very good	Poor
5.	Bandwidth required	$1/T_b$	$1/2\ T_b$	$1/2\ T_b$	$1/T_b$	$1/2\ T_b$
6.	Crosstalk	High	High	Low	Low	Low



(E-737) **Fig. P. 3.11.1**

1. **Power spectra of NRZ unipolar :**
   - Curve "a" of Fig. 3.11.3 shows a normalized graph. Both the power spectral density and frequency are normalized.
   - Most of the power of the NRZ unipolar format is concentrated between dc (f = 0) and the bit rate of input data ($f = 1/T_b$).
2. **Power spectra of NRZ polar format :**
   - Curve "b" in Fig. 3.11.3 shows a normalized power spectra of NRZ polar format.
   - Here also like unipolar NRZ format, most of the power lies in the main lobe of the sinc. shaped curve. This main lobe extends upto a bit rate of $1/T_b$.
3. **NRZ bipolar (AMI) format :**
   - Curve "c" in Fig. 3.11.3 shows the normalized power spectra for the AMI format. It can be seen that almost all the power lies inside the bandwidth equal to $1/T_b$
   - Similar to the two NRZ formats discussed earlier. However the spectral content of the NRZ bipolar format is relatively small around the zero frequency.
4. **Manchester format :**
   - Curve "d" in Fig. 3.11.3 shows the normalized power spectra of Manchester format.
   - Here most of power lies inside the bandwidth equal to $2/T_b$ which is twice the bandwidth of unipolar, polar and bipolar NRZ formats.

3.11.6 Comparison of Line Codes :

SPPU : May 09

University Questions

Q. 1 What are desirable properties of line codes ? Compare RZ and NRZ line coding formats on the basis of above properties along with their merits and demerits. **(May 09, 6 Marks)**

Refer Table A.

3.11.7 Solved Examples on Line Codes :

Ex. 3.11.1 : Encode the following binary data stream into Return to zero (RZ), Non-return to zero (NRZ), AMI and Manchester codes.

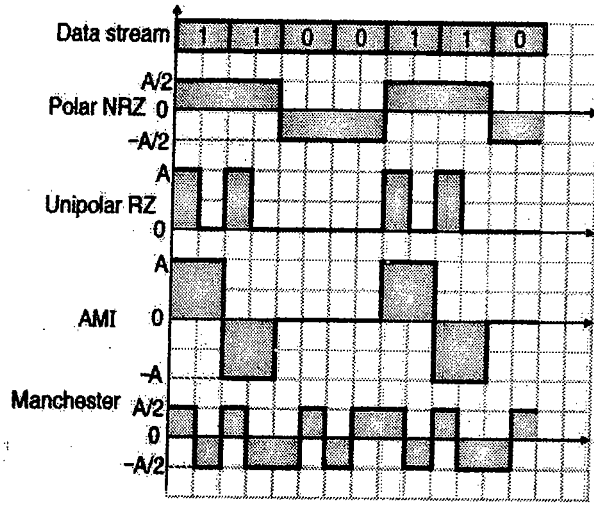
Data stream : 1 1 0 0 0 0 1 0

Soln. : The line coding formats for the given data stream are as shown in Fig. P. 3.11.1.

Ex. 3.11.2 : Draw the following data formats for the bit stream 1 1 0 0 1 1 0 :

1. Polar NRZ2. Unipolar RZ
3. AMI 4. Manchester.

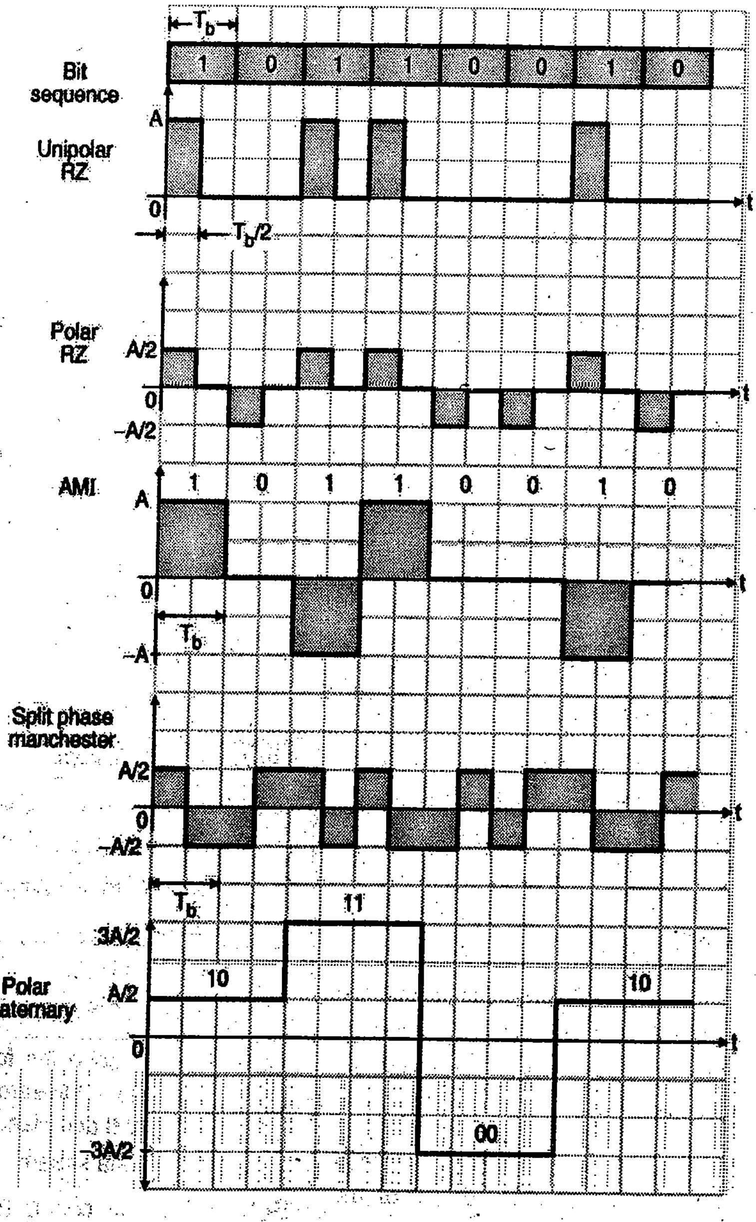
Soln. : The data formats are as shown in Fig. P. 3.11.2.



(E-738) **Fig. P. 3.11.2**

3.11.8 Difference between Source Coding and

- Earlier we have discussed various source coding techniques such as PCM, Delta modulation, ADM etc. Source coding techniques are used in order to convert an analog signal to its digitally coded equivalent. The output of a source coder is a train of binary digits i.e. "0s" and "1s".
- The line coding techniques discussed in section 3.10 convert the stream of binary digits into a format or code which is more suitable for transmission over a cable or any other medium.
- This is because the data transmission must satisfy certain requirements such as synchronization, elimination of dc component, low cross talk etc.
- The line code format should be such that the bandwidth requirements be as low as possible. Examples of line codes are RZ, NRZ, AMI, Manchester and other formats.



(E-739) **Fig. P. 3.11.3**

Ex. 3.11.3 : Consider that the bit sequence given below is to be transmitted. Bit sequence = 10110010

Draw the resulting waveform, if the sequence is transmitted using :

1. Unipolar RZ
2. Polar RZ
3. AMI
4. Split Phase Manchester
5. M-ary where M = 4. (Polar quaternary).

May 2000, 5 Marks

Soln. :

The required waveforms are as shown in Fig. P. 3.11.3.

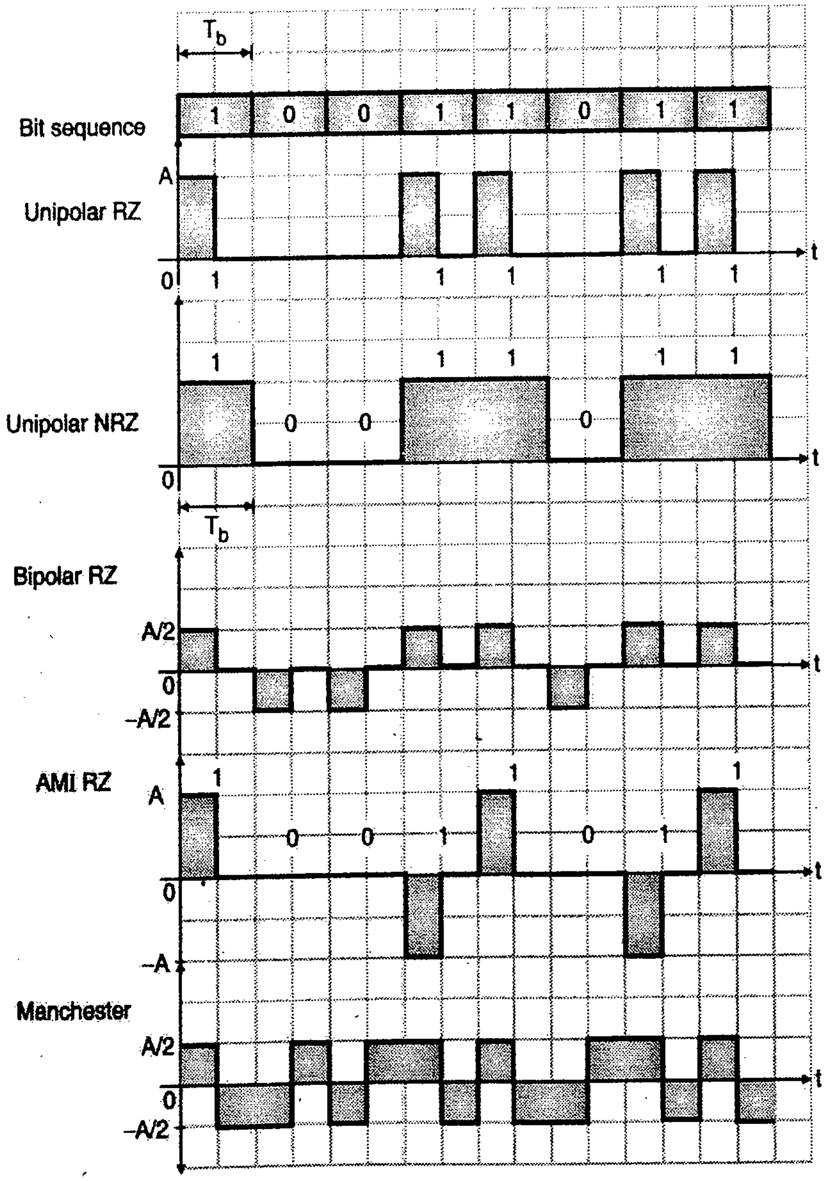
Ex. 3.11.4 : To transmit a bit sequence 10011011, draw the resulting waveforms using :

1. Unipolar RZ
2. Unipolar NRZ
3. Bipolar RZ
4. AMI RZ
5. Manchester.

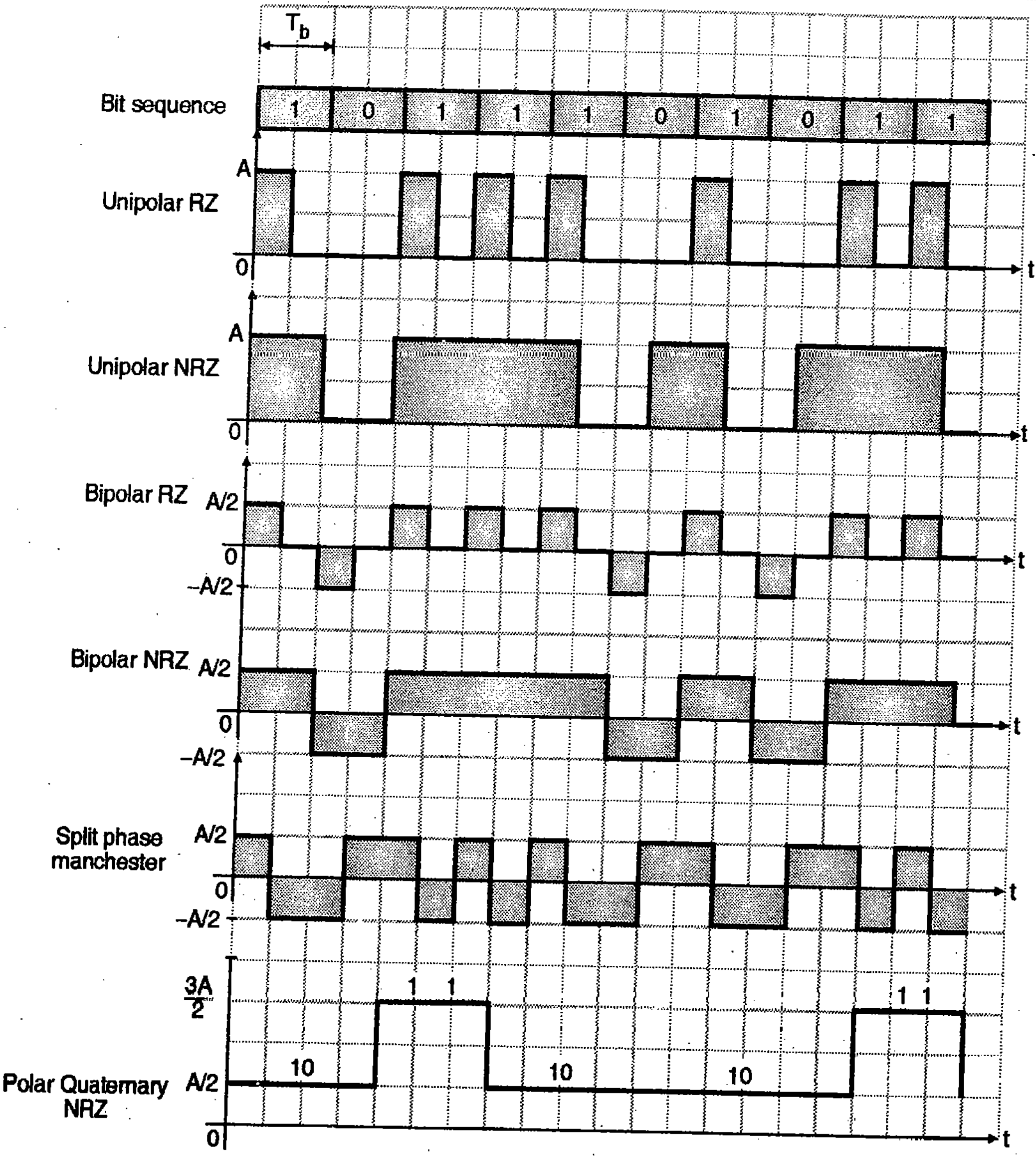
Dec. 2000. 8 Marks

Soln. :

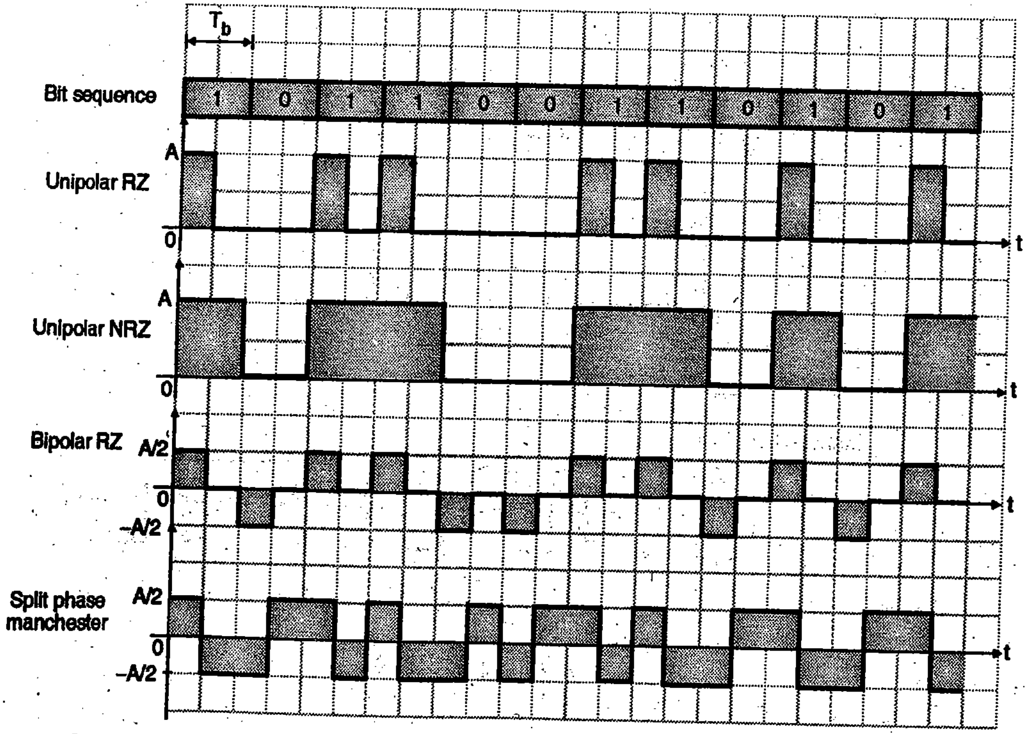
For the required waveforms refer Fig. P. 3.11.4.



(E-740) Fig. P. 3.11.4



(E-741) Fig. P. 3.11.5



(E-742) Fig. P. 3.11.6

Ex. 3.11.5 : The bit sequence 1 0 1 1 1 0 1 0 1 1 is to be transmitted using following formats :

1. Unipolar RZ and NRZ
2. Bipolar RZ and NRZ
3. Split-phase Manchester
4. Polar quaternary NRZ.

Draw all the waveforms. **May 01, 8 Marks**

Soln. :

The required waveforms are as shown in Fig. P. 3.11.5.

Ex. 3.11.6 : The binary data 101100110101 is transmitted over a baseband channel. Draw the waveform for the transmitted data using following formats :

1. Unipolar RZ
2. Unipolar NRZ
3. Bipolar RZ
4. Split-phase manchester.

Compare above schemes for their BW requirements. **Dec. 01, 8 Marks**

Soln. :

The required waveforms are as shown in Fig. P. 3.11.6.

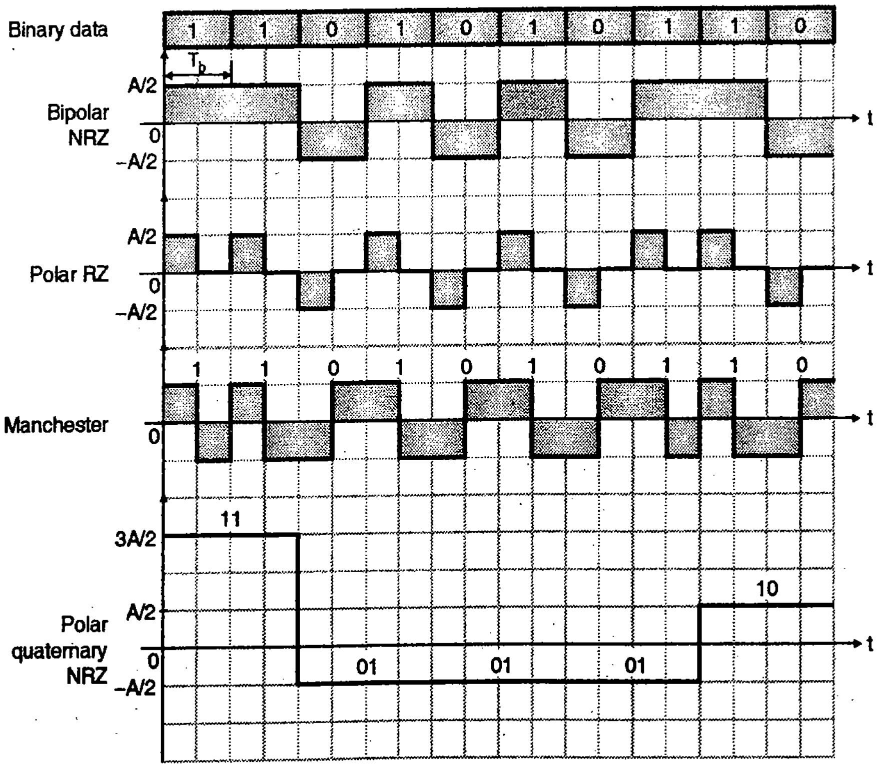
Ex. 3.11.7 : The binary data 1101010110 is transmitted over a baseband channel.

Draw the waveform for transmitted data using following format :

1. Bipolar NRZ
2. Polar RZ
3. Split phase Manchester
4. Polar quaternary NRZ signalling.

Compare above scheme for their bandwidth requirement. **Dec. 02, 8 Marks**

Soln. : The required waveforms are as shown in Fig. P. 3.11.7.



(E-743) **Fig. P. 3.11.7 : Waveforms**

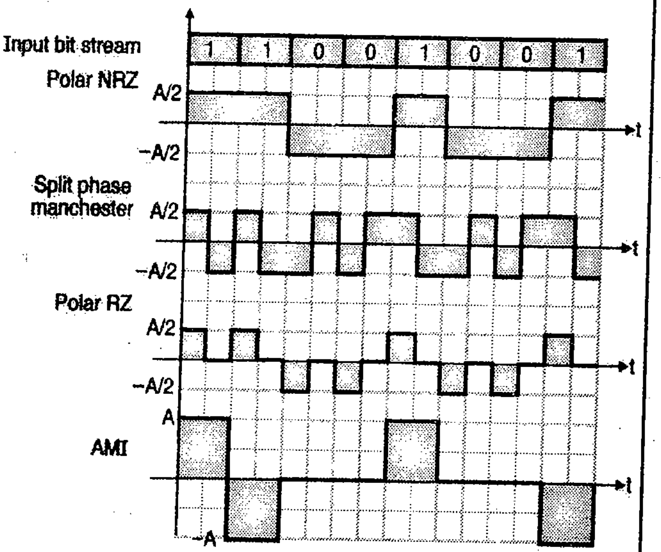
Ex. 3.11.8 : Draw the following line coding formats for the bit stream 1100 1001 :

1. Polar NRZ
2. Split phase Manchester
3. Polar RZ
4. AMI.

Sketch the power spectral density functions for 1. and 3. formats and compare them with respect to their suitability to line.

Dec. 99, 8 Marks

Soln. :



(E-804) **Fig. P. 3.11.8**

For the comparison and power spectral density refer to section 3.11.1.

Ex. 3.11.9 : In unipolar signaling (line code) scheme logical '1' is transmitted as + p (t) pulse and logical '0' as no pulse. Estimate the spectrum Y(ω). Assume p(t) is half width rectangular pulse i.e. RZ. Plot the p.s.d $S_y(\omega)$. Comment on transmission bandwidth, error detection, capability, favorable power spectral density or not, timing content, and transparency. Given that for a train of unit strength impulses but coded according to digital data sequence x(t).

$$\text{PSD } S_x(\omega) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} R_n e^{-jn\omega T_0}$$

$$= \frac{1}{T_0}\left[R_0 + Z \sum_{n=1}^{\infty} R_n \cos n\omega T_0\right]$$

Where T_0 = bit duration

$R_x(\hat{c})$ = correlation

$$= \frac{1}{T_0} \sum_{n=-\infty}^{\infty} R_n \delta(\hat{c} - nT_0)$$

$$R_n = \lim_{N\to\infty} \frac{1}{N} \sum_k a_k \cdot a_{k+n}$$

$$= \lim_{T\to\infty} \frac{T_0}{T} \sum a_k \cdot a_{k+n}$$

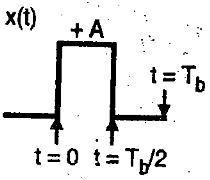
$a_k = k^{th}$ impulse, $a_{k+n} = k + n^{th}$ impulse.

Assume input sequence (data) equiprobable logical '1' and '0'.

Dec. 05, 18 Marks, Dec. 06, 8 Marks, Dec. 11, 2 Marks

Soln. :

Power spectral density of an RZ signal AS Fig. P. 3.11.9(a) shows the RZ signal.



(E-805) **Fig. P. 3.11.9(a) : RZ signal**

Mathematically this signal can be represented as,

$x(t) = +A$ for $t = 0$ to $T_b/2$

$= 0$ ... for $t = T_b/2$ to T_b.

The Fourier transform of x(t) is given by,

$$X(f) = \int_0^{T_b/2} A e^{-j2\pi ft} dt$$

$$= \frac{A}{-j2\pi f}[e^{-j2\pi ft}]_0^{Tb/2} = \frac{-A}{j\pi f}[e^{-j\pi fT_b} - 1]$$

But $1 = e^{-j\pi fT_b/2} \times e^{+j\pi fT_b/2}$ and $e^{-j\pi fT_b}$

$$= e^{-j\pi fT_b/2} \times e^{-j\pi fT_b/2}$$

$$\therefore X(f) = \frac{-A}{j2\pi f}$$

$$[e^{-j\pi fT_b/2} \times e^{-j\pi fT_b/2} - e^{-j\pi fT_b/2} \times e^{+j\pi fT_b/2}]$$

$$= \frac{+A}{j2\pi f} e^{-j\pi fT_b/2} [e^{j\pi fT_b/2} - e^{-j\pi fT_b/2}]$$

$$= \frac{A e^{-j\pi fT_b/2}}{\pi f} \frac{[e^{j\pi fT_b/2} - e^{-j\pi fT_b/2}]}{2j}$$

$$= \frac{T_b}{2} A e^{-j\pi fT_b/2} \frac{\sin(\pi f T_b/2)}{\pi f T_b/2}$$

$$X(f) = \frac{A T_b}{2} e^{-j\pi fT_b/2} \operatorname{sinc}(f T_b/2)$$

$$\therefore |X(f)| = \frac{A^2 T_b^2}{4} \operatorname{sinc}^2(f T_b/2)$$

The power spectral density is given by,

$$\text{PSD} = S(f) = \frac{|X(f)|^2}{T_b}$$

$$\therefore S(f) = \frac{A^2 T_b^2}{4 T_b} [\operatorname{sinc}(f T_b/2)]^2$$

$$\therefore S(f) = \frac{A^2 T_b}{4} \operatorname{sinc}^2(f T_b/2)$$

Plotting of PSD :

The PSD is a sinc function. So it has a maximum at f = 0 as sinc (0) = 1.

$$\therefore \; S(f) = \frac{A^2 T_b}{4} \; \text{... at } f = 0.$$

S(f) will pass through zero when $\frac{f T_b}{2} = \pm 1, \pm 2, \pm 3 \ldots$

$$\therefore \; S(f) = 0$$

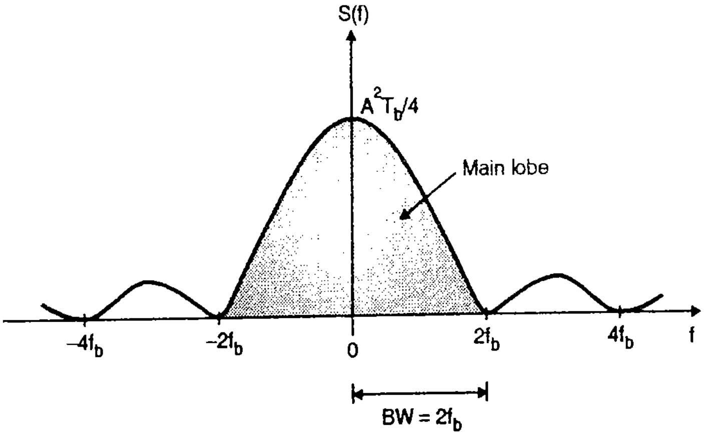
$$\text{for } \; f T_b = \pm 2, \pm 4, \pm 6 \ldots$$

$$\text{or} \quad f = \pm \frac{2}{T_b}, \pm \frac{4}{T_b}, \pm \frac{6}{T_b} \ldots$$

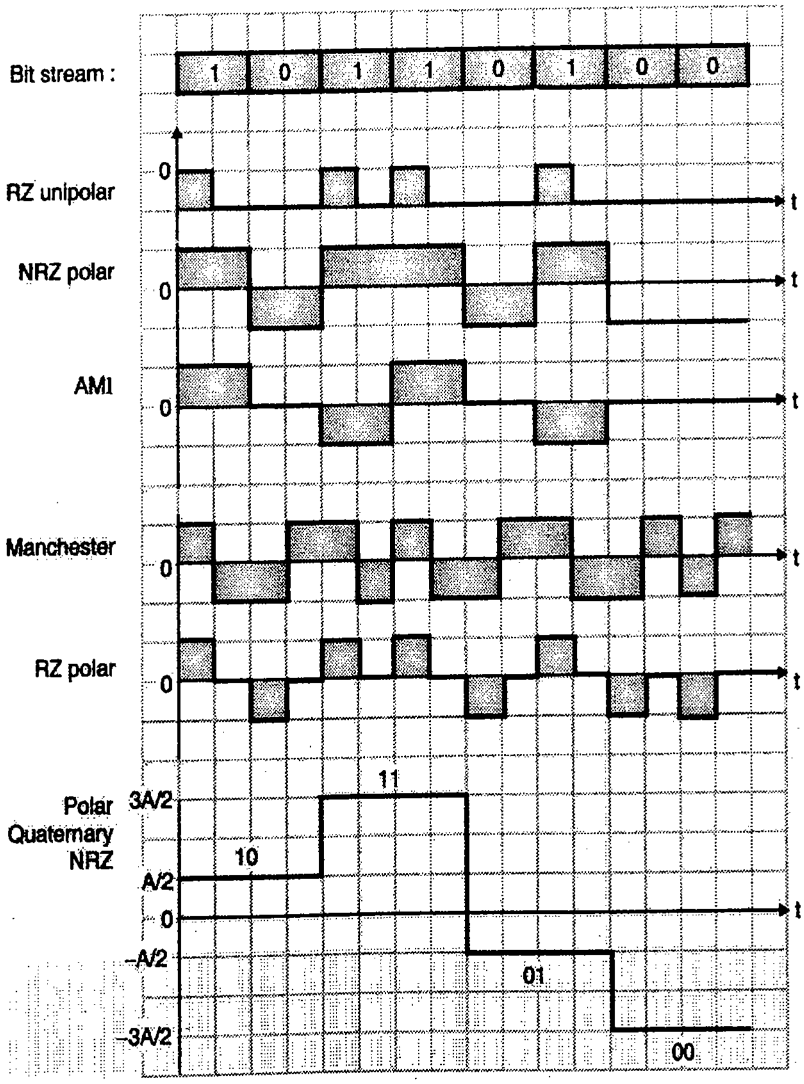
$$\text{But} \quad 1 / T_b = f_b$$

$$\therefore \; S(f) = 0 \text{ for } f = \pm 2 f_b, \pm 4 f_b, \pm 6 f_b \ldots$$

The PSD of RZ signal is plotted as shown in Fig. P. 3.11.9(b).



(E-806) **Fig. P. 3.11.9(b) : PSD of RZ signal**



(E-807) **Fig. P. 3.11.10**

The bandwidth corresponds to the main lobe which extends for from $f = -2 f_b$ to $f = +2 f_b$.

But the spectrum of Fig. P. 3.11.9(b) is double sided one so the bandwidth $BW = 2 f_b$ Hz.

The signal power $P = S(0) = A^2 T_b / 4$. In order to improve the noise immunity, the signal power needs to be increased by increasing either A or T_b. But with increase in T_b, the signaling rate would reduce.

Ex. 3.11.10 : Draw the line code formats for 10110100 :

1. RZ unipolar 2. NRZ polar

3. AMI 4. Manchester

5. RZ polar 6. Polar Quaternary (NRZ)

Dec. 2000. 8 Marks, May 03, May 06, 6 Marks

Soln. : The required line code formats are as shown in Fig. P. 3.11.10.

Ex. 3.11.11 : For the sequence 10111001, sketch the waveform using the following data formats :

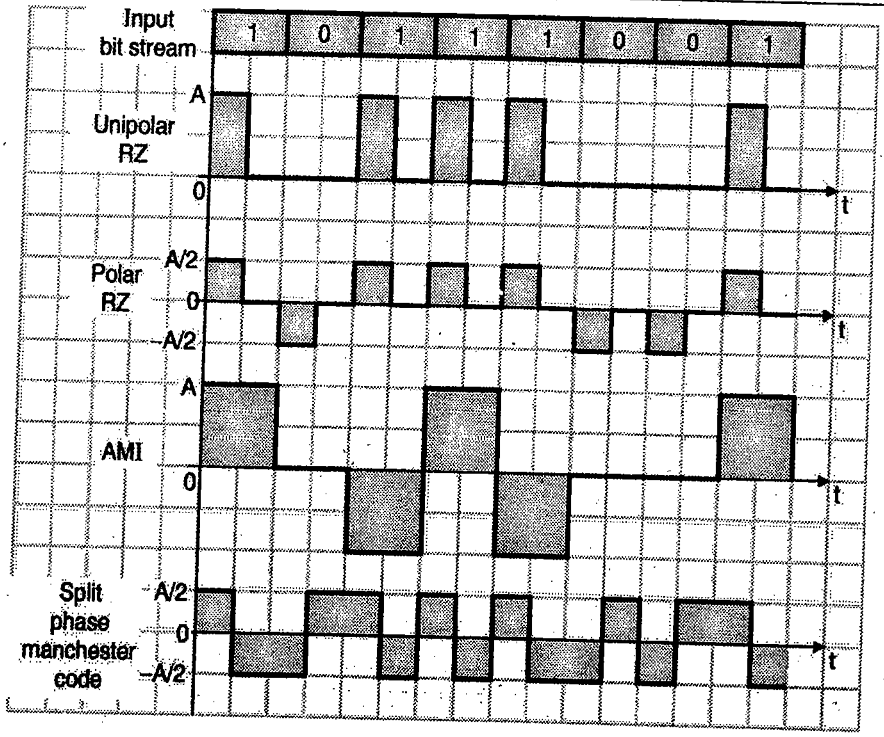
1. Unipolar RZ
2. Polar NRZ
3. Alternate mark inversion
4. Split phase manchester coding.

Draw the corresponding spectrum of the above formats and explain.

Dec. 11. 10 Marks

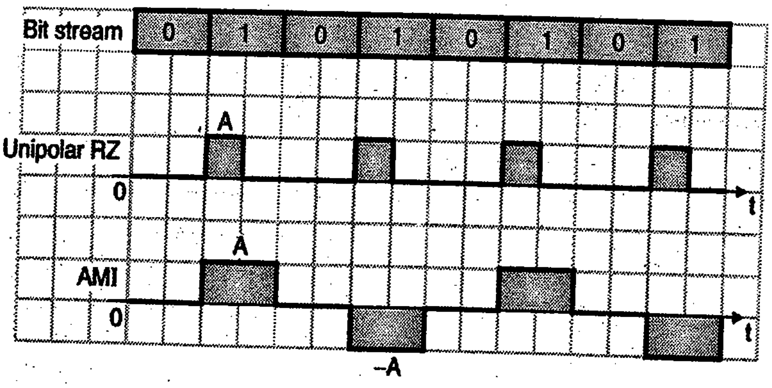
Soln. :

For spectrum refer Ex. 3.11.9 and sections 3.11.2 and 3.11.4.



(E-1150) Fig. P. 3.11.11

1. **Alternate sequence of 1's and 0's :**



(E-1354) Fig. P. 3.11.12(a)

Ex. 3.11.12 : Consider the following sequences of 1's and 0's :

1. An alternate sequence of 1's and 0's.
2. A long sequence of 1's followed by a long sequence of 0's.
3. A single "0" and then a long sequence of 1's. Sketch the waveform for each of these sequences using unipolar RZ and alternate mark inversion signaling.

Dec. 12, 8 Marks

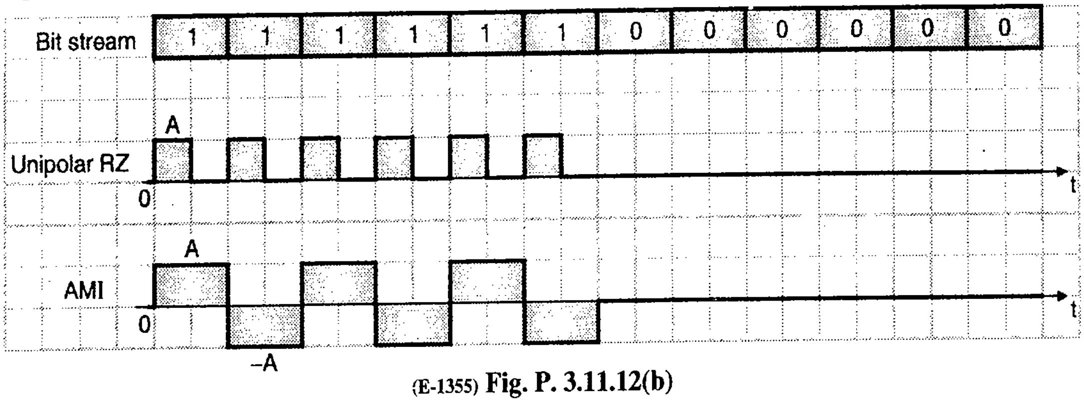
Soln. : The required waveforms are as shown in Fig. P. 3.11.12.

Ex. 3.11.13 : Draw the line code formats and PSD waveform for 11110000 and comment on power spectral density. **May 13, 8 Marks**

1. RZ polar
2. NRZ polar
3. AMI
4. Manchester.

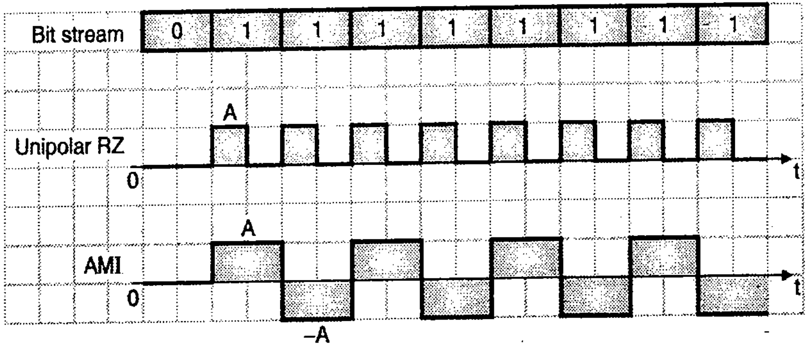
Soln. : Refer sections 3.11.1, 3.11.2, 3.11.3 and 3.11.4 for PSD.

2. A long sequence of 1's followed by along sequence of 0's :

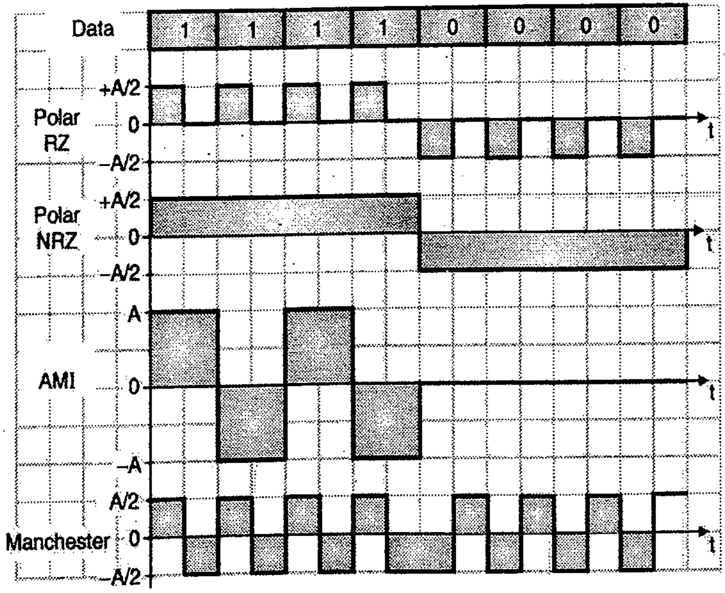


(E-1355) Fig. P. 3.11.12(b)

3. A single 0 and long sequence of 1's :



(E-1356) Fig. P. 3.11.12(c)



(E-1361) Fig. P. 3.11.13

3.12 Synchronization :

SPPU : May 06, Dec. 08, May 09, May 12, May 16

University Questions

Q. 1 Explain the need of a synchroniser in digital multiplexing. Draw and explain frame synchroniser. **(May 06, 10 Marks)**

Q. 2 What is a synchronizer ? Explain any one type of bit synchronizer. **(Dec. 08, 6 Marks, May 09, 8 Marks)**

Q. 3 Why synchronization is necessary in digital communication ? Explain bit and frame synchronization using suitable sketch. **(May 12, 8 Marks)**

Q. 4 What is equalizer ? Explain Adaptive equalizers. **(May 16, 7 Marks)**

- In the digital communication system, synchronization plays a very important role.
- At the receiver we need to achieve the frame, bit and carrier synchronization to recover the original digital data signal.
- When the signal travels from transmitter to receiver over a communication channel, then due to imperfections of the channel, the digital pulses tend to spread out.
- The resulting interference is known as intersymbol interference (ISI).
- Equalizers are the circuits which are used to counteract the intersymbol interference.

3.13 Frame and Bit Synchronization :

SPPU : Dec. 07, May 08, Dec. 08, May 09, Dec. 10, May 12, Dec. 12

University Questions

Q. 1 What are the different synchronization techniques ? Draw the block diagram of synchronization in Binary receiver and explain closed loop bit synchronization technique. **(Dec. 07, 8 Marks)**

Q. 2 Explain the need of a synchronizer in digital multiplexing. Draw and explain frame synchroniser. **(May 08, 8 Marks)**

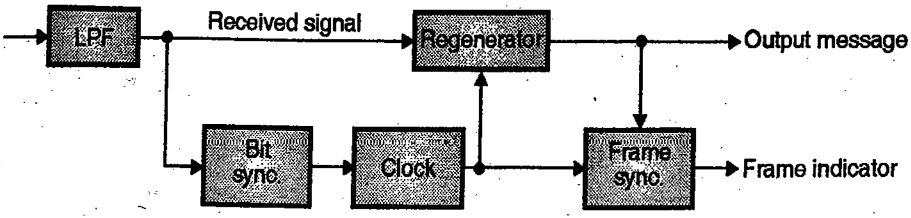
Q. 3 What is a synchronizer ? Explain any one type of bit synchronizer. **(Dec. 08, 6 Marks, May 09, 8 Marks)**

Q. 4 What is bit synchronization ? Explain closed loop bit synchronizer. **(Dec. 10, 8 Marks)**

Q. 5 Why synchronization is necessary in digital communication ? Explain bit and frame synchronization using suitable sketch. **(May 12, 8 Marks)**

Q. 6 What is a synchronizer ? Explain any one type of bit synchronizer and need for frame synchronization with relevant diagram. **(Dec. 12, 8 Marks)**

- Synchronization is a technique which can make the clocks at the transmitter and receiver, operate at the same rate. The clocks in a digital communication system should be operated at same rate in order to have synchronization. The synchronization is of following types :
  1. Symbol synchronization
  2. Frame synchronization
  3. Carrier synchronization.
- In this section we are going to discuss only about the symbol and frame synchronization.
- The positions of the symbol and bit synchronizers in a binary receiver are shown in Fig. 3.13.1.
- The bit synchronizer will extract synchronization from the received signal itself, while the frame synchronizer uses the output message of the regenerator and clock to derive the framing information.



(E-744) **Fig. 3.13.1 : Binary receiver with bit and frame synchronizers**

3.13.1 Closed - Loop Bit Synchronization :

SPPU : Dec. 07, Dec. 10, Dec. 12

University Questions

Q. 1 What are the different synchronization techniques ? Draw the block diagram of synchronization in Binary receiver and explain closed loop bit synchronization technique. **(Dec. 07, 8 Marks)**

Q. 2 What is bit synchronization ? Explain closed loop bit synchronizer. **(Dec. 10, 8 Marks)**

Q. 3 What is a synchronizer ? Explain any one type of bit synchronizer and need for frame synchronization with relevant diagram. **(Dec. 12, 8 Marks)**

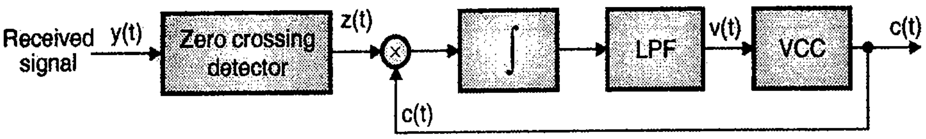
- A closed-loop bit synchronizer using a voltage controlled clock is shown in Fig. 3.13.2(a). This scheme provides a very reliable synchronization. The waveforms for the closed loop bit synchronizer is as shown in Fig. 3.13.2(b).

Operation of bit synchronizer :

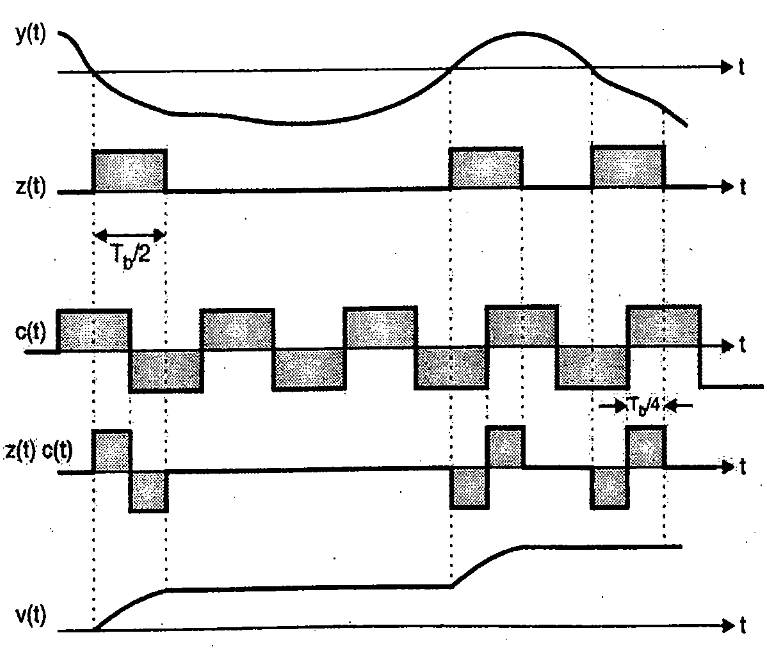
- The received signal y (t) is applied at the input of a zero crossing detector. The zero crossing detector (ZCD) produces positive pulses of duration T_b / 2 corresponding to every zero crossing instant of received signal y (t) .
- The ZCD output z (t) is then multiplied with c (t) which is a square wave clock coming from a voltage controlled clock (VCC).
- The control voltage v (t) is obtained by integrating and low pass filtering the product z (t) c (t). This control voltage is used for adjusting the clock frequency c (t) .
- The loop reaches a steady state condition when the edges of c (t) and z (t) are synchronized and are shifted by T_b / 4 because then the product waveform c (t) z (t) will have a zero area under it and v (t) remains constant.
- While implementing this system practically, the analog multiplier and integrators are replaced by equivalent digital components.

Disadvantages :

1. This circuit will suffer from timing jitter if the zero crossings of y (t) are not spaced at the instant which are integer multiples of T_b.
2. The synchronization will be lost if the message y (t) consists of long strings of 1s and 0s, because then y (t) does not have zero crossing frequent enough.

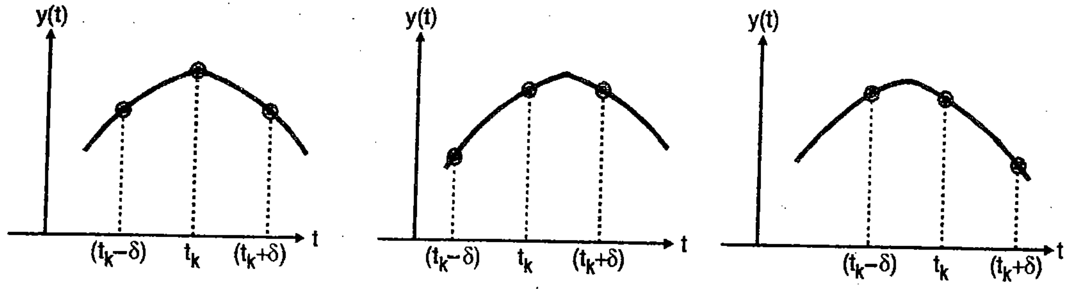


(a) Block diagram of closed loop bit synchronizer



(b) Waveforms of closed loop bit synchronizer

(E-745) **Fig. 3.13.2**

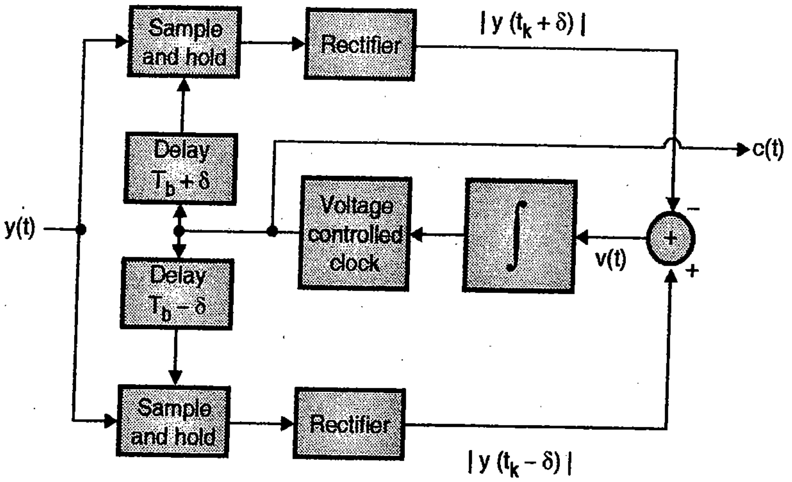


(a) Sync signal exactly at optimum sampling instant

(b) Early sync signal

(c) Late sync signal

(E-746) Fig. 3.13.3 : Principle of early-late synchronizer



(E-747) Fig. 3.13.4 : Early-late bit synchronizer

3.13.2 Early - Late Synchronizer :

SPPU : Dec. 05, May 07, May 11, Dec. 12, Dec. 14

University Questions

Q. 1 Describe Early - late bit synchronizer for polar NRZ signaling with the help of neat diagram. **(Dec. 05, 10 Marks)**

Q. 2 Explain early late synchronizer. **(May 07, 8 Marks)**

Q. 3 Describe Early-late Synchronizer for polar NRZ signaling with the help of neat diagram. **(May 11, 6 Marks)**

Q. 4 What is a synchronizer ? Explain any one type of bit synchronizer and need for frame synchronization with relevant diagram. **(Dec. 12, 8 Marks)**

Q. 5 With the help of neat schematic, explain early-late synchronizer. **(Dec. 14, 6 Marks)**

- The second disadvantage of closed-loop bit synchronizer forces us to search for another technique of synchronization which is completely independent of zero crossings of the message signal y (t). The synchronizer which uses this technique is called as early-late synchronizer.

Principle of early-late synchronizer :

- The early-late synchronizer operates on the principle that a properly filtered digital signal has peaks at the optimum sampling times and that the filtered signal is reasonably symmetric on either sides as shown in Fig. 3.13.3(a).
- In Fig. 3.13.3(a) t_k is the optimum sampling instant at which the signal amplitude is maximum. $|y(t_k)|$ is the sampled value at the optimum sampling instant. Due to the symmetry of y (t) on either sides of t_k we can write that,

$$|y(t_k-\delta)| \approx |y(t_k+\delta)| \qquad ...(3.13.1)$$

and $|y(t_k-\delta)|$ and $y|(t_k+\delta)|$ both are smaller than $|y(t_k)|$.

where t_k = Optimum sampling instant and $\delta < (T_b/2)$

- Fig. 3.13.3(b) describes the situation of early sampling. As shown in Fig. 3.13.3(b) if the sampling is done earlier than the optimum sampling instant t_k then,

$$|(y(t_k+\delta)| > |y(t_k-\delta)| \qquad ...(3.13.2)$$

- The situation of late sampling is described in Fig. 3.13.3(c). As shown in Fig. 3.13.3(c), the sampling is done later than the optimum sampling instant t_k, then

$$|(y(t_k+\delta)| > |y(t_k-\delta)| \qquad ...(3.13.3)$$

Early-late synchronizer block diagram :

- The block diagram of the early-late synchronizer is shown in Fig. 3.13.4. This synchronizer uses the principle discussed earlier to generate the control voltage for the voltage controlled clock (VCC) in a feedback loop.

Operation of the synchronizer :

- From the discussion till now we have understood that,
- For early sampling : $|(y(t_k - \delta)| > |y(t_k + \delta)|$ and
- For late sampling : $|(y(t_k - \delta)| > |y(t_k + \delta)|$
- In Fig. 3.13.4 the incoming signal y(t) is applied to two sample and hold circuits. The sampling signals for these S and H circuits are obtained from two delay producing blocks which are driven by the voltage controlled clock (VCC).
- The input signal y (t) is sampled early at $(t_k - \delta)$ by the lower sample and hold circuit and late at $(t_k + \delta)$ by the upper sample and hold circuit.
- At the outputs of the S and H blocks we obtain the sampled values of y (t). These sampled values are rectified and subtracted from each other to produce the correcting voltage v (t).
- When the sampling takes place exactly at the optimum sampling instant as shown in Fig. 3.13.3(a), the values of $|y(t_k - \delta)|$ and $|y(t_k + \delta)|$ will be exactly equal.

$$\therefore |y(t_k - \delta)| - |y(t_k + \delta)| = 0 \text{ i.e. } v(t) = 0$$

i.e. the correcting voltage v (t) = 0. Hence the output of the voltage controlled clock (VCC) will remain constant (unchanged). The bit synchronization is then said to have been achieved.

- If the sync signal appears early then,

$$v(t) = |y(t_k - \delta)| - |y(t_k + \delta)| < 0 \qquad ...(3.13.4)$$

This will decrease the net output voltage of the integrator which will slow down the voltage controlled clock therefore the synchronization will be achieved.

- An exactly opposite process will take place if the sync signal appears late, and VCC will speed up to achieve synchronization.

3.13.3 Frame Synchronization :

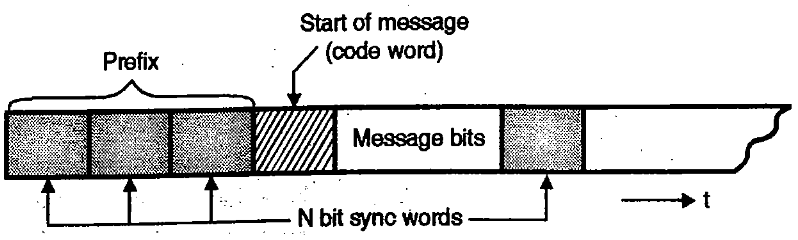
SPPU : May 06, May 08, Dec. 12

University Questions

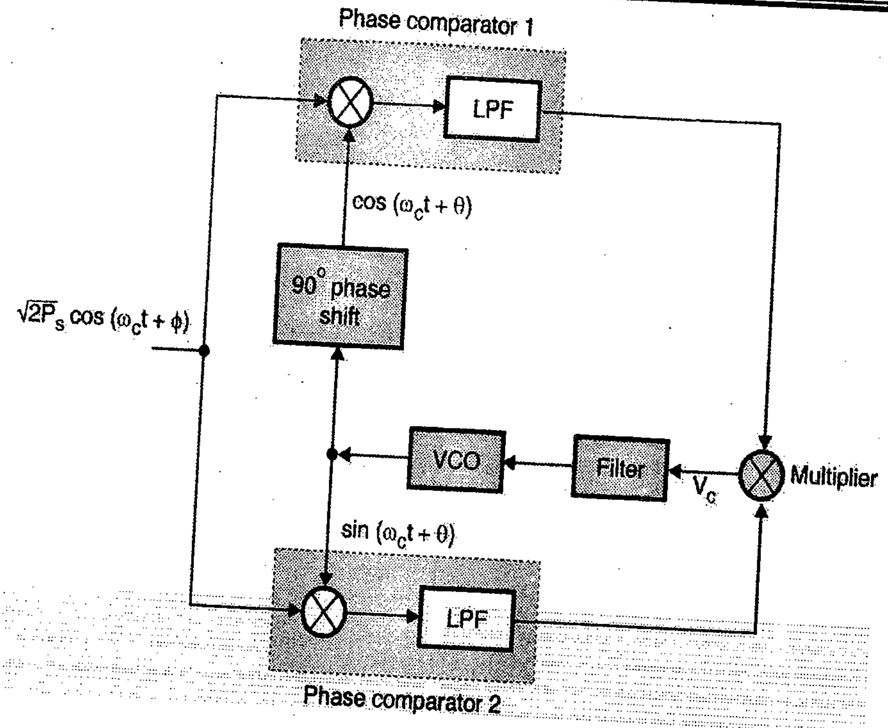
Q. 1 Explain the need of a synchronizer in digital multiplexing. Draw and explain frame synchroniser. **(May 06, 10 Marks, May 08, 8 Marks)**

Q. 2 What is a synchronizer ? Explain any one type of bit synchronizer and need for frame synchronization with relevant diagram. **(Dec. 12, 8 Marks)**

- A digital receiver is supposed to make a decision of whether a logic 1 or logic 0 is received.
- In order to make such a decision and produce an appropriate output voltage it must know whether the voltage at its input corresponds to the received signal or noise.
- Otherwise it will produce random output bits corresponding to the input noise. To avoid this, it is necessary to identify the beginning of a message.
- Another important thing is to identify the subdivisions or frames within the message. If a message which contains all this information is transmitted and received by the receiver, then we can say that "frame synchronization" has been done.
- To facilitate the frame synchronization, the binary transmission usually includes special N-bit sync words as shown in Fig. 3.13.5.
- The initial "prefix" consists of several repetitions of the sync word, in order to indicate the beginning of transmission and allows time for bit-sync acquisition.
- After prefix follows a different codeword which indicates the start of the message itself. Frames are labelled by inserting sync words periodically in the bit stream, as shown in Fig. 3.13.5. We have already discussed frame synchronization for PAM-TDM system and for the PCM-TDM system i.e. T_1 system.



(E-748) **Fig. 3.13.5 : Frame synchronization**



(E-749) **Fig. 3.13.6 : Block diagram of Costas loop**

3.13.4 Costas Loop for Carrier Synchronization :

SPPU : Dec. 06, May 08, Dec. 12

University Questions

Q. 1 Explain the operation of costas loop synchronization to recover the carrier. **(Dec. 06, 8 Marks)**

Q. 2 Explain the need of a synchronizer in digital multiplexing. Draw and explain frame synchroniser. **(May 08, 8 Marks)**

Q. 3 What is a synchronizer ? Explain any one type of bit synchronizer and need for frame synchronization with relevant diagram. **(Dec. 12, 8 Marks)**

The Costas loop is used for carrier synchronization in the BPSK receiver, where BPSK is the short form of binary phase shift keying. Fig. 3.13.6 shows the block diagram of Costas loop.

Operation :

- Fig. 3.13.6 shows that the Costas loop consists of two phase comparators and a VCO alongwith a 90° phase shifter.
- The VCO generates a carrier frequency f_c with an arbitrary phase angle θ. This carrier signal is applied to the phase comparator 1 (PC1) through a 90° phase shifting network whereas it is applied directly to PC2. Both the phase comparators receive the BPSK input signal i.e. $\sqrt{2P_s}\cos\omega_c t$.
- The output of multiplier in phase comparator 1 is given by,

$$\text{Output of multiplier } 1 = \sqrt{2\,P_s}\cos\omega_c t\cos(\omega_c t+\theta)$$

$$= \frac{1}{2}\times\sqrt{2P_s}\;[\cos(2\,\omega_c t+\theta+\phi)+\cos(\phi-\theta)] \quad ...(3.13.5)$$

- The low pass filter in the phase comparator 1 will eliminate the second harmonic term in Equation (3.13.5) to yield,

$$\text{Output of PC1} \;=\; \frac{1}{2}\times\sqrt{2P_s}\cdot\cos(\phi-\theta) \quad ...(3.13.6)$$

- Similarly the output of the phase comparator 2 is given by :

$$\text{Output of PC2} \;=\; \frac{1}{2}\times\sqrt{2\,P_s}\sin(\phi-\theta) \quad ...(3.13.7)$$

- The outputs of the two phase comparator are applied to a multiplier which produces the control voltage V_c. The control voltage is given by :

$$V_c \;=\frac{1}{4}2\,P_s\sin(\phi-\theta)\cos(\phi-\theta)$$

$$= \frac{P_s}{2}\sin(\phi-\theta)\cos(\phi-\theta)$$

$$\therefore\; V_c \;= \frac{P_s}{4}\sin 2(\phi-\theta) \quad ...(3.13.8)$$

- Thus the control voltage V_c is proportional to the phase difference $(\phi-\theta)$ between the received carrier and the carrier generated by VCO.

- The control voltage is then filtered and used for correcting the frequency of VCO. Thus the Costas loop can be used for the purpose of carrier synchronization.

3.13.5 Carrier Synchronization Using PLL :

SPPU : Dec. 12

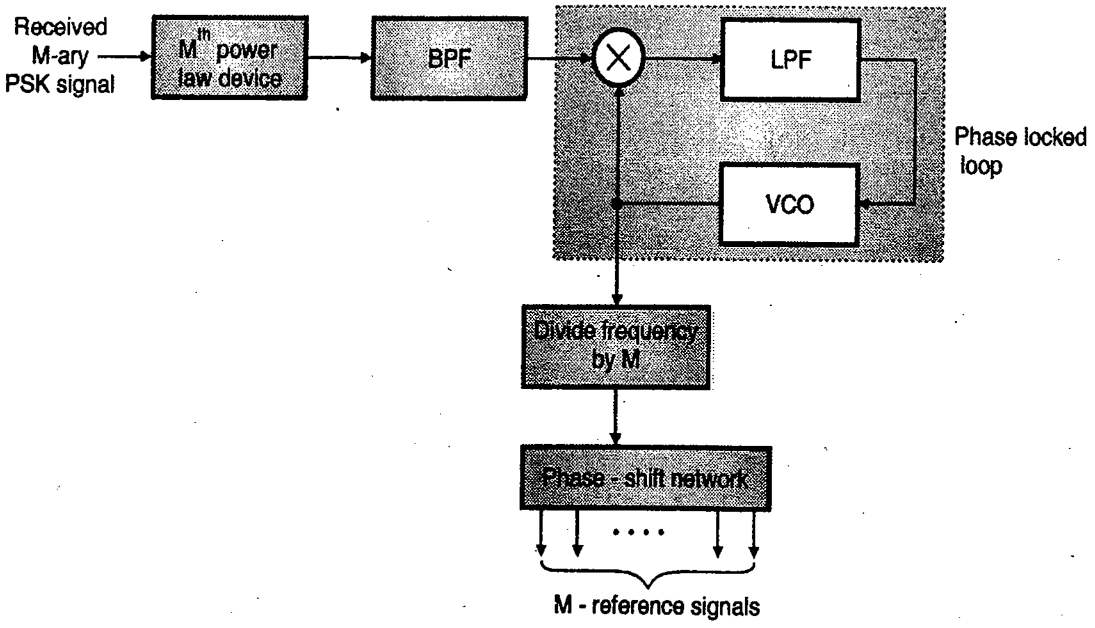
University Questions

Q. 1 What is a synchronizer ? Explain any one type of bit synchronizer and need for frame synchronization with relevant diagram. **(Dec. 12, 8 Marks)**

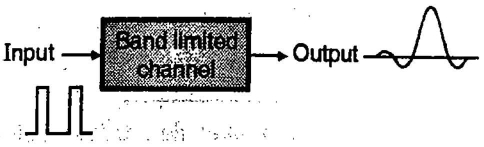
- When coherent detection is used, it is necessary to know about frequency and phase of the carrier.
- The estimation of carrier frequency and phase at the receiver is called as **carrier recovery** or **carrier synchronization.**
- In order to perform the demodulation, the receiver is supposed to know the time instants at which the modulation changes its state.
- These instants can be estimated by using the **clock recovery or symbol synchronization.**
- Synchronization is not required for the noncoherent systems.
- In practice, we can use the phase locked loop (PLL) for the purpose of carrier synchronization.
- Fig. 3.13.7 shows the block diagram of carrier recovery circuit for the M-ary PSK system. A similar circuit is used in the receivers of binary PSK, QPSK and M-ary PSK systems.
- This circuit is called as the M^{th} power loop and for BPSK where M = 2, it is called as a squaring loop.
- The disadvantage of this method, is that we encounter a phase ambiguity problem, as discussed in BPSK receiver.
- Another method of carrier synchronization is the use of Costa's loop.

3.14 Pulse Transmission through a Bandlimited Channel :

- In all the digital systems, the pulses are transmitted from the transmitter to receiver.
- But the channels carrying these pulses never have enough bandwidth required to ensure the preservation of the shape of the pulse.
- Such channels are called as the **band limited** channels. Fig. 3.14.1(a) shows the output of a bandlimited channel when a narrow pulse is applied at its input.

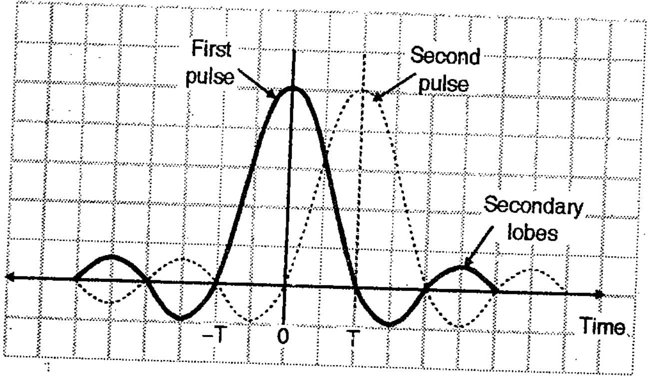


(E-750) **Fig. 3.13.7 : M^{th} power loop**



(E-295) **Fig. 3.14.1(a) : Band limited channel**

- Fig. 3.14.1(b) shows that the bandlimited channel tends to spread the narrow pulses.
- This produces the secondary lobes as shown in Fig. 3.14.1(b). The secondary lobes are called as **ringing tails.**
- Fig. 3.14.1(b) shows the energy distribution in the output of a bandlimited channel. It shows that almost 90% energy is contained in the main lobe.



(E-296) **Fig. 3.14.1(b) : Output of the channel**

- So we can recover almost all the information even if we allow the main lobe to pass through. That means even if the bandwidth is equal to $\frac{1}{T}$.

$$\therefore \quad B = \frac{1}{T}$$

- The corresponding maximum bit rate that is supported by this channel is,

$$R = 2B$$

3.15 Intersymbol Interference (ISI) :

SPPU : Dec. 10, May 12, Dec. 12, May 13, Dec. 13, Dec. 16

University Questions

Q. 1 What is Inter Symbol Interference (ISI) ? Explain the ideal solution to control ISI. **(Dec. 10, 8 Marks)**

Q. 2 What is intersymbol interference ? Explain its causes and remedies to avoid it. **(May 12, 8 Marks)**

Q. 3 Explain Inter Symbol Interference. With the help of baseband binary data transmission system derive expression of ISI. **(Dec. 12, 10 Marks)**

Q. 4 What is ISI ? Hence explain the methods to eliminate the same. **(May 13, 8 Marks)**

Q. 5 Explain Inter symbol interference (ISI) with help of block diagram of a binary baseband transmission system. Also explain Nyquist solution used for curing ISI. **(Dec. 13, 8 Marks)**

Q. 6 Explain inter symbol interference. Explain its causes and remedies to avoid it. **(Dec. 16, 6 Marks)**

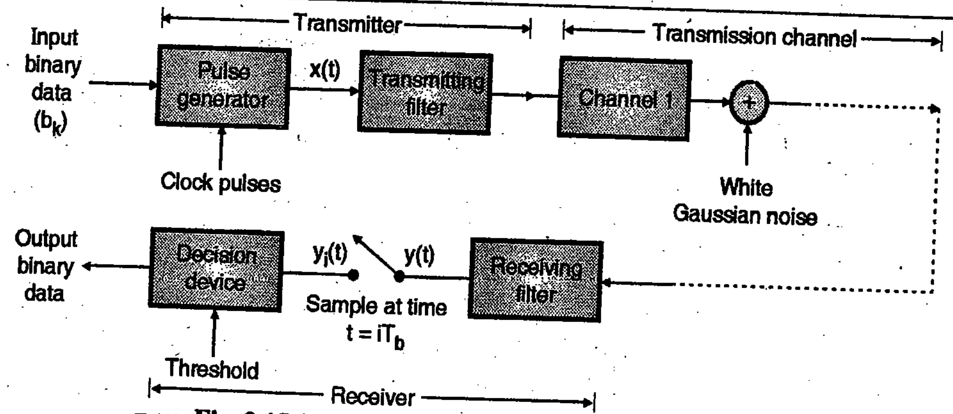
- In a communication system when the data is being transmitted in the form of pulses (bits), the output produced at the receiver due to the other bits or symbols interferes with the output produced by the desired bit.
- This is called as intersymbol interference (ISI). The intersymbol interference will introduce errors in the detected signal at the receiver.
- Consider Fig. 3.15.1 which shows the baseband binary PAM system. The input signal consists of a binary data sequence $\{b_k\}$ with a bit duration of T_b seconds.
- This sequence is applied to a pulse generator which produces a discrete PAM signal (line code) given by :

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, v(t-kT_b) \quad ...(3.15.1)$$

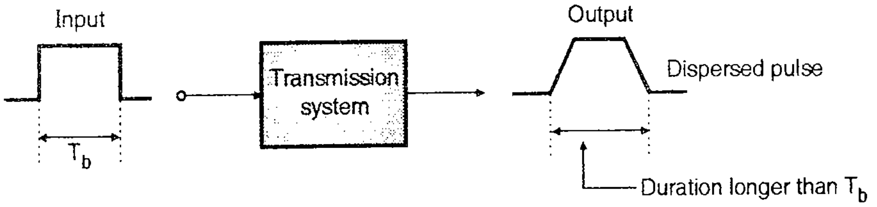
where v (t) denotes the basic pulse, normalized such that v (0) = 1. The first block of the system i.e. pulse amplitude modulator converts this input sequence into polar form as follows :

if $b_k = 1$ then $a_k = 1$
and $b_k = 0$ then $a_k = -1$

- The PAM signal x (t) is then passed through a transmitting filter. The output of the transmitting filter is then transmitted over transmission channel. The impulse response of the channel is h (t).
- A random noise is then added to the transmitted signal when it travels over the transmission channel. Thus the signal received at the receiving end is contaminated with noise.



(E-297) **Fig. 3.15.1 : Baseband binary data transmission system**



(E-298) **Fig. 3.15.2 : Cause of ISI**

- The channel output is applied to a receiving filter. This filter output is sampled synchronously with the transmitter. The sampling instants are determined by a clock or timing signal which is extracted from the receiving filter output.
- A sequence of samples is obtained at the output of receiving filter which is used to reconstruct the original data sequence with the help of a decision making device.
- Each sample is compared with a predetermined threshold level in the decision making device. If the amplitude of the sample is higher than the threshold level then it is decided that a symbol "1" is received. On the other hand if the signal has an amplitude lower than the threshold, then the decision is that a "0" is received.
- The receiving filter output can be written as,

$$y(t) = \mu \sum_{k=-\infty}^{\infty} a_k \, p(t - k T_b) + n(t) \qquad ...(3.15.2)$$

where μ is a scaling factor and the noise n (t) is the noise produced at the output of the receiving filter due to the channel added noise. The term p (t – k T_b) represents the combined impulse response of the receiving filter.

- The receiving filter output y (t) is sampled at the time instant $t_i = i\,T_b$ with i = 0,1,2 ……. This results in the sampled version of y(t) as follows :

$$y(t_i) = \mu \sum_{k=-\infty}^{\infty} a_k \, p(i T_b - k T_b) + n(t_i) \qquad ...(3.15.3)$$

$$\therefore\ y(t_i) = \mu a_i + \mu \sum_{\substack{k=-\infty \\ k \neq i}}^{\infty} a_k \, p(i T_b - k T_b) + n(t_i) \qquad ...(3.15.4)$$

- This is the receiver output y (t) at instant $t = t_i$.

Equation (3.15.4) has two terms :

1. The first term "$\mu\, a_i$" is produced by the i^{th} transmitted bit. Theoretically, only this term should be present at the receiver output but practically it is not so.
2. The second term represents the collective residual effect of all the transmitted bits, corresponding to the sampling time instant $t = t_i$. This residual effect is known as the "intersymbol interference". (ISI).

- The ISI results because the overall frequency response of the system is never perfect and pulse spreading is bound to take place.
- When a short pulse of duration T_b seconds is transmitted through a bandlimited transmission system, then various frequency components present in the input pulse are **differentially attenuated** and more importantly **differentially delayed** by the system.
- Due to this the pulse appearing at the output of the system will be "dispersed" over an interval which is longer than "T_b" seconds as shown in Fig. 3.15.2.
- Due to this dispersion, the adjacent symbols will interfere with each in time domain other when transmitted over the communication channel. This will result in the intersymbol interference (ISI).
- The transmitted pulse of duration T_b seconds and the dispersed pulse of duration more than T_b seconds are shown in Fig. 3.15.2.

3.15.1 Effect of ISI :

- If the ISI and noise are absent totally then, the transmitted bit can be decoded correctly at the receiver. However errors will be introduced due to presence of ISI at the receiver output.
- Due to this the receiver can make an error in deciding whether it has received a logic 1 or a logic 0.
- Another effect of ISI is the cross talk which may take place due to overlapping of the adjacent pulses due to spreading.
- It is necessary to use the special filters called equalizers in order to reduce ISI and its effect.

3.15.2 Causes of ISI :

SPPU : May 12, Dec. 16

University Questions

Q. 1 What is intersymbol interference ? Explain its causes and remedies to avoid it.

(May 12, 8 Marks)

Q. 2 Explain inter symbol interference. Explain its causes and remedies to avoid it.

(Dec. 16, 6 Marks)

- The four important causes for ISI are as follows :

1. Timing inaccuracies
2. Insufficient bandwidth
3. Amplitude distortion
4. Phase distortion.

1. **Timing inaccuracies :**

   The ISI will take place if the transmitter rate of transmission is not same as the ringing frequency of the given channel.

2. **Insufficient bandwidth :**

- If the transmission rate is less than the channel bandwidth then there is a very small possibility of timing error.
- But if the channel bandwidth is reduced, then the possibility of timing error will increase and the possibility of ISI also will increase.

3. **Amplitude distortion :**

- Generally filters are used in the communication systems in order to bandlimit the signals and reduce the noise.
- But the frequency response of the communication channels can not be accurately predicted.
- When the frequency characteristics of a communication channel differs from the expected one, the pulse distortion is likely to take place.
- The pulse distortion results in reduction of the peaks of the pulses i.e. amplitude distortion. In order to compensate for this, we have to use the amplitude equalization.

4. **Phase distortion :**

- If various frequency components in the input pulse undergo different amounts of time delay while travelling through the channel, then the phase distortion is bound to take place.
- This will cause the ISI. Special delay equalizers are required to be used to reduce the phase distortion and the associated ISI.

3.15.3 Remedy to Reduce the ISI :

SPPU : Dec. 10. May 12. May 13. Dec. 16

University Questions

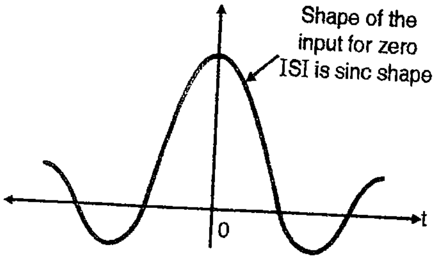
Q. 1 What is Inter Symbol Interference (ISI) ? Explain the ideal solution to control ISI. **(Dec. 10, 8 Marks)**

Q. 2 What is intersymbol interference ? Explain its causes and remedies to avoid it. **(May 12, 8 Marks)**

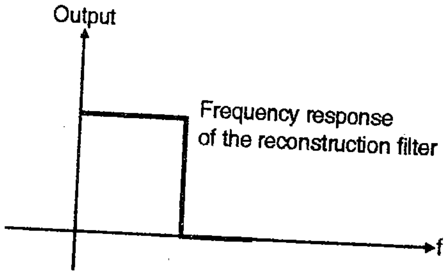
Q. 3 What is ISI ? Hence explain the methods to eliminate the same. **(May 13, 8 Marks)**

Q. 4 Explain inter symbol interference. Explain its causes and remedies to avoid it. **(Dec. 16, 6 Marks)**

- It has been proved that the function which produces a zero intersymbol interference is a "sinc function". Thus instead of a rectangular pulse if we transmit a sinc pulse then the ISI can be reduced to zero.
- Using the since pulse for transmission is known as "Nyquist Pulse Shaping". The sinc pulse transmitted to have a zero ISI is shown in Fig. 3.15.3(a).



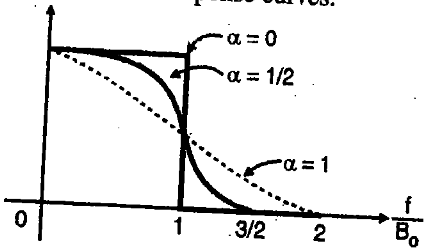
(a) Ideal pulse shape for zero ISI



(b) Frequency response of the filter

(E-299) Fig. 3.15.3

- We know that Fourier transform of a sinc pulse is a rectangular function. Therefore to preserve all the frequency components, the frequency response of the filter must be exactly flat in the pass band and zero in the attenuation band as shown in Fig. 3.15.3(b).
- This type of filter is practically not available. Therefore practically the frequency response of the filter is modified as shown in Fig. 3.15.4 with different roll off factors "α" to obtain the practically achievable filter response curves.



(E-300) Fig. 3.15.4 : Practical filter characteristics

3.16 Nyquist's Criterion for Distortionless Baseband Binary Transmission :

SPPU : Dec. 13

University Questions

Q. 1 Explain Inter symbol interference (ISI) with help of block diagram of a binary baseband transmission system. Also explain Nyquist solution used for curing ISI. **(Dec. 13, 8 Marks)**

In the previous section, we have seen that in absence of the ISI,

$$y(t_i) = \mu a_i \qquad ...(3.16.1)$$

- This expression shows that under these conditions, the ith transmitted bit can be decoded correctly.
- In order to minimize the effects of ISI, we have to design the transmitting and receiving filters properly.

- The transfer function of the channel and the shape of transmitted pulse are generally specified. So it becomes the first step towards design of filters.
- From this information we have to determine the transfer functions of the transmitting and receiving filters, to reconstruct the transmitted data sequence $\{b_k\}$.
- This is achieved by first "extracting" and then "decoding" the corresponding sequence of weights from the output y(t).

We have $$y(t) = \mu \sum_{k=-\infty}^{\infty} a_k\, p(t - k T_b) \quad ...(3.16.2)$$

This shows that output y(t) is dependent on a_k, the received pulse p(t) and the scaling factor μ.

- Extraction : Extraction is basically the process of sampling. The signal y(t) is sampled at instants $t = i\,T_b$, where i is an integer.
- Decoding : The decoding should be such that, the contribution of the weighted pulse i.e. $a_k\, p(i\,T_b - k\,T_b)$ for i = k be free from ISI. This can be stated mathematically as,

$$p(i\,T_b - k\,T_b) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \quad ...(3.16.3)$$

where p(0) = 1 due to normalizing.

- This is the condition for zero ISI.
- If p(t) i.e. received pulse satisfies the above equation, then the receiver output given by Equation (3.16.2) reduces to

$$y(t_i) = \mu\, a_i \quad ...(3.16.4)$$

which indicates zero ISI in the absence of noise.

Frequency domain representation :

- To obtain the transfer function, we have to transform the condition stated in Equation (3.16.3) into frequency domain.
- Consider that the sequence of samples is represented by $p(n\,T_b)$ where $n = 0, \pm 1, \pm 2$ .... after the process of extraction (sampling).
- The Fourier transform of such a sequence of samples can be obtained, as we did for the sampling process of a low pass signal.
- So the frequency domain representation of $p(n\,T_b)$ is as follows :

$$P_\delta(f) = F[p(n T_b)]$$

$$= R_b \sum_{n=-\infty}^{\infty} P(f - n R_b) \quad ...(3.16.5)$$

where $R_b = 1/T_b$ i.e. bit rate.

- But P_δ (f) also represents the Fourier transform of the sampled version of p (t). Hence $P_\delta(f)$ can be written as

$$P_\delta(f) = \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} [p(m T_b)\, \delta(t - m T_b)]\, e^{-j2\pi f t}\, dt \quad ...(3.16.6)$$

where $\sum_{m=-\infty}^{\infty} p(m T_b)\, \delta(t - m T_b)$ represents the sampled version of p (t).

- Let m = i − k. So if i = k then m = 0 and if $i \neq k$ then $m \neq 0$.
- So for m = 0 let us apply condition of Equation (3.16.3) to Equation (3.16.6) to get,

$$P_\delta(f) = \int_{-\infty}^{\infty} p(0)\, \delta(t)\, e^{-j2\pi f t}\, dt$$

Using the shifting property of delta function

$$\therefore \quad P_\delta(f) = p(0) \quad ...(3.16.7)$$

- But p(0) = 1 due to normalization, hence $P_\delta(f) = 1$.
- Substituting this into Equation (3.16.5) we get,

$$1 = R_b \sum_{n=-\infty}^{\infty} P(f - n R_b)$$

$$\therefore \quad \sum_{n=-\infty}^{\infty} P(f - n R_b) = 1/R_b = T_b \quad ...(3.16.8)$$

- This expression is called as the Nyquist criterion for distortionless baseband transmission in absence of noise.

3.16.1 Ideal Solution :

- The L.H.S. of Equation (3.16.8) represents a series of shifted spectrums.
- For n = 0, the LHS is equal to P(f) and it represents a frequency function with the narrowest band which satisfies Equation (3.16.8). The range of frequencies over which P (f) extends is from $-B_0$ to B_0 where B_0 corresponds to half the bit rate.

$$\therefore \quad B_0 = R_b/2 \quad ...(3.16.9)$$

- So P(f) can be specified as

$$P(f) = \frac{1}{2 B_0} \text{rect}\left(\frac{f}{2 B_0}\right) \quad ...(3.16.10)$$

and it is shown graphically in Fig. 3.16.1(a). This shows that P (f) is a rectangular function in frequency domain.

- This is the spectrum of a signal which produces zero ISI. So the signal in the time domain that produces zero ISI can be obtained by taking the IFT of P (f).

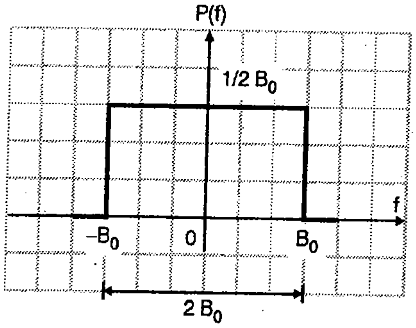
$$\therefore \quad p(t) = F^{-1}[P(f)]$$

$$= F^{-1}\left[\frac{1}{2 B_0} \text{rect}\left(\frac{f}{2 B_0}\right)\right]$$

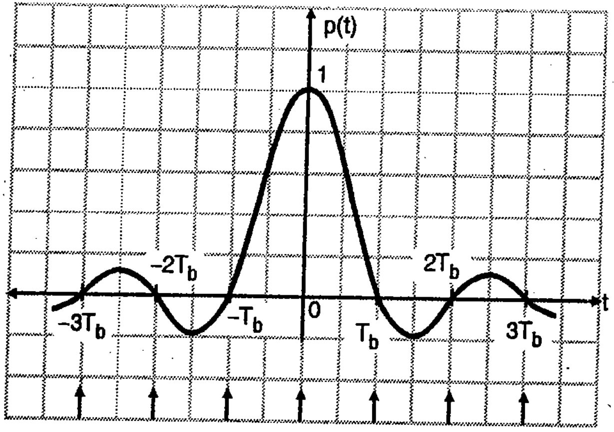
$$\therefore \quad p(t) = \text{sinc}(2 B_0 t) \quad ...(3.16.11)$$

This is a sinc signal.

- Fig. 3.16.1(b) shows the plot of this function. This function p(t) can be considered as the impulse response of an ideal low pass filter with bandwidth B_0.
- Thus the shape of a pulse should be a sinc pulse and not a rectangle in order to eliminate the ISI.



(a) Graphical representation of P(f)



Sampling instants

(b)

(E-301) **Fig. 3.16.1**

Advantages of using the sinc pulse :

1. Bandwidth requirement (of the channel) is reduced.
2. ISI is reduced to zero.

Possible Difficulties :

1. It is necessary that the amplitude characteristics of P(f) should be flat from $-B_0$ to B_0 and zero outside this band as shown in Fig. 3.16.1(a). But practically it is difficult or impossible to obtain such a sharp spectrum.
2. Due to sharp changes in P(f) at $\pm B_0$ there is practically no margin of error in sampling instants at the receiver.

3.16.2 Raised Cosine Spectrum :

- The two difficulties experienced by the ideal Nyquist channel can be overcome by increasing the bandwidth from its maximum value $B_0 = R_b/2$ to an adjustable value between B_0 and $2\ B_0$, in other words by allowing the edges of the rectangular frequency spectrum to be less abrupt.
- A condition is put on the overall frequency response P (f) to satisfy the given condition. As per Equation (3.16.8) –

$$\sum_{n=-\infty}^{\infty} P(f-NR_b) = T_b = \frac{1}{R_b}$$

- Expanding the summation sign we get

$$\ldots P(f+R_b)+P(f)+P(f-R_b)+P(f-2R_b)=T_b$$

$$\text{But } B_0 = \frac{R_b}{2} \quad \therefore R_b = 2B_0$$

$$\therefore \ldots P(f+R_b)+P(f)+P(f-2B_0)+P(f-4B_0)+\ldots = \frac{1}{2B_0}$$

- Allow only the three terms on LHS which correspond to n = – 1, n = 0 and n = 1 and limit the frequency band of interest to $[-B_0, B_0]$ to get,

$$P(f+2B_0)+P(f)+P(f-2B_0) = \frac{1}{2B_0} \text{ and given } -B_0 \le f \le B_0 \qquad \ldots(3.16.12)$$

- It is possible to device many bandlimited functions which will satisfy Equation (3.16.12) one of them is called as the **raised cosine spectrum.**
- This spectrum consists of a flat portion and a roll off portion. The raised cosine spectrum is expressed mathematically as follows :

$$P(f) = \begin{cases} 1/2\ B_0 \text{ (Flat portion)} & \ldots 0 \le |f| < f_1 \\ \frac{1}{4B_0}\left\{1-\sin\left[\frac{\pi(|f|-B_0)}{2B_0-2f_1}\right]\right\} & \ldots f_1 \le |f| < 2B_0-f_1 \\ 0 & \ldots |f| \ge 2B-f_1 \end{cases}$$

$$\ldots(3.16.13)$$

- The relation between frequency parameter f_1 and the bandwidth B_0 are related as follows :

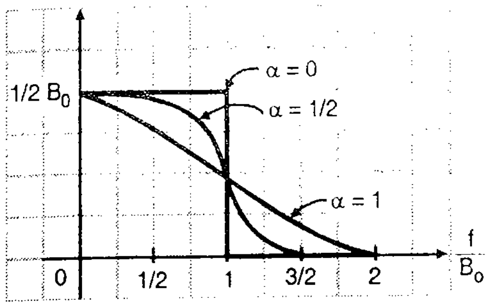
$$\alpha = 1-\frac{f_1}{W} \qquad \ldots(3.16.14)$$

where α is called as the roll off factor. It indicates the increase in bandwidth over the ideal value of bandwidth i.e. B_0.

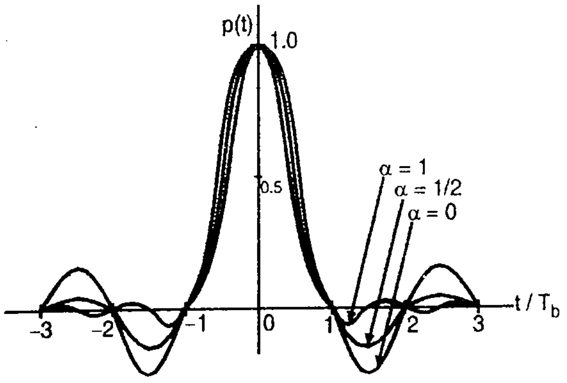
- The transmission bandwidth B_T is defined as follows :

$$B_T = 2B_0 - f_1$$
$$= B(1+\alpha) \qquad \ldots(3.16.15)$$

- The normalized frequency response of raised cosine function is obtained if we multiply P (f) with $2B_0$ and it is plotted in Fig. 3.16.2(a), for different values of α.
- The corresponding time domain signal p (t) is shown in Fig. 3.16.2(b).



(a) Frequency response



(b) Time response

(E-302) **Fig. 3.16.2 : Responses for different roll-off factors**

Observations :

1. For $\alpha = 0.5$ and 1 the characteristics of P(f) changes gradually with respect to frequency. Therefore it is easier to realize this characteristics practically.
2. The time response has the shape of a sinc function and all the sinc function pass through zero at $t = \pm T_b, \pm 2 T_b, ...$
3. The amplitude of side lobes increases with reduction in the value of α.
4. With $\alpha = 0$, the bandwidth requirement is maximum equal to $2 B_0$.

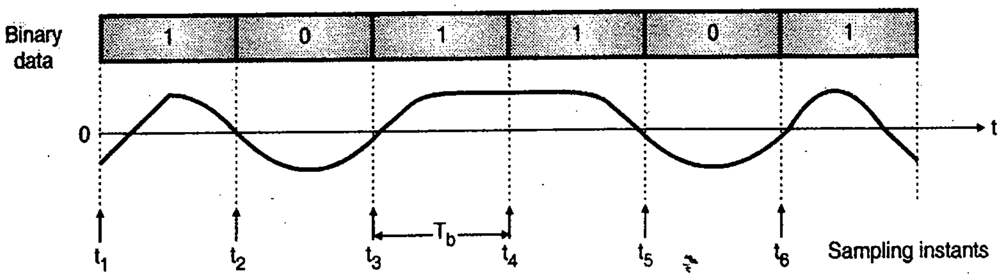
3.17 Eye Pattern : SPPU : May 11, Dec. 13

University Questions

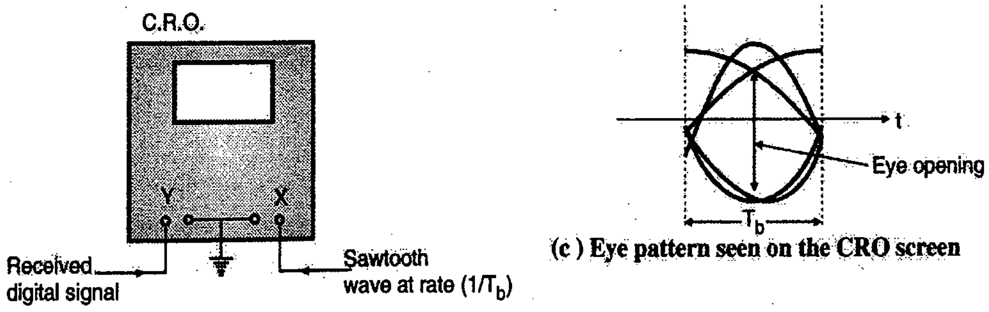
Q. 1 Explain eye diagram. **(May 11, 4 Marks)**

Q. 2 Explain the use of eye diagram to measure ISI. **(Dec. 13, 4 Marks)**

- Eye pattern is a pattern displayed on the screen of a cathode ray oscilloscope (C.R.O.). The shape of this pattern is very similar to the shape of human eye. Therefore it is called as **eye pattern**.
- Eye pattern is used for studying the intersymbol interference (ISI) and its effects on various communication systems.
- The eye pattern is obtained on the C.R.O. by applying the received signal to vertical deflection plates (Y-plates) of the C.R.O. and a sawtooth wave at the transmission symbol rate i.e. $(1 / T_b)$ to the horizontal deflection plates (X-plates) as shown in Fig. 3.17.1(c).
- The received digital signal and the corresponding oscilloscope display are as shown in Figs. 3.17.1(a) and (c) respectively.
- The resulting oscilloscope display shown in Fig. 3.17.1(c) is called as the "eye pattern". This is due to its resemblance to the human eye.



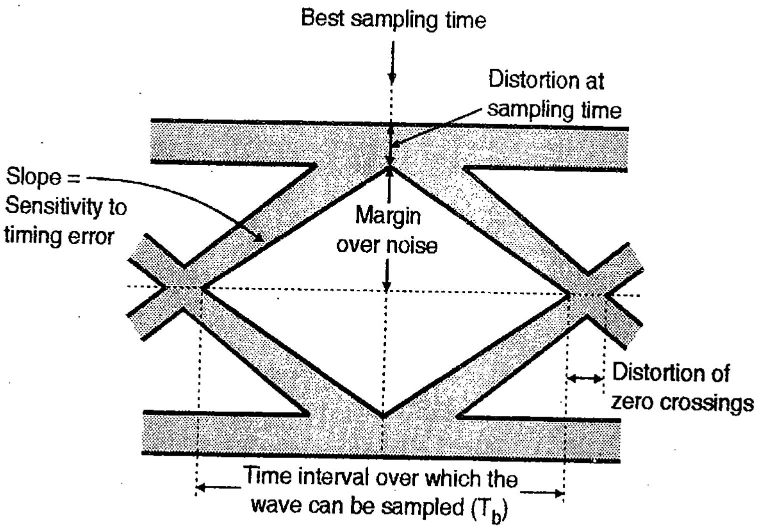
(a) Distorted binary wave



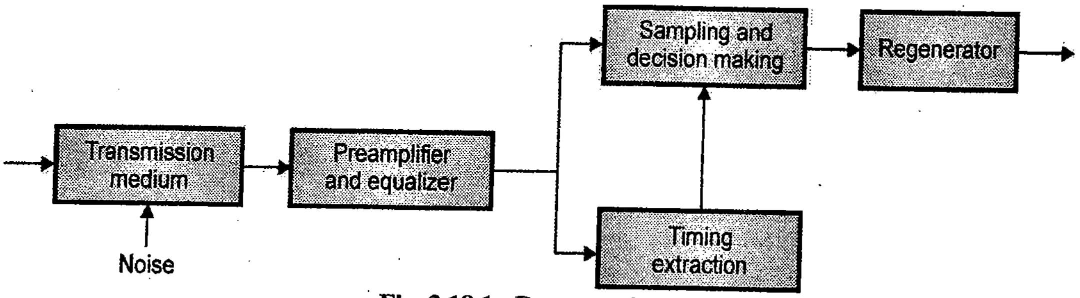
(c) Eye pattern seen on the CRO screen

(b) Oscilloscope connections

(E-353) **Fig. 3.17.1 : Obtaining eye pattern**



(E-354) **Fig. 3.17.2 : Interpretation of eye pattern**



(E-1040) **Fig. 3.18.1 : Regenerative repeater**

- The region inside the eye pattern is called as the eye opening.
- The eye pattern provides very important information about the performance of the system. The information obtainable is as follows (See Fig. 3.17.2).

Information obtained from eye pattern :

1. The width of the eye opening defines the time interval over which the received wave can be sampled, without an error due to ISI. The best instant of sampling is when the eye opening is maximum.
2. The sensitivity of the system to the timing error is determined by observing the rate at which the eye is closing as the sampling rate is varied.
3. The height of eye opening at a specified sampling time defines the margin over noise.
4. When the effect of ISI is severe, the eye is completely closed and it is impossible to avoid errors due to the combined effect of ISI and noise in the system.

3.18 Digital Receivers and Regenerative Repeaters :

- The functions of a receiver or regenerative repeater are as follows :
  1. To reshape the incoming pulses using an equalizer.
  2. To extract the timing information.
  3. To make the decision about the symbol detection.
- Fig. 3.18.1 shows a repeater which consists of a receiver and a regenerator.

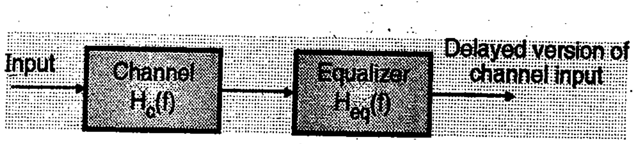
3.18.1 Equalization :

SPPU : May 16

University Questions

Q. 1 What is equalizer ? Explain Adaptive equalizers. **(May 16, 7 Marks)**

Whenever a signal is passed through a communication channel, distortion will be introduced. To compensate for the linear distortion, we can use a network called "equalizer" connected in cascade with the channel or system as shown in Fig. 3.18.2.



(E-797) **Fig. 3.18.2 : Block diagram of equalization**

Principle of Equalizer :

- The equalizer is designed in such a way that within the operating frequency band, the overall amplitude and phase responses of the cascade system shown in Fig. 3.18.2, are approximately equal to amplitude and phase responses for the distortionless transmission.
- Consider a communication channel with a transfer function $H_c (f)$. Let the transfer function of an equalizer be $H_{eq} (f)$. Then the overall transfer

function of the cascade connection is given by $H_c(f)\,H_{eq}(f)$. For the overall transmission through the cascaded connection of Fig. 3.18.2 to be distortionless, the overall transfer function should satisfy the following expression :

$$H_c(f)\,H_{eq}(f) = k\,e^{-j2\pi f t_0} \qquad ...(3.18.1)$$

where k = Scaling factor and t_0 = Constant time delay

- Therefore the transfer function of the equalizer is given by,

$$H_{eq}(f) = \frac{k\,e^{-j2\pi f t_0}}{H_c(f)} \qquad ...(3.18.2)$$

- Equation (3.18.2) shows the ideal value of the equalizer transfer function. Practically the equalizer transfer function should be as close as possible to the expression given in Equation (3.18.2).

Practical realization of an equalizer :

The equalizers can be practically realized using the structure of the "tapped-delay-line filter", discussed below.

3.18.2 Tapped Delay Line Filter :

- Consider a time invariant filter with an impulse response h (t). We assume that, $h(t) = 0$ for $t < 0$ i.e. the filter is causal.
- The impulse response of the filter is of finite duration i.e. $h(t) = 0$ for $t \geq T_f$.

1. We can express the filter output y (t) produced when input x (t) is applied to it as follows :

$$y(t) = \int_0^{T_f} h(t)\cdot x(t-\tau)\,d\tau \qquad ...(3.18.3)$$

2. Let the input x (t), impulse response h (t) and output y (t) be uniformly sampled, at a rate of $1/\Delta\tau$ samples per second.

$$\therefore \quad t = n\,\Delta\tau \qquad ...(3.18.4)$$

$$\text{and} \quad \tau = k\,\Delta\tau \qquad ...(3.18.5)$$

3. where n and k both are integers and $\Delta\tau$ is the sampling period.
4. If the value of $\Delta\tau$ is very small then the product $h(t)\,x(t-\tau)$ of Equation (3.18.3) will remain constant in the range $k\,\Delta\tau \leq \tau \leq (k+1)\,\Delta\tau$ for all values of k and t. Then we can approximate Equation (3.18.3) by the convolution sum as follows :

$$y(n\,\Delta\tau) = \sum_{k=0}^{N-1} h(k\,\Delta\tau)\cdot x(n\,\Delta\tau - k\,\Delta\tau)\,\Delta\tau \qquad ...(3.18.6)$$

$$\text{when, } N\,\Delta\tau = T_f$$

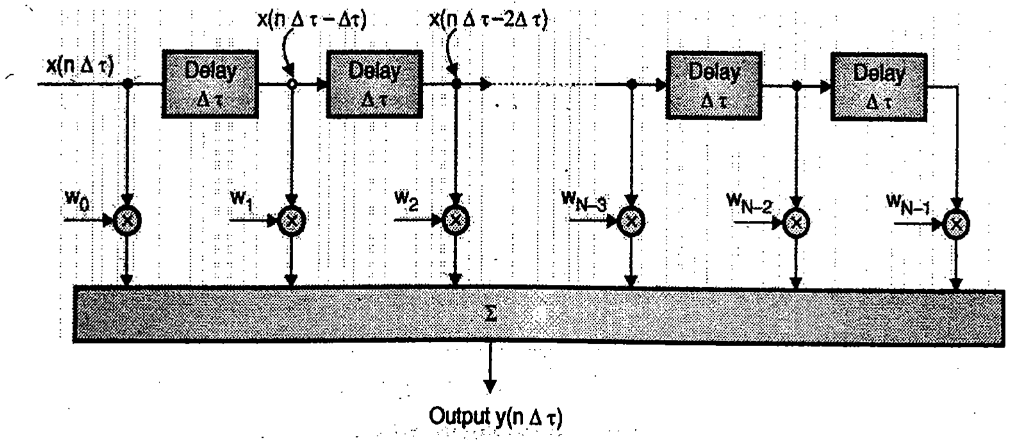
Substitute, $h(k\,\Delta\tau)\cdot\Delta\tau = w_k$ into Equation (3.18.6) to get,

$$y(n\,\Delta\tau) = \sum_{k=0}^{N-1} w_k\cdot x(n\,\Delta\tau - k\,\Delta\tau)\cdot \qquad ...(3.18.7)$$

Equation (3.18.7) can be realized using the circuit shown in Fig. 3.18.3 which is called as a tapped-delay-line filter or transversal filter. Because if we expand Equation (3.18.7) then we get,

$$\begin{aligned} y(n\,\Delta\tau) = {} & w_0\,x(n\,\Delta\tau) + w_1\,x(n\,\Delta\tau - \Delta\tau) \\ & + w_2\,x(n\,\Delta\tau - 2\,\Delta\tau) + \ldots\ldots + w_{N-1}\,x \\ & [n\,\Delta\tau - (N-1)\,\Delta\tau] \end{aligned} \qquad ...(3.18.8)$$

This expression is realized as shown in Fig. 3.18.3.



(E-798) **Fig. 3.18.3 : Tapped-delay-line filter or transversal filter**

3.19 Automatic Equalizers :

- In order to solve a set of simultaneous equations the tap gains of the zero forcing equalizer described in the previous section are adjusted. This is called as the "trimming" of the equalizers. The process of equalizer trimming involves the following steps.

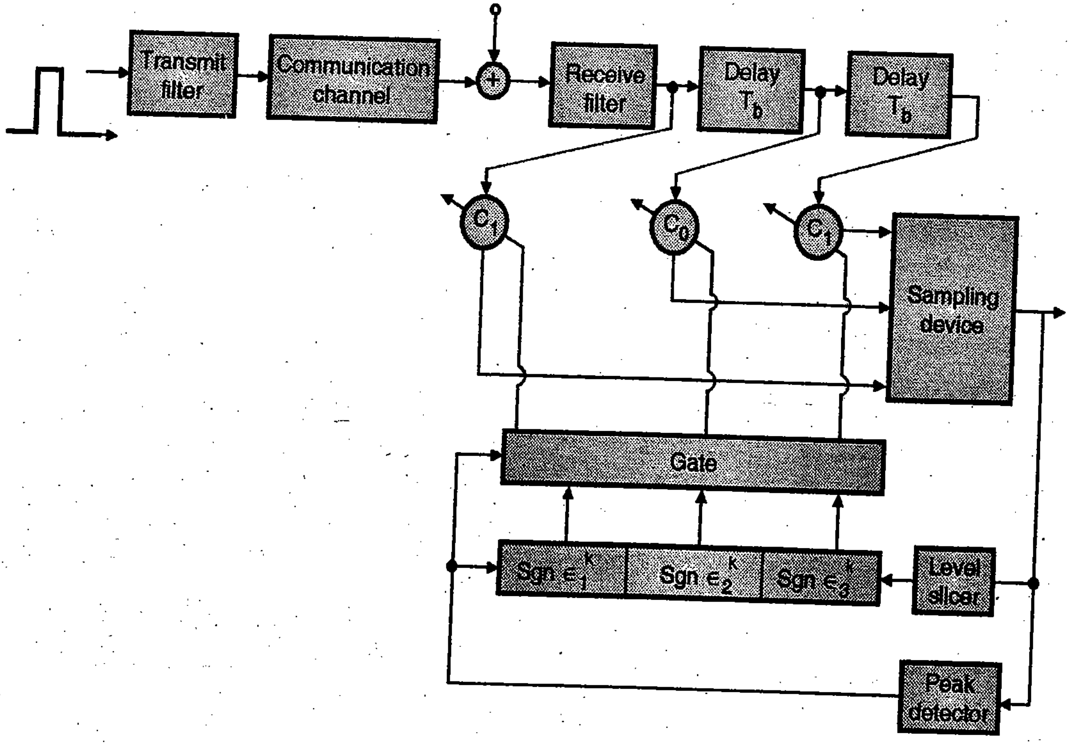
Procedure of trimming :

1. Send a pulse through the system.
2. Measure the output at the receiving filter output at the proper sampling instants.
3. Calculate the values of the tap gains.
4. Accordingly set the gains on the taps of the equalizer.

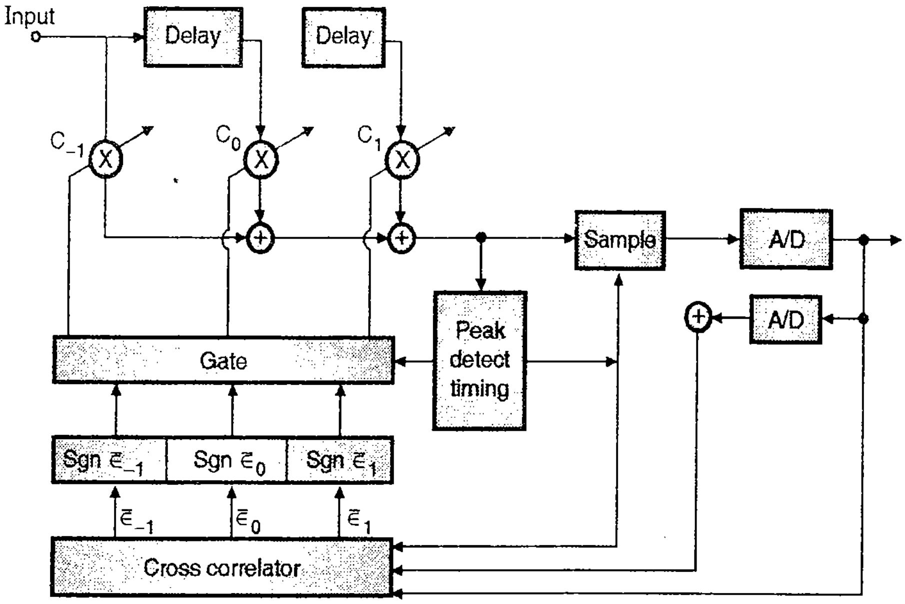
- In the automatic equalizers, the tap gains are adjusted automatically with the help of accurate and simple automatic system.
- These automatic systems are classified into two types :
  1. Preset type equalizers
  2. Adaptive equalizers.
- The preset type equalizers use a special sequence of pulses either before or during the gaps in the data transmission.
- In the adaptive type equalizers the adjustments take place continuously during the transmission of data by operating on the data signal itself.
- In the automatic equalizers the tap gains are optimally adjusted by using the iterative technique.

3.19.1 Preset Equalizer :

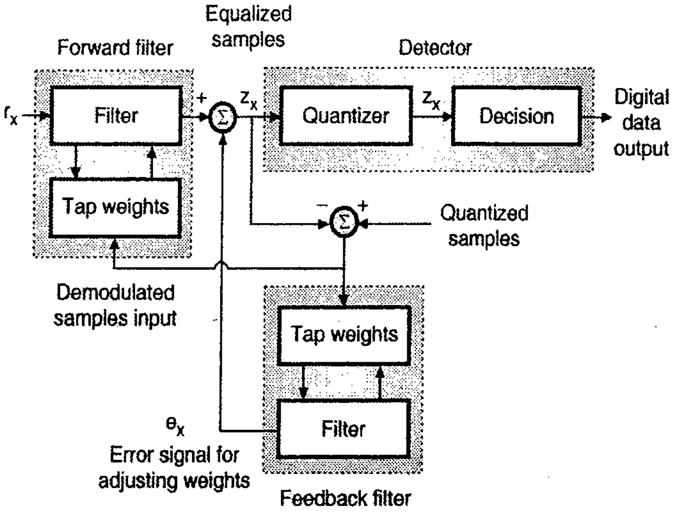
- The block diagram of the preset equalizer is shown in Fig. 3.19.1. Here the components of the error vector are measured by transmitting a sequence of widely separated pulses through the system.
- Then the output of the equalizer is measured at the proper instants of time.
- The tap gains are adjusted with the help of fixed iterations of the step size Δ.
- The sampling of the filter output at proper instants is done by using a timing circuit. This timing circuit is triggered by the peak detector.
- The center sample is compared with + 1 (or sliced at + 1) and the polarity of the error component ϵ_0^k is obtained. The polarities of the remaining error components are obtained by using the value of the filter output at $t = \pm j\, T_s$.
- The gate is opened at the end of the k^{th} test pulse. Depending on the polarity of the error component ϵ^k the tap gains are increased or decreased by Δ accordingly.
- This procedure is called as the "training procedure". This procedure can take hundreds of pulses.



(E-799) **Fig. 3.19.1 : A three tap preset equalizer**



(E-800) **Fig. 3.19.2 : A three tap adaptive equalizer**



(E-801) **Fig. 3.19.3**

3.19.2 Adaptive Equalizer :

SPPU : May 16

University Questions

Q. 1 What is equalizer ? Explain Adaptive equalizers. (May 16, 7 Marks)

The block diagram of an adaptive equalizer is as shown in Fig. 3.19.2.

- In the three tap adaptive equalizer of Fig. 3.19.2 the error vector $\in^k$ is estimated continuously when the normal data transmission is going on.
- This is known as the adaptive equalizer because this scheme has the ability to change itself even when the data transmission is on. This eliminates the need for long training sessions; which is there for the preset equalizer.
- The advantages of the adaptive equalizers are that they are more versatile, more accurate and cheaper than the preset equalizers.

3.19.3 Decision Feedback Equalizer :

- The basic limitation of a linear equalizer, such as the transversal filter, is that is exhibits a poor perform on channel having spectral nulls.
- A decision feedback equalizer (DFE) is a nonlinear equalizer that makes use of the detector previous decisions and eliminates the ISI on pulses that are currently being demodulated.
- In other words, the effects corresponding to the previous pulses on a current pulse is subtracted.
- Fig. 3.19.3 shows a simplified block diagram of a DFE where the forward filter and the feedback filter both are linear filters such as transversal filter.
- The DFE is nonlinear because of the nonlinear characteristic of the detector that provides an input to the feedback filter.

- The principle of a DFE is as follows : If we know the values of the symbols previously detected, then ISI due to these symbols can be canceled out completely at the output of the forward filter by subtracting values of the past symbols with appropriate weighting.
- The forward and feedback tap weights can be adjusted simultaneously to fulfill a criterion such as minimizing the MSE.
- The advantage of a DFE implementation is the feedback filter, not only removes ISI, but operates on noiseless quantized levels, as well and thus its output is free of channel noise.

3.20 Scramblers and Unscramblers :

SPPU : May 06, Dec, 08, May 09

University Questions

Q. 1 Explain the need of scrambler. Show that the unscrambler reproduces original sequence given to the scrambler. **(May 06, 10 Marks)**

Q. 2 Define Scrambling and its importance in digital communication. **(Dec. 08, May 09, 4 Marks)**

- In the communication applications sometimes it is necessary to transmit the message in such a form that only the desired receiver should be able to understand it.
- For this it is necessary to perform coding. Scrambling is a coding operation applied to the message at the transmitter.
- The process of scrambling will "randomise" the bit stream eliminating the long strings of 0s and 1s which might affect the receiver synchronization.
- The scrambling process also eliminates most periodic bit patterns which can produce undesirable discrete-frequency components in the power spectrum.
- The scrambled sequence is transmitted by the transmitter and received by the receiver. At the receiver it is unscrambled to get the original bit sequence back.

3.20.1 Tapped Shift Register :

Simple but effective scramblers and unscramblers use the tapped shift register as basic building block. A tapped shift register of generic form is shown in Fig. 3.20.1.

Operation :

- Fig. 3.20.1 shows an n-stage shift register. b_k is a binary input sequence applied to this shift register. Successive bits from b_k enter the register and shift from one stage to the next corresponding to each clock pulse.
- The output of each stage is applied to a binary tap gain $(\alpha_1, \alpha_2 \ldots \text{etc.})$
- The output b_k' is formed by combining the bits in the register through a set of binary tap gains and modulo 2 adders.

$$\therefore \text{ Output } b_k' = \alpha_1 b_{k-1} \oplus \alpha_2 b_{k-2} \oplus \ldots \oplus \alpha_n b_{k-n} \quad \ldots(3.20.1)$$

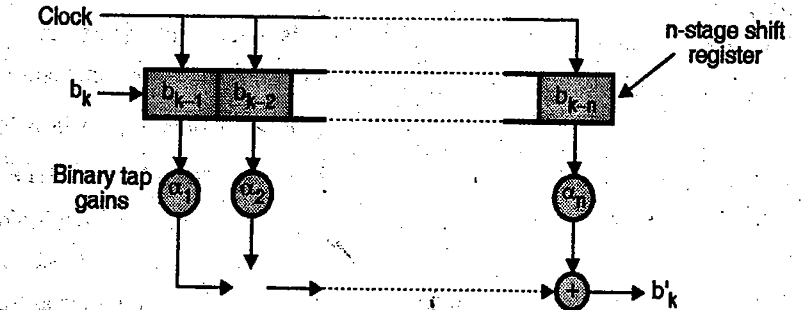
where the binary tap gains $\alpha_1, \alpha_2, \ldots$ are binary digits i.e. 0 or 1.

So $\alpha_1 = 1$ simply means a direct connection and $\alpha_1 = 0$ means no connection.

- The symbol ⊕ represents a modulo-2 addition. The properties of modulo-2 addition are as follows :

1. $b_1 \oplus b_2 = 0 \quad \ldots \text{for } b_1 = b_2$
   $= 1 \quad \ldots \text{for } b_1 \neq b_2 \quad \ldots(3.20.2)$
2. $b_1 \oplus b_2 \oplus b_3 = (b_1 \oplus b_2) \oplus b_3 = b_1 \oplus (b_2 \oplus b_3) \quad \ldots(3.20.3)$

where b_1, b_2 and b_3 are arbitrary binary digits. We can implement the modulo - 2 addition with exclusive – OR gates.



(E-802) **Fig. 3.20.1 : Tapped shift register**

3.20.2 Scrambler :

SPPU : May 06, May 07, Dec. 08, May 09

University Questions

Q. 1 Explain the need of scrambler. Show that the unscrambler reproduces original sequence given to the scrambler. **(May 06, 10 Marks)**

Q. 2 Explain scramblers and unscramblers. **(May 07, 8 Marks)**

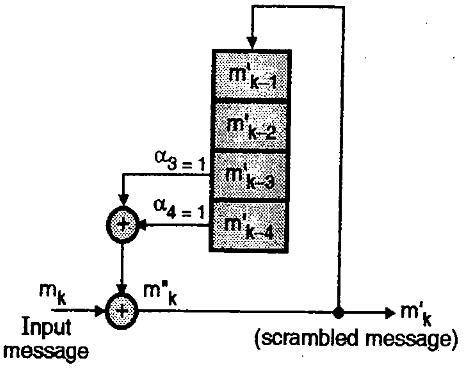
Q. 3 Define Scrambling and its importance in digital communication. **(Dec. 08, May 09, 4 Marks)**

- Fig. 3.20.2(a) shows the schematic of a scrambler. It uses a 4-stage shift register with tap gains $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = \alpha_4 = 1$.
- The clock line has not been shown for our convenience but it is very much present.
- m_k is the binary message sequence at the input to the scrambler. It is added to the register output, m''_k to obtain the scrambled message m'_k.
- The scrambled message m'_k is also fedback to the shift register input. Looking at Fig. 3.20.2(a) we can write that,

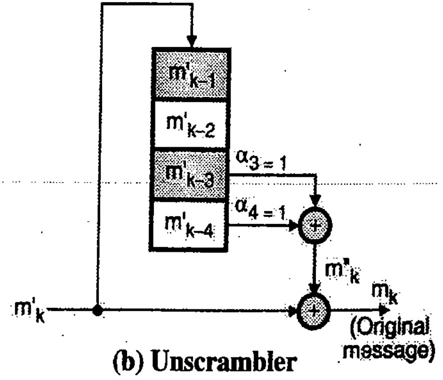
$$m''_k = m'_{k-3} \oplus m'_{k-4} \quad \text{...(3.20.4)}$$

$$\text{and} \quad m'_k = m_k \oplus m''_k \quad \text{...(3.20.5)}$$

- Here m'_k represents a scrambled version of m_k. The scrambled version is completely different from the original message m_k and hence the message m_k cannot be recovered unless we have a proper unscrambler at the receiver.
- The scrambled signal m'_k is then transmitted over a suitable communication medium.



(a) Binary scrambler



(b) Unscrambler

(E-803) Fig. 3.20.2

3.20.3 Unscrambler :

SPPU : May 06, May 07

University Questions

Q. 1 Explain the need of scrambler. Show that the unscrambler reproduces original sequence given to the scrambler. **(May 06, 10 Marks)**

Q. 2 Explain scramblers and unscramblers. **(May 07, 8 Marks)**

- Fig. 3.20.2(b) shows the schematic of an unscrambler. It uses a 4-stage shift register with tap gains $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = \alpha_4 = 1$. The unscrambler is used to recover the original message signal m_k from the scrambled signal m'_k.
- The unscrambler structure is exactly reverse as compared to the structure of the scrambler. It reproduces the original message signal m_k at its output. This is because :

$$m'_k \oplus m''_k = (m_k \oplus m''_k) \oplus m''_k$$

$$\because \quad m'_k = m_k \oplus m''_k$$

$$= m_k \oplus (m''_k \oplus m''_k) \quad \text{...using Equation (3.20.3)}$$

$$= m_k \oplus 0 \quad \text{... since } m''_k \oplus m''_k = 0$$

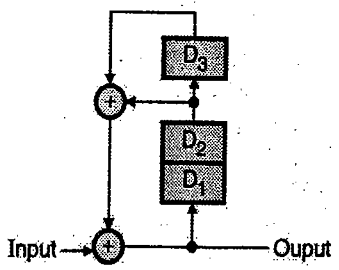
$$= m_k$$

- Thus by adding (modulo-2 addition), the scrambled signal m'_k with m''_k it is possible to obtain the original signal m_k.
- Note that due to the identical structures of shift registers and same input m'_k for scrambler as well as unscrambler, we will get the same sequence m''_k for the scrambler as well as unscrambler.

Disadvantages :

- The most serious problem in the unscrambler is that one erroneous bit in m'_k will cause several output bit errors. And these errors will keep propagating. The error propagation will stop only when the unscrambler register is full of correct bits.
- There can be some specific input message sequences (m_k) to the scrambler which will result in long strings of 0s or 1s in the scrambled sequence m'_k. This will result in loss of synchronization.

Ex. 3.20.1 : The data stream 10101010001 is an input to a scramble shown in Fig. P. 3.20.1(a). Obtain the scrambled output assuming initial content of all registers to be zero. **Dec. 06, 8 Marks**



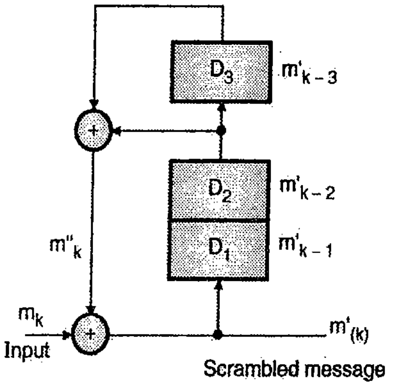
(E-808) Fig. P. 3.20.1(a)

Soln. : Refer Fig. P. 3.20.1(b).

$$m''_k = m'_{k-3} \oplus m'_{k-2}$$

$$\text{and} \quad m'_k = \text{Output} = m_k \oplus m''_k$$

$$\therefore \quad \text{Output} = m'_{k-3} \oplus m'_{k-2} \oplus m_k$$



(E-809) **Fig. P. 3.20.1(b)**

Initially assume that all the flipflops are in the reset condition.

$$\therefore \quad m'_{k-2} = m'_{k-3} = 0 \text{ ...initially}$$

$$\therefore \quad m''_k = 0 \oplus 0 = 0 \text{ ...initially}$$

Refer Table P. 3.20.1 for the output of the scrambler for the given input sequence.

(E-810) **Table P. 3.20.1**

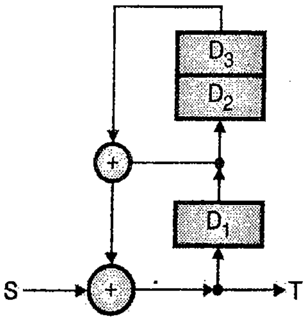
Input	Output	F/F Outputs				
m_k	m'_k	m'_{k-1}	m'_{k-2}	m'_{k-3}	m''_k	
-	0	0	0	0	0	Initially
1	1	0	0	0	0	
0	0	1	0	0	0	
1	1	0	1	0	1	
0	1	1	0	1	1	
1	0	1	1	0	1	
0	1	0	1	1	0	
1	1	0	0	1	1	
0	1	1	0	0	0	
0	0	1	1	0	1	
0	1	0	1	0	1	
1	0	1	0	1	1	

So the scrambled output is,

$$m'_k = 010110111010 \qquad \text{...Ans.}$$

Ex. 3.20.2 A scrambler is shown in Fig. P. 3.20.2. Design the corresponding descrambler. If a sequence S = 101010100000111 is applied to the input of this scrambler, determine the output sequence T. Verify that if this T is applied to the input of the descrambler, the output is the sequence S.

Dec. 07, May 13, 8 Marks



(E-811) **Fig. P. 3.20.2**

Soln. :

Step 1 : Determine the output sequence :

Refer Fig. P. 3.20.2 (a)

$$m''_k = m'_{k-3} \oplus m'_{k-1} \qquad \text{...(1)}$$

and scrambled output

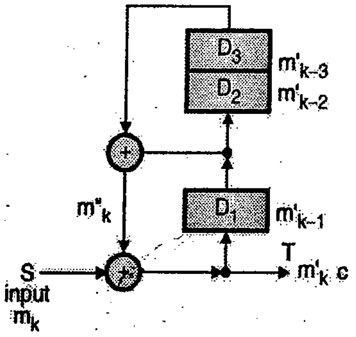
$$m'_k = m_k \oplus m''_k \qquad \text{...(2)}$$

$$= m_k \oplus m'_{k-3} \oplus m'_{k-1} \qquad \text{...(3)}$$

Initially assume that all the flip flops are in reset condition

$$\therefore \quad m'_{k-1} = m'_{k-3} = 0$$

$$\therefore \quad m''_k = 0 \oplus 0 = 0 \qquad \text{...initially}$$



(E-812) **Fig. P. 3.20.2(a) : Scrambler**

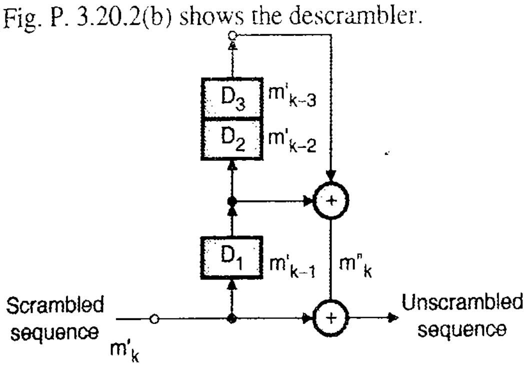
Refer Table P. 3.20.2 for the scrambler output for the given input sequence.

(E-813) **Table P. 3.20.2**

	Input	Output	F/F outputs			
	m_k	m'_k	m'_{k-1}	m'_{k-2}	m'_{k-3}	m''_k
Initially	–	0	0	0	0	0
	1	1	0	0	0	0
	0	1	1	0	0	1
	1	0	1	1	0	1
	0	1	0	1	1	1
	1	1	1	0	1	0
	0	1	1	1	0	1
	1	1	1	1	1	0
	0	0	1	1	1	0
	0	1	0	1	1	1
	0	0	1	0	1	0
	0	0	0	1	0	0
	0	1	0	0	1	1
	1	0	1	0	0	1
	1	0	0	1	0	0

Step 2 : Descrambler :

Fig. P. 3.20.2(b) shows the descrambler.



(E-814) **Fig. P. 3.20.2(b) : Descrambler**

From Fig. P. 3.20.2(b)

$$m_k'' = m'_{k-1} \oplus m'_{k-3} \qquad ...(4)$$

$$\therefore \quad \text{Output} = m_k' \oplus m_k'' \qquad ...(5)$$

From Equation (2) $m_k' = m_k \oplus m_k''$

Substituting in Equation (5) we get,

$$\text{Output} = m_k \oplus m_k'' \oplus m_k''$$

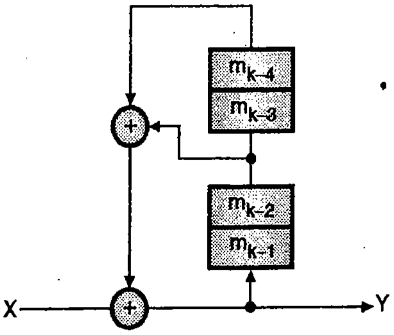
$$\text{But} \quad m_k'' \oplus m_k'' = 0$$

$$\therefore \quad \text{Output} = m_k \oplus 0 = m_k$$

Thus we get back the original sequence.

Ex. 3.20.3 : A scrambler is shown in Fig. P. 3.20.3. Design the corresponding unscramble if a sequence is 1010110 applied to the scrambler input, determine the output.

May 11, 6 Marks



(E-1147) **Fig. P. 3.20.3**

Soln. :

Step 1 : Determine the output sequence of the scrambler :

Refer Fig. P. 3.20.3(a), to write

$$\text{Output } m_k' = m_k \oplus m_k''$$

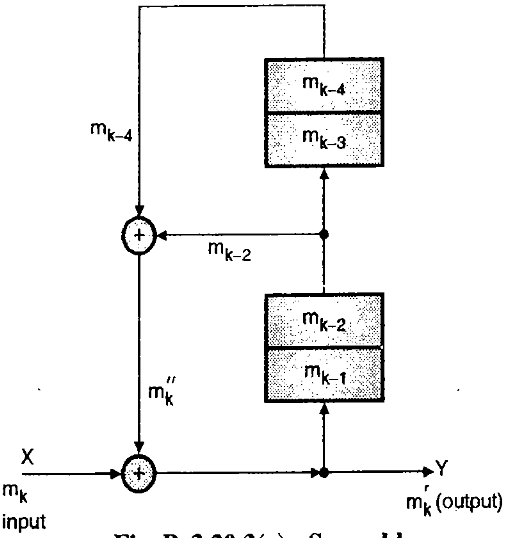
$$\text{But } m_k'' = m_{k-4} \oplus m_{k-2}$$

$$\therefore \quad \text{Output } m_k' = m_k \oplus m_{k-2} \oplus m_{k-4} \qquad ...(1)$$

- Initially assume that all the flip-flops are in Reset condition.

$$\therefore \quad m_{k-2} = m_{k-4} = 0$$

$$\therefore \quad m_k'' = 0 \oplus 0 = 0 \qquad ...\text{Initially}$$



(E-1148) **Fig. P. 3.20.3(a) : Scrambler**

Refer Table P. 3.20.3 for scrambler output in response to the given input sequence.

Hence the scrambled output is given by,

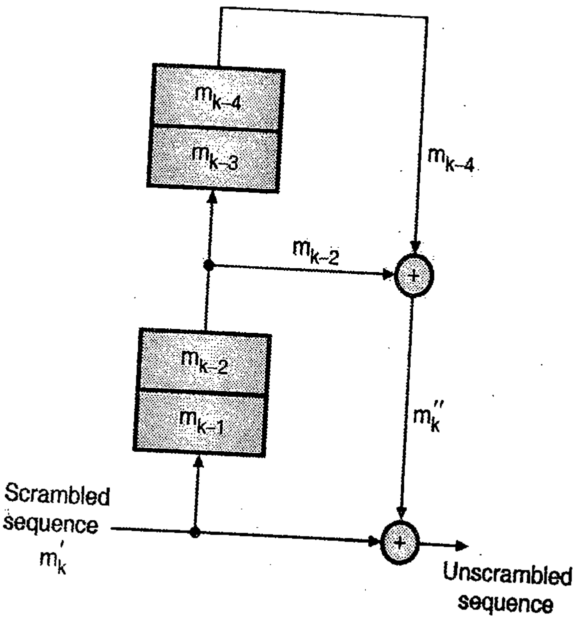
$$m_k' = 01000010 \qquad ...\textbf{Ans.}$$

(E-1148(a)) **Table P. 3.20.3**

Input m_k	Output $m_k' = m_k \oplus m_k''$	F/F Outputs m_{k-1}	m_{k-2}	m_{k-3}	m_{k-4}	$m_k'' = m_{k-2} \oplus m_{k-4}$
-	0	0	0	0	0	0 (Initially)
1	1	0	0	0	0	0
0	0	1	0	0	0	0
1	0	0	1	0	0	1
0	0	0	0	1	0	0
1	0	0	0	0	1	1
1	1	0	0	0	0	0
0	0	1	0	0	0	0

Step 2 : Descrambler :

The required descrambler is as shown in Fig. P. 3.20.3(b).



(E-1149) **Fig. P. 3.20.3(b) : Descrambler**

From Fig. P. 3.20.3(b) we get,

$$m_k'' = m_{k-2} \oplus m_{k-4}$$

$$\therefore \text{ Output } = m_k' \oplus m_k''$$

$$\text{But } \quad m_k' = m_k + m_k''$$

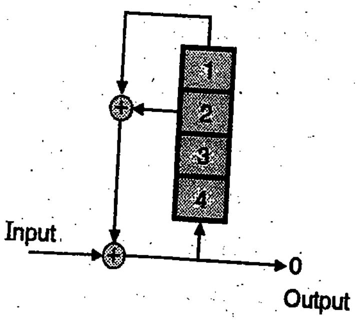
$$\therefore \text{ Output } = m_k \oplus (m_k'' \oplus m_k'')$$

$$\text{But } m_k'' \oplus m_k'' = 0$$

$$\therefore \text{ Output } = m_k \oplus 0 = m_k$$

Ex. 3.20.4 : Consider a sequence 10100001000000010 is applied to the scrambler shown in Fig. P. 3.20.4. Determine the output sequence 'O'. Design the corresponding descrambler and verify whether its output matches with the given sequence.

Dec. 11, 10 Marks



(E-1151) **Fig. P. 3.20.4**

Soln. :

Step 1 : Determine the output sequence :

Refer Fig. P. 3.20.4(a) to write

$$\text{Output } \quad m_k' = m_k \oplus m_k'' \quad \text{...(1)}$$

$$\text{But } \quad m_k'' = m_{k-4}' \oplus m_{k-3}' \quad \text{...(2)}$$

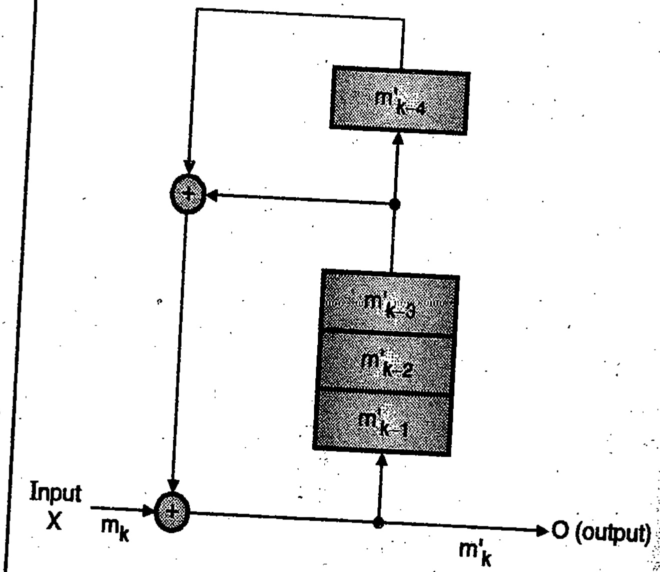
Therefore the output is given by,

$$m_k' = m_k \oplus m_{k-4}' \oplus m_{k-3}' \quad \text{...(3)}$$

Initially assume that all the flip-flops are in the reset condition

$$\therefore \quad m_{k-4}' = m_{k-3}' = 0$$

$$\therefore \quad m_k'' = 0 \oplus 0 = 0 \quad \text{...initially}$$



(E-1152) **Fig. P. 3.20.4(a)**

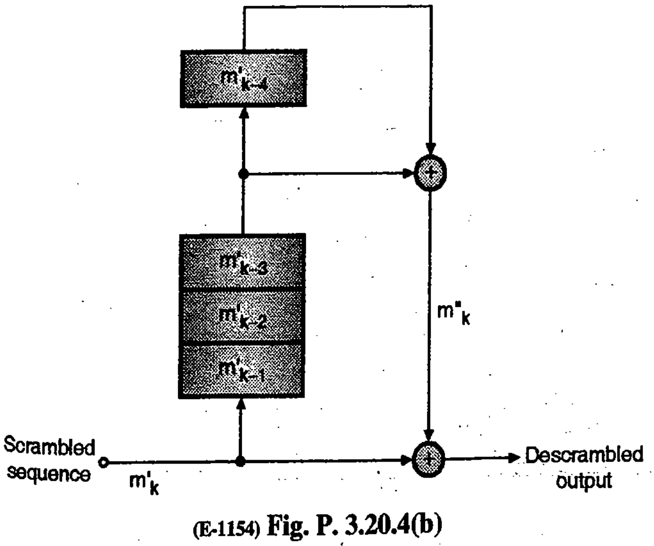
(E-1153) **Table P. 3.20.4**

	Input	Output	F/F outputs				$m''_k =$
	m_k	m'_k	m'_{k-1}	m'_{k-2}	m'_{k-3}	m'_{k-4}	$m'_{k-3} \oplus m'_{k-4}$
Initially_	-	0	0	0	0	0	0
	1	1	0	0	0	0	0
	0	0	1	0	0	0	0
	1	1	0	1	0	0	0
	0	0	1	0	1	0	1
	0	1	0	1	0	1	1
	0	1	1	0	1	1	0
	0	0	1	1	0	1	1
	1	0	0	1	1	0	1
	0	0	0	0	1	1	0
	0	0	0	0	0	1	1
	0	1	0	0	0	0	0
	0	0	1	0	0	0	0
	0	0	0	1	0	0	0
	0	0	0	0	1	0	1
	0	1	0	0	0	1	1
	1	0	1	0	0	0	0
	0	0	0	1	0	0	0

Refer Table P. 3.20.4 for the scrambler output corresponding to the given input sequence.

Step 2 : Descrambler :

Fig. P. 3.20.4(b) shows the descrambler.



(E-1154) **Fig. P. 3.20.4(b)**

From Fig. P. 3.20.4(b), we can write that,

$$m''_k = m'_{k-3} \oplus m'_{k-4}$$

$$\therefore \text{ output } = m'_k \oplus m''_k$$

$$\text{But } m'_k = m_k \oplus m''_k$$

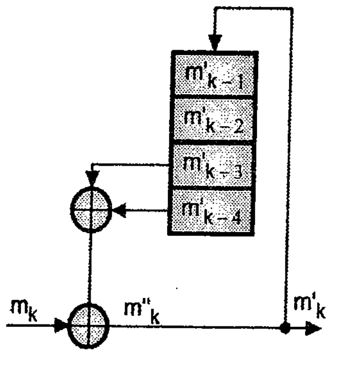
$$\therefore \text{ Output } = m_k \oplus m''_k \oplus m''_k$$

$$\text{But } m''_k \oplus m''_k = 0$$

$$\therefore \text{ Output } = m_k \oplus 0 = m_k$$

Thus we get the original signal back at the output of the descrambler circuit.

Ex. 3.20.5 : A scrambler is shown in Fig. P. 3.20.5. Design the corresponding descrambler. If a sequence m_k = 10110000000001 is applied to the input of this scrambler, determine the output sequence m'_k. Verify that if this m'_k is applied to the input of the scrambler, the output sequence m_k. **Dec. 13, 8 Marks**



(E-1367) **Fig. P. 3.20.5**

Soln. :

Part I : Generation of Scrambled sequence.

- Fig. P. 3.20.5 shows the schematic of a scrambler. It uses a 4-stage shift register with tap gains $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = \alpha_4 = 1$.
- The clock line has not been shown for our convenience but it is very much present.
- m_k is the binary message sequence at the input to the scrambler. It is added to the register output, m''_k to obtain the scrambled message m'_k.
- The scrambled message m'_k is also feedback to the shift register input. Looking at Fig. P. 3.20.5 we can write that,

$$m''_k = m'_{k-3} \oplus m'_{k-4} \quad \text{...(1)}$$

and $$m'_k = m_k \oplus m''_k \quad \text{...(2)}$$

Given : m_k = 1 0 1 1 0 0 0 0 0 0 0 0 0 1

(E-1371) **Table P. 3.20.5(a)**

Input m_k	Output m'_k	FF outputs				m''_k
		m'_{k-1}	m'_{k-2}	m'_{k-3}	m'_{k-4}	
1	1	0	0	0	0	0
0	0	1	0	0	0	0
1	1	0	1	0	0	0
1	0	1	0	1	0	1
0	1	0	1	0	1	1
0	1	1	0	1	0	1
0	1	1	1	0	1	1
0	1	1	1	1	0	1
0	0	1	1	1	1	0
0	0	0	1	1	1	0
0	0	0	0	1	1	0
0	1	0	0	0	1	1
0	0	1	0	0	0	0
1	1	0	1	0	0	0

Thus the scrambled sequence output is as follows :

$$m_k' = 0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0$$

Part II : Verification (Unscrambler).

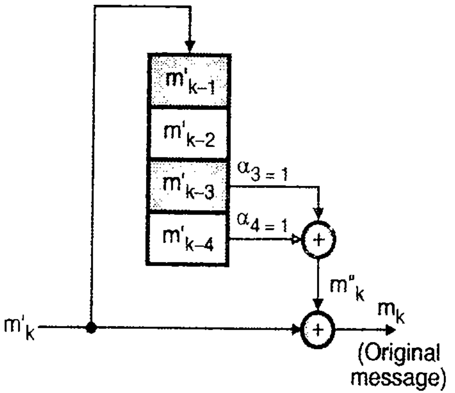
- Fig. P. 3.20.5(a) shows the schematic of an unscrambler. It uses a 4-stage shift register with tap gains $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = \alpha_4 = 1$. The unscrambler is used to recover the original message signal m_k from the scrambled signal m_k'.
- The unscrambler structure is exactly reverse as compared to the structure of the scrambler. It reproduces the original message signal m_k at its output. This is because :

$$m_k' \oplus m_k'' = (m_k \oplus m_k'') \oplus m_k'' \quad \because \quad m_k' = m_k \oplus m_k''$$

$$= m_k \oplus (m_k'' \oplus m_k'')$$

$$= m_k \oplus 0 \qquad \text{... since } m_k'' \oplus m_k'' = 0$$

$$= m_k$$



(E-803(a)) **Fig. P. 3.20.5(a) : Unscrambler**

(E-1372) **Table P. 3.20.5(b) : Unscrambler**

Input m'_k	FF Inputs m'_{k-1}	m'_{k-2}	m'_{k-3}	m'_{k-4}	m''_k	Output m'_k
1	0	0	0	0	0	1
0	1	0	0	0	0	0
1	0	1	0	0	0	1
0	1	0	1	0	1	1
1	0	1	0	1	1	0
1	1	0	1	0	1	0
1	1	1	0	1	1	0
1	1	1	1	0	1	0
0	1	1	1	1	0	0
0	0	1	1	1	0	0
0	0	0	1	1	0	0
1	0	0	0	1	1	0
0	1	0	0	0	0	0
1	0	1	0	0	0	1

The output sequence of the unscrambler is given by,

m_k = 1 0 1 1 0 0 0 0 0 0 0 0 0 1

This is same as the input to the scrambler.

Review Questions

Q. 1 Why do we need to use the line code formats ?

Q. 2 State the important properties of line codes.

Q. 3 What is the meaning of word "RZ".

Q. 4 How to improve the synchronizing capacity of a code ?

Q. 5 What is the difference between line coding and source coding ?

Q. 6 What is ISI ?

Q. 7 What is the cause of ISI ?

Q. 8 What is the remedy to ISI ?

Q. 9 What is the shape of transmitted pulse to avoid ISI ?

Q. 10 Explain how the sync pulse can avoid ISI.

Q. 11 How to measure ISI on CRO ?

Q. 12 What is the significance of eye opening ?

Q. 13 What is the effect of ISI ?

Q. 14 State various types of line codes.

Q. 15 What is the importance of dc component in line codes ?

Q. 16 Explain with the help of waveforms, the term self synchronization.

Q. 17 Give the classification of line codes.

Q. 18 Define the term line codes.

Q. 19 Draw the eye diagram and explain it.

Q. 20 Draw the power spectras of various line codes.

Q. 21 Compare the line codes.

Q. 22 Explain the Nyquists criteria for distortionless baseband binary transmission.

Q. 23 Derive the condition for zero ISI.

Q. 24 Explain the ideal solution to the problem of ISI and state various problems associated with it.

Q. 25 Write a note on : Raised Cosine Spectrum.

❑❑❑

CHAPTER 4

Random Signal & Noise

Unit III

Syllabus :

Introduction, Mathematical definition of a random process, Stationary processes, Mean, Correlation and Co-variance function, Ergodic processes, Transmission of a random process through a LTI filter, Power spectral density, Gaussian process, Noise, Narrowband noise, Representation of narrowband noise in terms of in phase and Quadrature components.

4.1 Introduction :

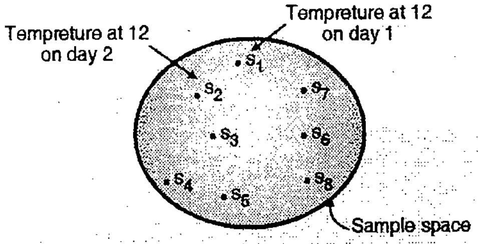
- In many practical applications we need to statistically analyze a communication system. In order to do so we should have the knowledge of random variables, mean, varience, standard deviation, various probability models, random processes etc.
- We need to perform the statistical analysis because, the signals used in communication systems are random in nature.
- The most important part in the statistical analysis of a communication system is how to characterize the random signals such as voice signals, TV signals, electrical noise, computer data etc.

4.1.1 Properties of Random Signals :

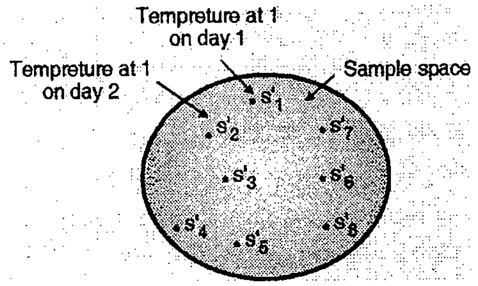
- The random signals mentioned earlier have the following two properties :
  1. These signals are functions of time and defined over a finite time interval.
  2. It is not possible to exactly describe the waveform of these signals with respect to time.

4.2 Random Process or Stochastic Process :

- The concept of a random process is an extension of the random variable (RV).
- Consider a random variable X which represents the temperature of a city at 12 O'clock. The temperature X is a random variable and can take on different values on different day.
- In order to get the complete idea about the random variable X, we will have to record values of X at 12 O'clock over many days.
- After doing so the sample space of this random variable would look as shown in Fig. 4.2.1(a). Each sample point corresponds to the temperature at 12 O'clock on a particular day.
- Sample point s_1 corresponds to temperature at 12 O'clock on day 1, s_2 for day 2 and so on.
- Then we can plot the PDF $f_X(x)$ of X which represents temperature of the city of 12 O'clock.



(a) Sample space of a random variable X



(b) Sample space of a random variable representing temperature at 12 O'clock representing temperature at 1 O'clock

(E-817) **Fig. 4.2.1 : Relation between sample space and random variable**

- But the temperature is not just the function of day on which it is being recorded, it is also a function of time. That means the temperature at 1 P.M. will have a completely different distribution from the distribution of temperature at 12 O'clock.
- We can define another random variable to represent the city temperature at 1 O'clock as shown in Fig. 4.2.1(b). This sample space has the sample points s_1', s_2', ...etc. which are completely different from s_1, s_2 ...
- Thus, in general we can say that sample space and sample points correspond to the random variables.

4.2.1 Sample Functions :

- Now take the next step and record the city temperature for each value of t (12 O'clock, 1 O'clock, 2 O'clock etc.) on every day and plot the waveforms x (t, λ_i) as shown in Fig. 4.2.2 where λ_i indicates the day i.e. λ_1 is day 1, λ_2 is day 2 and so on as shown in Fig. 4.2.2.
- Each waveform in Fig. 4.2.2 is called as a sample function.
- Thus sample function 1 is a waveform showing the temperature variation with respect to time "t" on day 1, sample function 2 is the waveform showing temperature variation with time on day 2 and so on.

4.2.2 Ensemble :

Ensemble means family or collection. So collection of all the possible sample functions is called as an ensemble.

Similarity between ensemble and sample space :

- Sample space is the collection of all possible sample points whereas ensemble is the collection of all possible sample functions. This is the similarity between them.

4.2.3 Definition of Random Process :

SPPU : Dec. 10, Dec. 14

University Questions

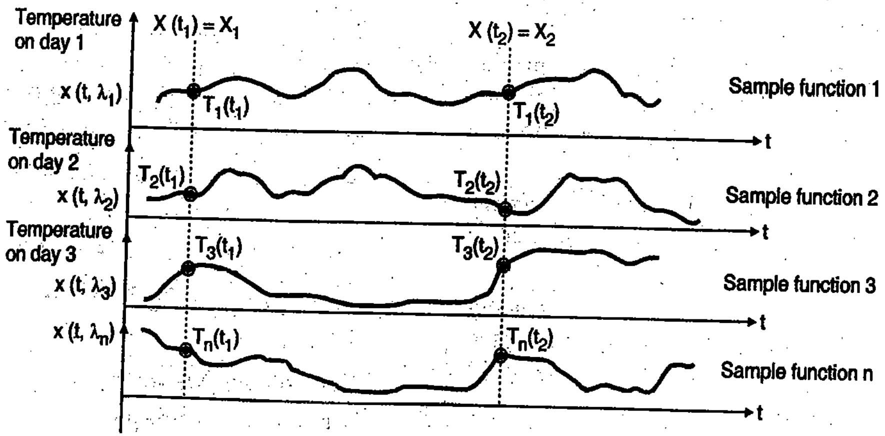
Q. 1 Define random process. What are time averages associated with random process ? **(Dec. 10, 8 Marks)**

Q. 2 Define random process. Explain various time averages associated with the random process. **(Dec. 14, 6 Marks)**

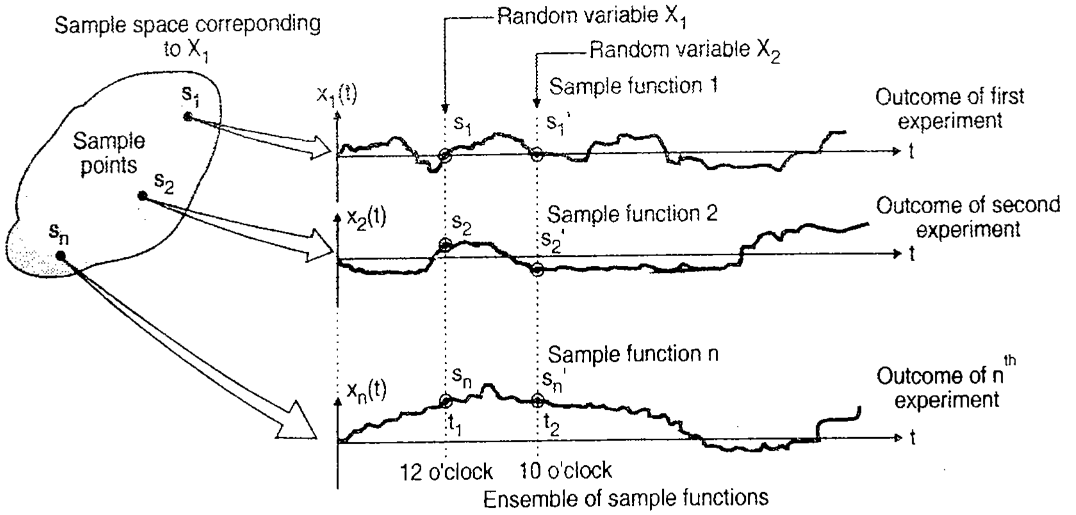
- The ensemble comprised of functions of time is called as a random or stochastic process.
- Note that each sample function consists of infinite sample points. So ensemble of sample functions will consist of infinite sample point.
- Since a group of sample points (say s_1, s_2 ...) corresponds to one random variable, the ensemble will correspond to infinite number of random variables.
- Thus a random process is a collection of an infinite number of random variables. A random process is also called as a "stochastic" process.
- The word "stochastic" comes from Greek which means "to aim (guess) at". A random process is denoted by X (t) and if we denote a random variable by X (s), then the random process is denoted by X (t, s).
- We can also define the random process as ensemble of random variables which are functions of time and hence denote the random process as X (t).

4.2.4 Relation between a Random Variable and a Random Process :

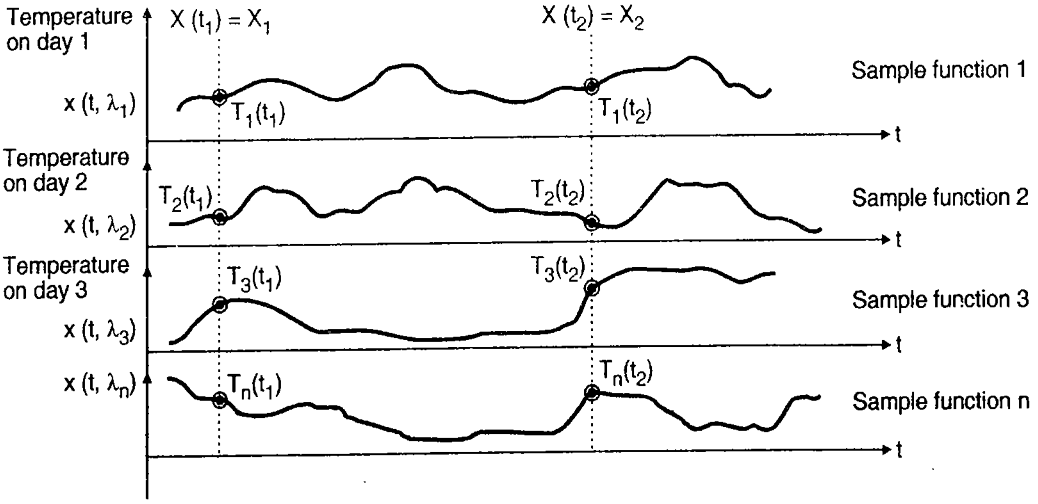
- We specify a random variable X by repeating an experiment a large number of times and from the outcomes of the experiment we determine f_X (x).
- We do the same thing at each value of t, to specify a random process, X (t). For example for a random process X (t) which represents the temperature of the city, we will have to note down the temperatures for each value of t (for each time of a day). This will give us a waveform x (t, λ_i) as shown in Fig. 4.2.2 where λ_i indicates the day for which the temperatures have been noted.
- Thus for a random variable, we map the outcome of a random experiment into a real number.
- Whereas for a random process we map the outcome of a random experiment into a sample function.
- This shows that sample points are equivalent to sample functions and sample space is equivalent to ensemble of sample functions.
- The relation between random variables and random process is shown in Fig. 4.2.3.
- In Fig. 4.2.3, X_1 is a random variable generated by the amplitudes of the random process at instant t = t_1 and X_2 is a random variable generated by the amplitudes of the random process at instant t = t_2.



(E-818) **Fig. 4.2.2 : An ensemble of sample function**



(E-819) **Fig. 4.2.3 : Relation between a random variable and a random process**



(E-820) **Fig. 4.2.4 : An ensemble of sample functions**

4.2.5 Ensemble Mean or Ensemble Average :

SPPU : May 08

University Questions

Q. 1 Define the terms related to random processes : Mean **(May 08, 2 Marks)**

- Ensemble mean or ensemble average is taken over the ensemble (collection) of waveforms at a fixed instant of time. Thus it is an average value obtained at a particular value of time t.
- For example, the ensemble mean value taken at $t = t_1$ will consist of all the values taken at $t = t_1$, that means $T_1(t_1)$, $T_2(t_1)$, $T_3(t_1)$, ..., $T_n(t_1)$ as shown in Fig. 4.2.4.

Mathematically the ensemble mean $\overline{X(t)}$ or m_x is defined as :

$$\text{Ensemble mean } m_x = \int_{-\infty}^{\infty} X f_X(x;t)\,dx \quad ...(4.2.1)$$

- In Equation (4.2.1), time t is treated as a constant because we are obtaining the ensemble mean at a fixed time instant. Therefore the only variable in this equation is x.
- The "ensemble mean" can be obtained at other time instants i.e. at t_2, t_3 ... etc. also. The values of ensemble mean at different time instants will be different from each other.
- Therefore the ensemble mean is a function of time.

4.2.6 Time Averages :

SPPU : Dec. 10, Dec. 14

University Questions

Q. 1 Define random process. What are time averages associated with random process ? **(Dec. 10, 8 Marks)**

Q. 2 Define random process. Explain various time averages associated with the random process. **(Dec. 14, 6 Marks)**

- The ensemble averages are obtained at constant values of t, whereas the time averages are obtained by changing time t.

- Time average of a random process is defined as the statistical average obtained by considering time t as a variable. The time average of any sample function x (t) can be defined as,

Time average of a sample function = $m_x(T)$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t)\, dt \qquad ...(4.2.2)$$

- The time average of sample function 1 in Fig. 4.2.4 may or may not be identical to that of the sample function 2 or 3 and so on.
- Thus in the time average we have to treat time "t" as a variable. This is the difference between time average and ensemble average.
- The autocorrelation function for a random process X (t) is defined as follows.

$$R_X(t_1, t_2) = E[X(t_1)\, X(t_2)] \qquad ...(4.2.3)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_{X_1 X_2}(x_1, x_2)\, dx_1\, dx_2 \qquad ...(4.2.4)$$

- The autocorrelation function of Equation (4.2.4) can be used to measure the similarity between the amplitudes $X(t_1)$ and $X(t_2)$ of the random process X (t) at time instants t_1 and t_2.
- The value of $R_X(t_1, t_2)$ is obtained by taking the product of the values of the sample functions at instants t_1 and t_2 and then taking the mean of this product.
- $f_{X_1 X_2}(x_1, x_2)$ is the second order probability density function of the random process X (t).

4.2.7 Classification of Random Processes :

SPPU : May 11, Dec. 11, Dec. 12

University Questions

Q. 1 Explain classification of random processes with mathematical expressions. **(May 11, 6 Marks)**

Q. 2 Explain in brief the different types of random processes with suitable examples. **(Dec. 11, 8 Marks)**

Q. 3 Classify and explain different types of random processes. **(Dec. 12, 8 Marks)**

The random processes are broadly categorized into the following categories :

1. Stationary and Non-stationary random processes.
2. Wide-sense (or weakly) stationary processes.
3. Ergodic processes.

4.2.8 Characteristics of a Random Processes :

- Now let us see how to characterize or describe a random process. Sometimes we may be able to describe it analytically using a mathematical expression.
- For example the random process X (t) is given by,

$$X(t) = A \sin(2\pi f_c t + \theta) \qquad ...(4.2.5)$$

Where θ = R. V. distributed over the range $(0, 2\pi)$.

- This analytical expression can completely describe the random process X (t) and its ensemble. Such an analytical description is always based on the well defined models.
- However it is not always possible to describe a random process analytically. Without a specific mode, we may have just an ensemble obtained experimentally. Such an ensemble has the complete information about that particular random process.
- Then we consider such a random process as equivalent to a Random Variable X which is the function of time.
- So this random process is a collection of an infinite number of RVs which are generally dependent on each other.
- The complete information about such dependent RVs can be obtained from the joint PDF of those RVs.
- For n samples, the n RVs are fully characterised by the n^{th} order joint PDF or by an n^{th} order joint CDF.
- But determining such a PDF with $n \to \infty$ is extremely difficult.

4.3 Stationary and Non-stationary Random Processes :

SPPU : Dec. 08, May 09, May 11, Dec. 11, May 12, Dec. 12, May 13, May 16

University Questions

Q. 1 With help of mathematical expression explain stationary random processes non-stationary random processes, and wide sense stationary processes and Ergodic processes. **(Dec. 08, 8 Marks)**

Q. 2 Explain stationary random processes, non-stationary random processes, and wide sense stationary processes and Ergodic processes, with help of mathematical expression. **(May 09, 8 Marks)**

Q. 3 Explain classification of random processes with mathematical expressions. **(May 11, 6 Marks)**

Q. 4 Explain in brief the different types of random processes with suitable examples. **(Dec. 11, 8 Marks)**

Q. 5 Explain in detail about stationary, non stationary, wide sense stationary and ergodic processes with suitable mathematical expressions and examples. **(May 12, 8 Marks)**

Q. 6 Classify and explain different types of random processes. **(Dec. 12, 8 Marks)**

Q. 7 Explain stationary, non stationary, wide sense stationary and Ergodic processes with the help of mathematical expression. **(May 13, 8 Marks)**

Q. 8 Explain in detail about stationary, wide sense stationary and ergodic process with suitable mathematical expressions. **(May 16, 6 Marks)**

Definition :

- A random process whose statistical characteristics do not change with time is known as a stationary random process.
- Therefore the shift of time origin will not have any effect on the stationary random process.

Let us understand this in detail.

Description :

- Consider a random process X (t) which is initiated at t = – ∞. Let X (t_1), X (t_2) ... X (t_n) denote the random variables obtained by observing the random process X (t) at different time instants t = t_1, t_2 ..., t_n etc. as shown in Fig. 4.3.1.
- For the sake of simplicity we can denote these random variables as X_1, X_2, ... X_n. So the joint cumulative distribution function (joint CDF) of this set of random variables is denoted by

$$F_{X(t_1), X(t_2) ... X(t_n)} (x_1, x_2, ... x_n)$$

or

$F_{X_1, X_2 ... X_n} (x_1, x_2 ..., x_n)$ in the simplified notation.

- Now suppose that each instant of observation i.e. t_1 t_2, ... etc. is shifted by a fixed duration of time say τ.
- So a new set of random variables containing X (t_1 + τ), X (t_2 + τ), ... X (t_n + τ) is obtained (See Fig. 4.3.1).
- The joint CDF of this new set of random variables is

$$F_{X(t_1+\tau), X(t_2+\tau) ..., X(t_n+\tau)} (x_1, x_2, ..., x_n)$$

4.3.1 Definition of a Strictly Stationary Process :

- The random process X (t) is said to be stationary in the strict sense if the joint CDF of the original set of random variables is equal to the joint CDF of the new set of random variable obtained after a time shift of τ.
- That means

$$F_{X(t_1), X(t_2) ... X(t_n)} (x_1, x_2, ..., x_n)$$
$$= F_{X(t_1+\tau), X(t_2+\tau), ... X(t_n+\tau)} (x_1, x_2, ... x_n) \quad ...(4.3.1)$$

for all time shifts τ , all n and all the possible choices of t_1, t_2, ..., t_n.

In other words a random process X (t) is strictly stationary if the joint distribution of any set of random variables obtained by observing the random process X (t) is invariant (remains the same) with respect to the location of the origin t = 0.

4.3.2 Jointly Stationary Process :

- The two stationary processes X (t) and Y (t) are initiated at t = – ∞ are called as jointly stationary if the joint distribution function of the random variables X (t_1), X (t_2) ... X (t_n) and Y (t_1'), Y (t_2') ... Y (t_j') are invarient with respect to the location of origin t = 0 for all n and j and for all the choices of t_1, t_2 ... t_n and t_1',t_2', ..., t_j'.

4.3.3 Properties of Strictly Stationary Process :

Consider Equation (4.3.1) which states that the joint CDF is given by,

$$F_{X(t_1), X(t_2) ... X(t_n)} (x_1, x_2 ..., x_n) = F_{X(t_1+\tau), X(t_2+\tau), ... X(t_n+\tau)}, ..., (x_1, x_2, ..., x_n)$$

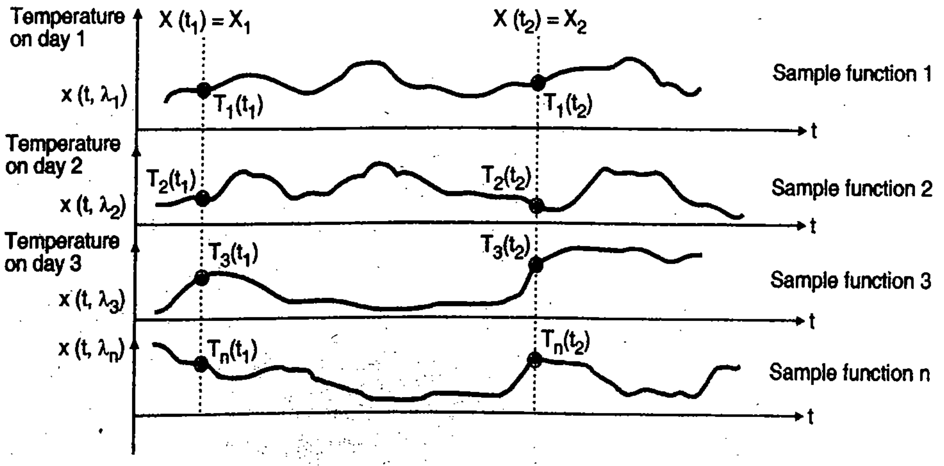
Property 1 :

For n = 1 the above expression gets converted as,

$$F_{X(t_1)}, (x_1) = F_{X(t_1+\tau)} (x_1) = F_X (x)$$

... for all t and τ ...(4.3.2)

So the first property says that the first order distribution function (n = 1) of a stationary random process is independent of time.



(E-821) **Fig. 4.3.1 : Stationary process**

Property 2 :

For $n = 2$ and $\tau = -t_1$, the expression in Equation (4.3.1) gets converted as,

$$F_{X(t_1)X(t_2)}(x_1, x_2) = F_{X(t_1-t_1)X(t_2-t_1)}(x_1, x_2)$$

$$\therefore F_{X(t_1),X(t_2)}(x_1, x_2) = F_{X(0)X(t_2-t_1)}(x_1, x_2)$$

$$\text{for all } t_1 \text{ and } t_2 \quad ...(4.3.3)$$

- So the second property states that the second order distribution function (for $n = 2$) of a stationary random process depends only on the time difference between the observation times (t_1 and t_2) and not dependent on the particular times at which the random process is observed.
- The random process X (t) representing the temperature of a city is a "non-stationary" system because the mean value etc. (temperature statistics) depends on the time of the day.
- The process of noise generation is a stationary process because its statistics such as mean, mean square value etc. does not change with time.
- It is not easy to determine whether a process is stationary because we have to carry out investigation of n^{th} order statistics for the same.

4.4 Mean, Correlation and Covariance Functions of a Stationary Random Process :

4.4.1 Mean of a Stationary Random Process :

SPPU : May 07, May 13, Dec. 15, Dec. 16

University Questions

Q. 1 Define mean, correlation and covariance function for random process. **(May 07, 6 Marks)**

Q. 2 Define mean, correlation, standard deviation and variance of random process. **(May 13, 8 Marks)**

Q. 3 Define mean, correlation, standard deviation of a random process. **(Dec. 15, 6 Marks)**

Q. 4 Define mean, correlation and covariance function for random process. Write down mathematical expression for the same. **(Dec. 16, 6 Marks)**

- Consider a stationary random process X (t).
- The mean value of such a random process is given by,

$$m_X(t) = \text{Expected value of } X(t)$$

$$= E[(X(t))] = \int_{-\infty}^{\infty} x\, f_{X(t)}(x)\, dx \quad ...(4.4.1)$$

Where $f_{X(t)}(x)$ is the first order PDF of the process.

- Referring to the first property of stationary processes (Equation (4.3.2)), we can say that if the given process is a stationary process then $f_{X(t)}(x)$ is independent of time "t".
- Therefore the mean of a stationary process is always constant and given by,

$$m_X(t) = m_X \quad \text{...for all values of t} \quad ...(4.4.2)$$

4.4.2 Autocorrelation of Process X (t) :

SPPU : May 07, May 08, May 12, May 13, Dec. 15, Dec. 16

University Questions

Q. 1 Define mean, correlation and covariance function for random process. **(May 07, 6 Marks)**

Q. 2 Define the terms related to random processes : Autocorrelation **(May 08, 2 Marks)**

Q. 3 Define the power spectral density and autocorrelation function of periodic signals. Show that both are related in frequency domain. **(May 12, 8 Marks)**

Q. 4 Define mean, correlation, standard deviation and variance of random process. **(May 13, 8 Marks)**

Q. 5 Define mean, correlation, standard deviation of a random process. **(Dec. 15, 6 Marks)**

Q. 6 Define mean, correlation and covariance function for random process. Write down mathematical expression for the same. **(Dec. 16, 6 Marks)**

- The autocorrelation of process X (t) is defined as the expected value of the product of two random variables X (t_1) and X (t_2). These two variables are obtained by observing the given process at $t = t_1$ and t_2 respectively.
- The autocorrelation function is denoted by $R_X(t_1, t_2)$ and it is expressed mathematically as follows :

$$R_X(t_1, t_2) = E[X(t_1)\, X(t_2)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1\, x_2\, f_{X(t_1)}\, f_{X(t_2)}(x_1, x_2)\, dx_1\, dx_2 \quad ...(4.4.3)$$

Where $f_{X(t_1)}\, f_{X(t_2)}(x_1, x_2)$ = Second order probability density function.

- From property 2 of the stationary random process (Equation (4.3.3)) we can say that for a stationary random process, $f_{X(t_1)}\, f_{X(t_2)}(x_1, x_2)$ which is called as the second order PDF will be dependent only on the difference between observation times t_1 and t_2. Therefore we come to a conclusion that the autocorrelation function of a stationary random process is dependent only on the time difference $(t_2 - t_1)$. That means

$$R_X(t_1, t_2) = R_X(t_2 - t_1) \quad \text{...for all } t_1 \text{ and } t_2 \quad ...(4.4.4)$$

4.4.3 Autocovariance Function of a Stationary Process :

SPPU : May 07, May 13, Dec. 15, Dec. 16

University Questions

Q. 1 Define mean, correlation and covariance function for random process. **(May 07, 6 Marks)**

Q. 2 Define mean, correlation, standard deviation and variance of random process. **(May 13, 8 Marks)**

Q. 3 Define mean, correlation, standard deviation of a random process. **(Dec. 15, 6 Marks)**

Q. 4 Define mean, correlation and covariance function for random process. Write down mathematical expression for the same. **(Dec. 16, 6 Marks)**

- The autocovariance function of a stationary process is mathematically expressed as follows,

$$C_X(t_2 - t_1) = E[(X(t_1) - m_X)(X(t_2) - m_X)]$$

$$= R_X(t_2 - t_1) - m_X^2 \quad \text{...(4.4.5)}$$

- Here $C_X(t_2 - t_1)$ represents the autocovariance function.
- From Equation (4.4.5) we conclude that similar to the autocorrelation function, the autocovariance function of a stationary process X (t) is dependent only on the time difference $(t_2 - t_1)$.
- From this equation it is possible to calculate the autocovariance if the mean and autocorrelation of the random process are known.
- The mean and autocorrelation of a random process are thus sufficient to describe the first two moments of a random process but they only provide a partial description of the distribution of a random process X (t).
- It is important to note that the mean and autocorrelation function are not sufficient to guarantee that the given random process X (t) is stationary.

4.4.4 Wide Sensed Stationary Process :

SPPU : Dec. 06, Dec. 10, May 11, Dec. 11, May 12, Dec. 12, May 13, May 16

University Questions

Q. 1 What are the conditions for a random process to be wide sense stationary ? What is ergodicity ? **(Dec. 06, 6 Marks)**

Q. 2 What are the conditions for a random process to be wide sense stationary ? **(Dec. 10, 8 Marks)**

Q. 3 Explain classification of random processes with mathematical expressions. **(May 11, 6 Marks)**

Q. 4 Explain in brief the different types of random processes with suitable examples. **(Dec. 11, 8 Marks)**

Q. 5 Explain in detail about stationary, non stationary, wide sense stationary and ergodic processes with suitable mathematical expressions and examples. **(May 12, 8 Marks)**

Q. 6 Classify and explain different types of random processes. **(Dec. 12, 8 Marks)**

Q. 7 Explain stationary, non stationary, wide sense stationary and Ergodic processes with the help of mathematical expression. **(May 13, 8 Marks)**

Q. 8 Explain in detail about stationary, wide sense stationary and ergodic process with suitable mathematical expressions. **(May 16, 6 Marks)**

- A given process may not be stationary in the "strict" sense, yet it may have mean value $m_X(t)$ and an autocorrelation function which are independent of the shift in the time origin.
- That means as we have mentioned earlier,

$$m_X(t) = \text{Constant and } R_X(t_1, t_2)$$

$$= R_X(t_2 - t_1).$$

- Such a random process is known as a "Wide-sense stationary" or Weakly stationary process.
- All the stationary processes are wide-sense stationary but every wide-sense stationary process may not be strictly stationary.
- However it is important to know that a "truly stationary" process is only a concept because it cannot occur in real life.

4.5 Autocorrelation Function of a Random Process :

SPPU : Dec. 06, Dec. 15

University Questions

Q. 1 Define autocorrelation function. State and explain any three properties of auto correlation function. **(Dec. 06, 6 Marks)**

Q. 2 Define mean, correlation, standard deviation of a random process. **(Dec. 15, 6 Marks)**

- For carrying out signal analysis, one of the most important statistical characteristics of a random process is its **auto correlation function.**
- This provides all the required spectral information about the random process.
- The autocorrelation function of a random process is defined as,

$$R_X(\tau) = E[X(t+\tau) X(\tau)] \quad \text{...(4.5.1)}$$

Let us now learn some of the important properties of autocorrelation function.

4.5.1 Properties of Autocorrelation Function :

SPPU : Dec. 06, Dec. 09

University Questions

Q. 1 Define autocorrelation function. State and explain any three properties of auto correlation function. **(Dec. 06, 6 Marks)**

Q. 2 State properties of auto-correlation function. Show that when wide sense stationary process passed through a LTI filter with impulse response h(t) produces constant mean-square value. **(Dec. 09, 9 Marks)**

Some of the important properties of autocorrelation function are as follows :

Property 1 :

The mean square value of a process is equal to the value of its autocorrelation at $\tau = 0$. That means,

$$R_X(0) = E[X^2(t)] \quad \text{...(4.5.2)}$$

Property 2 :

This property states that the autocorrelation function $R_X(\tau)$ of a random process X (t) is an even function of τ. That means,

$$R_X(\tau) = R_X(-\tau) \quad \text{...(4.5.3)}$$

Proof :

- We have defined the autocorrelation function of a random process as,

$$R_X(\tau) = E[X(t+\tau)X(t)] \quad \text{...for all t.}$$

- Substitute $(t-\tau)$ in place of t to get,

$$R_X(-\tau) = E[X(t)\cdot X(t-\tau)]$$

- Note that $X(t-\tau)$ represents the delayed version of x (t) and x $(t+\tau)$ represents the advanced version of x (t). Hence both of them will have same shape.

$$\therefore \quad R_X(\tau) = R_X(-\tau) \qquad \text{...Proved.}$$

Property 3 :

This property states that the autocorrelation function $R_X(\tau)$ is maximum at $\tau = 0$. That means.

$$|R_X(\tau)| \leq R_X(0) \qquad \text{...(4.5.4)}$$

Proof :

- Consider a non negative quantity.

$$E\left[\{X(t+\tau) \pm X(t)\}^2\right] \geq 0$$

$$\therefore E[X^2(t+\tau) \pm 2X(t+\tau)X(t) + X^2(t)] \geq 0$$

Since E is a linear operator we can write that,

$$\therefore \quad E[X^2(t+\tau)] \pm E[2X(t+\tau)X(t)] + E[X^2(t)] \geq 0$$

- But $R_X(0) = E[X^2(t)]$ and $E[X^2(t+\tau)] = R_X(0)$ and $E[X(t+\tau)X(t)] = R_X(\tau)$

$$\therefore 2R_X(0) \pm 2R_X(\tau) \geq 0$$

- In other words,

$$R_X(0) \geq |R_X(\tau)| \qquad \text{...Proved.}$$

4.5.2 Physical Significance of Autocorrelation Function :

- Fig. 4.5.1 illustrates the physical significance of the autocorrelation function.
- The physical significance of $R_X(\tau)$ is that it is a measure of interdependence of two random variables obtained by observing a random process X (t) at time instants that are τ seconds apart from each other.
- Therefore if the random process X(t) changes very rapidly with time, the autocorrelation function will decrease rapidly to zero as shown in Fig. 4.5.1.
- The decorrelation time τ_0 characterizes the reduction in $R_X(\tau)$ from $R_X(0)$ to 0. If $\tau > \tau_0$ then the magnitude of the autocorrelation function $R_X(\tau)$ remains below some prescribed value.
- Therefore the decorrelation time τ_0 of a wide-sense stationary random process X (t) of a zero mean is defined as the time taken for the magnitude of autocorrelation function $R_X(\tau)$ to decrease to 1 percent of the maximum value $R_X(0)$.

Ex. 4.5.1 : Show that the random process

$X(t) = A\cos(\omega_c t + \theta)$

where θ is a random variable uniformly distributed in the range $(0, 2\pi)$, is a wide-sense stationary process.

Dec. 05, 10 Marks, May 06, Dec. 07, May 08, 8 Marks

Soln. : In order to prove that the given random process X (t) is a wide-sense stationary process it is necessary to show that,

1. Ensemble mean of sample function amplitudes at any t is same.
2. Autocorrelation $R_\tau(t_1, t_2) = R_\tau(t_2 - t_1)$

As θ is uniformly distributed the PDF is given by,

$$f_\theta(\theta) = 1/2\pi \quad ... \quad 0 \leq \theta \leq 2\pi$$
$$= 0 \quad ... \text{ elsewhere} \qquad ...(1)$$

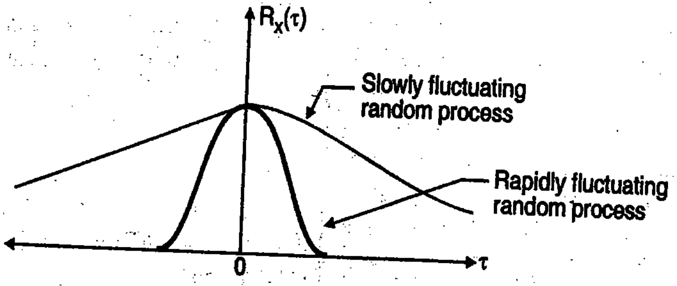
1. To obtain the ensemble average :

The ensemble of the random process X (t) consists of sinusoids of constant amplitude A and constant frequency ω_c but the phase θ will change in a random.

The phase θ will be equally likely and will have a value between $(0, 2\pi)$ for any sample function. The ensemble mean of this process is given by,

$$m_X(t) = \int_{-\infty}^{\infty} X f_X(x, t)\, dx \qquad ...(2)$$

$$\text{Here } X = X(t)$$
$$= A\cos(\omega_c t + \theta) \text{ and } f_X(x, t)$$
$$= f_\theta\,\theta = \frac{1}{2\pi}$$



(E-822) Fig. 4.5.1 : Physical significance of autocorrelation function

Substituting these values in Equation (2) we get,

$$m_X(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \cdot A\cos(\omega_c t+\theta)\, d\theta$$

$$= \frac{A}{2\pi}\left[\sin(\omega_c t+\theta)\right]_0^{2\pi} \qquad ...(3)$$

$\therefore\ m_X(t) = 0$ **...Ans.**

Thus the ensemble mean of the sample function amplitudes at any instant t is zero. i.e. independent of t.

2. To obtain the autocorrelation function :

From Equation (4.2.3), the expression for $R_\tau(t_1, t_2)$ is given by,

$$R_\tau(t_1, t_2) = E[X(t_1)\cdot X(t_2)]$$

$$= E[A\cos(\omega_c t_1+\theta)\, A\cos(\omega_c t_2+\theta)]$$

$$\therefore\ R_\tau(t_1, t_2) = E[A^2\cos(\omega_c t_1+\theta)\cdot\cos(\omega_c t_2+\theta)] \qquad ...(4)$$

But $\cos A \cos B = \frac{1}{2}\{\cos(A+B)+\cos(A-B)\}$

$$\therefore\ R_\tau(t_1, t_2) = E\left[\frac{A^2}{2}\left\{\cos[\omega_c(t_1-t_2)] + \cos[\omega_c(t_1+t_2+2\theta)]\right\}\right]$$

$$\therefore\ R_\tau(t_1, t_2) = \frac{A^2}{2}E\left\{\cos[\omega_c(t_1-t_2)]\right\} + \frac{A^2}{2}E\left\{\cos[\omega_c(t_1+t_2+2\theta)]\right\} \qquad ...(5)$$

The first term on the R.H.S. of Equation (5) does not contain the random variable θ. Hence

$$E\left\{\cos[\omega_c(t_1-t_2)]\right\} = \cos[\omega_c(t_1-t_2)] \qquad ...(6)$$

The second term on RHS of Equation (5) is a function of random variable θ. Therefore its expected value is given by,

$$E\left\{\cos[\omega_c(t_1+t_2+2\theta)]\right\} = \int_{-\infty}^{\infty}\cos[\omega_c(t_1+t_2+2\theta)]\cdot\frac{1}{2\pi}d\theta$$

$$\text{...... because } E[X] = \int_{-\infty}^{\infty} X f_X(x)\,dx$$

$$\therefore E\left\{\cos[\omega_c(t_1+t_2+2\theta)]\right\} = \int_0^{2\pi}\frac{1}{2\pi}\cdot\cos[\omega_c(t_1+t_2+2\theta)]\,d\theta$$

$$= \frac{1}{2\pi}\{\sin\omega_c(t_1+t_2+2\theta)\}_0^{2\pi}$$

$$= 0 \qquad ...(7)$$

Substituting Equations (6) and (7) into Equation (5) we get,

$$R_\tau(t_1, t_2) = \frac{A^2}{2}\{\cos[\omega_c(t_1-t_2)]\}$$ **...Ans.**

This expression shows that the autocorrelation function is the function of time difference $(t_1 - t_2)$.

Thus we have proved that :

1. The ensemble mean is independent of time.
2. Autocorrelation function is function of time difference $(t_1 - t_2)$.

   Hence the given process X (t) is wide-sense stationary process.

Ex. 4.5.2 : Let V(t) = X + 3t where X is a random variable with $\bar{X} = 0$ and $\overline{X^2} = 5$. Show that $\overline{V(t)} = 3t$ and $R_V(t_1, t_2) = 5 + 9\,t_1 t_2$ where $R_V(t_1, t_2)$ is autocorrelation function and $\overline{V(t)}$ is mean value of V(t).

May 06. 8 Marks

Soln. :

Step 1 : Obtain $\overline{V(t)}$:

$$\text{Mean } \overline{V(t)} = \int_{-\infty}^{\infty} V(t)\, f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty}(X+3t)\, f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} X f_X(x)\,dx + \int_{-\infty}^{\infty} 3t\, f_X(x)\,dx$$

$$\text{But } \int_{-\infty}^{\infty} X f_X(x)\,dx = \bar{X} = 0$$

$$\therefore\ \overline{V(t)} = 0 + 3t\int_{-\infty}^{\infty} f_X(x)\,dx$$

$$\text{But } \int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$\therefore\ \overline{V(t)} = 0 + (3t \times 1) = 3t$$

Step 2 : Find the autocorrelation R (t_1, t_2) :

$$R(t_1, t_2) = E[V(t_1)\cdot V(t_2)]$$

$$= E[(X+3t_1)\cdot(X+3t_2)]$$

$$= E[X^2 + 3Xt_1 + 3Xt_2 + 9t_1 t_2]$$

$$= E[X^2] + E[3Xt_1] + E[3Xt_2] + E[9t_1\ t_2]$$

$$\text{But} \quad E[X^2] = \overline{X^2} = 5$$

$$\therefore \quad R(t_1, t_2) = 5 + 3t_1\ E[X] + 3t_2\ E[X] + E[9t_1\ t_2]$$

$$\text{But} \quad E[X] = 0$$

$$\therefore \quad R(t_1, t_2) = 5 + (3t_1 \times 0) + (3t_1 \times 0) + 9t_1\ t_2$$

$$= 5 + 9t_1\ t_2$$

Ex. 4.5.3 : Show that the random process X(t) = A cos (ω_c t + ϕ) where ϕ is uniformly distributed random variable in the range ($-\pi$, $+\pi$) is a wide sense stationary process.

May 08, 8 Marks

Soln. :

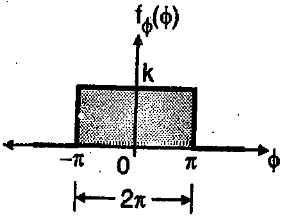
Step 1 : Find the PDF :

The random variable ϕ is uniformly distributed over the range ($-\pi$, π) as shown in Fig. P. 4.5.3.

$$\therefore \quad f_\phi(\phi) = k \qquad \ldots -\pi \le (\phi) \le \pi$$

$$= 0 \qquad \ldots \text{elsewhere}$$

We know that the area under the PDF curve is always equal to 1.



(E-854) **Fig. P. 4.5.3 : PSD of ϕ**

$$\therefore \quad k \times 2\pi = 1$$

$$\therefore \quad k = \frac{1}{2\pi}$$

$$\therefore \quad f_\phi(\phi) = \frac{1}{2\pi} \qquad \ldots -\pi \le (\phi) \le \pi$$

$$= 0 \qquad \ldots \text{elsewhere}$$

Step 2 : Find the mean of X(t) :

The mean m_X (t) of the process X(t) is given by,

$$m_X(t) = E[X(t)] = \int_{-\infty}^{\infty} A\cos(\omega t + \phi) \cdot f_\phi(\phi)\, d\phi$$

$$\therefore \quad m_X(t) = \frac{A}{2\pi} \int_{-\pi}^{\pi} \cos(\omega t + \phi)\, d\phi = 0 \qquad \ldots(1)$$

Step 3 : Find the auto correlation R_{XX} (t, t + τ) :

$$R_{XX}(t, t+\tau) = E[X(t) \cdot X(t+\tau)]$$

$$= \int_{-\infty}^{\infty} \{A\cos(\omega t + \phi) \cdot A\cos[\omega(t+\tau) + \phi]\}\ f_\phi(\phi)\, d\phi$$

$$= \frac{A^2}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2}\cos[\omega(t+\tau) + \phi - \omega t - \phi]$$

$$+ \frac{1}{2}\cos[\omega(t+\tau) + \phi + \omega t + \phi]\, d\phi$$

$$= \frac{A^2}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2}[\cos(\omega t) + \cos(2\omega t + 2\phi + \omega t)]\, d\phi$$

The integration of the second term in the equation above is zero.

$$R_{XX}(t, t+\tau) = \frac{A^2}{2\pi} \times \frac{1}{2}[\cos \omega t]\ [\phi]_{-\pi}^{\pi}$$

$$= \frac{A^2}{2\pi} \times \frac{1}{2}\cos \omega t \times [2\pi]$$

$$= \frac{A^2}{2}\cos \omega t \qquad \ldots(2)$$

Conclusion :

- From Equation (1), it is clear that the mean m_x (t) is constant and Equation (2) indicates that the auto correlation is a function of time difference τ only. Hence we conclude that the given system is a wide sensed stationary system (WSS).

Ex. 4.5.4 : A random process X (t) = A cos (ω_c t + θ), where 'A' and ω_c are constants while 'θ' is a random variable with uniform pdf.

$$f_\theta\,\theta = \frac{1}{2\pi}, -\pi < \theta < \pi$$

1. Find mean, auto correlation function and psd of X (t). (Show that X (t) is WSS before finding psd).
2. Find auto-correlation function by time averaging and show that

$R_{XX}(\tau) >= R_{XX}(\tau)$.

Dec. 09. 9 Marks

Soln. :

Part I : To prove that X(t) is WSS :

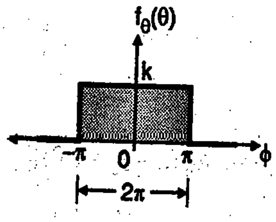
Step 1 : Find the PDF :

The random variable θ is uniformly distributed over the range ($-\pi$, π) as shown in Fig. P. 4.5.4.

$$\therefore \quad f_\theta(\theta) = k \quad \ldots -\pi \le \theta \le \pi$$

$$= 0 \quad \ldots \text{elsewhere}$$

We know that the area under the PDF curve is always equal to 1.



(E-855) **Fig. P. 4.5.4 : PSD of ϕ**

$$\therefore\ k \times 2\pi = 1$$

$$\therefore\ k = \frac{1}{2\pi}$$

$$\therefore\ f_\theta(\theta) = \frac{1}{2\pi} \qquad \ldots -\pi \le \theta \le \pi$$

$$= 0 \qquad \ldots \text{elsewhere}$$

Step 2 : Find the mean of X(t) :

The mean $m_x(t)$ of the process X(t) is given by,

$$M_X(t) = E[X(t)]$$

$$= \int_{-\infty}^{\infty} A\cos(\omega t + \theta) \cdot f_\theta(\theta)\, d\phi$$

$$\therefore\ m_X(t) = \frac{A}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t + \theta)\, d\theta$$

$$= 0 \qquad \ldots(1)$$

Step 3 : Find the auto correlation $R_{XX}(t, t+\tau)$:

$$R_{XX}(t, t+\tau) = E[X(t) \cdot X(t+\tau)]$$

$$= \int_{-\infty}^{\infty} \{A\cos(\omega t + \theta) \cdot A\cos[\omega(t+\tau) + \theta]\}\, f_\theta(\theta)\, d\phi$$

$$= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}\cos[\omega(t+\tau) + \theta - \omega t - \theta]$$

$$+ \frac{1}{2}\cos[\omega(t+\tau) + \theta + \omega t + \theta]\, d\theta$$

$$= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}[\cos(\omega t) + \cos(2\omega t + 2\theta + \omega t)]\, d\theta$$

The integration of the second term in the equation above is zero.

$$\therefore\ R_{XX}(t, t+\tau) = \frac{A^2}{2\pi} \times \frac{1}{2}[\cos \omega t]\,[\theta]_{-\pi}^{\pi}$$

$$= \frac{A^2}{2\pi} \times \frac{1}{2}\cos \omega t \times [2\pi]$$

$$= \frac{A^2}{2}\cos \omega t \qquad \ldots(2)$$

Conclusion :

- From Equation (1), it is clear that the mean $m_X(t)$ is constant and Equation (2) indicates that the auto correlation is a function of time difference τ only. Hence we conclude that the given system is a wide sensed stationary system (WSS).

Part II : Time averaged autocorrelation :

- The time averaged autocorrelation of the sample function is defined as,

$$\bar{R}_{XX}(\tau) = \langle X(t)\, X(t+\tau) \rangle$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, x(t+\tau)\, dt$$

$$= \frac{A^2}{T_0}\int_{-T_0/2}^{T_0/2} \cos(\omega t + \theta)\cos[\omega(t+\tau) + \theta]\, dt$$

$$= \frac{A^2}{2T_0}\int_{-T_0/2}^{T_0/2} [\cos(2\omega t + \omega\tau + 2\theta) + \cos(\omega\tau)]\, dt$$

$$= \frac{A^2}{2T_0}\int_{-T_0/2}^{T_0/2} \cos[2\omega t + \omega t + 2\theta]\, dt$$

$$+ \frac{A^2}{2T_0}\int_{-T_0/2}^{T_0/2} \cos(\omega\tau)\, dt$$

$$\therefore\ \bar{R}_{XX}(\tau) = \frac{A^2}{2T_0} \times 0 + \frac{A^2}{2T_0}\cos(\omega\tau)\,[t]_{-T_0/2}^{T_0/2}$$

$$= \frac{A^2}{2T_0}\cos(\omega\tau) \times T_0$$

$$\therefore\ \bar{R}_{XX}(\tau) = \frac{A^2}{2}\cos(\omega\tau) \qquad \ldots(3)$$

Comparing Equations (2) and (3), we get,

$$< R_{XX}(t) > = R_{XX}(t, t+\tau)$$

4.6 Cross Correlation Functions :

- Consider two random processes X (t) and Y (t). Let their autocorrelation functions be $R_X(t, u)$ and $R_Y(t, u)$ respectively.
- Then the two cross correlation functions of X (t) and Y (t) are defined as follows :

$$R_{XY}(t, u) = E[X(t)\, Y(u)] \qquad \ldots(4.6.1)$$

$$R_{YX}(t, u) = E[Y(t)\, X(u)] \qquad \ldots(4.6.2)$$

- Here t and u are the two instants of time at which the two processes are being observed.
- We can also display the correlation properties of the two random processes in the matrix form as follows :

$$R(t, u) = \underbrace{\begin{bmatrix} R_X(t, u) & R_{XY}(t, u) \\ R_{YX}(t, u) & R_Y(t, u) \end{bmatrix}}_{\text{Correlation matrix}} \qquad \ldots(4.6.3)$$

- This matrix is known as the correlation matrix, of the random processes X (t) and Y (t). Note that both these processes are wide sense stationary processes.
- In addition to this they are jointly wide sense stationary process as well. Then the correlation matrix can be written as follows :

$$R(\tau) = \begin{bmatrix} R_X(\tau) & R_{XY}(\tau) \\ R_{YX}(\tau) & R_Y(\tau) \end{bmatrix} \qquad \ldots(4.6.4)$$

Where $\tau = t - u$.

4.6.1 Properties of Cross Correlation Function :

Property 1 :

The cross correlation function is not generally an even function of τ.

Property 2 :

The cross correlation function does not have a maximum at $\tau = 0$.

Property 3 :

The cross correlation function obeys a certain symmetry relationship as given by the following expression.

$$R_{XY}(\tau) = R_{XY}(-\tau) \quad \text{...(4.6.5)}$$

4.6.2 Example of Cross Correlation :

- The example of cross correlation is the quadrature modulation process.
- Let $X_1(t)$ and $X_2(t)$ represent a pair of quadrature modulated processes. They are related to a wide sense stationary process $X(t)$ as shown below.

$$X_1(t) = X(t)\cos(2\pi f_c t + \theta)$$

$$X_2(t) = X(t)\sin(2\pi f_c t + \theta)$$

Where f_c is the carrier frequency and θ is a random variable which is uniformly distributed over the interval $(0, 2\pi)$. θ is independent of $X(t)$.

- One cross correlation function of $X_1(t)$ and $X_2(t)$ is expressed as,

$$R_{12}(\tau) = E[X_1(t)X_2(t-\tau)]$$

$$= E[X(t)X(t-\tau)\cos(2\pi f_c t + \theta)\sin(2\pi f_c t - 2\pi f_c \tau + \theta)]$$

$$= E[X(t)X(t-\tau)]\,E[\cos(2\pi f_c t + \theta)\sin(2\pi f_c t - 2\pi f_c \tau + \theta)] \quad \text{...(4.6.6)}$$

$$= \frac{1}{2}R_X(\tau)\,E[\sin(4\pi f_c t - 2\pi f_c \tau + 2\theta) - \sin(2\pi f_c \tau)]$$

$$= -\frac{1}{2}R_X(\tau)\sin(2\pi f_c \tau) \quad \text{...(4.6.7)}$$

- At $\tau = 0, \sin(2\pi f_c \tau) = 0$

$$\therefore R_{12}(0) = E[X_1(t)X_2(t)] = 0$$

- This means that the random variables which are obtained by observing the quadrature modulated process $X_1(t)$ and $X_2(t)$ at a certain value of t are orthogonal.

Ex. 4.6.1 : Two random processes X (t) and Y (t) are given by,

$$X(t) = A\cos(\omega t + \theta)$$

and $Y(t) = A\sin(\omega t + \theta)$

Where A and ω are constants and θ is a random variable having a uniform distribution over $[0, 2\pi]$. Find the cross correlation of X (t) and Y (t).

Dec. 06, Dec. 13, 8 Marks, May 11, 4 Marks

Soln. :

- The cross correlation is given by,

$$R_{XY}(\tau) = E[X(t)\,Y(t+\tau)]$$

$$= E[A\cos(\omega t + \theta)\cdot A\sin(\omega t + \omega\tau + \theta)]$$

$$= E[A^2\cos(\omega t + \theta)\sin(\omega t + \theta + \omega\tau)]$$

$$= \frac{A^2}{2}E[\sin(2\omega t + 2\theta + \omega\tau) - \sin(-\omega\tau)]$$

$$= \frac{A^2}{2}E[\sin(2\omega t + 2\theta + \omega\tau)] + \frac{A^2}{2}E[\sin(\omega\tau)]$$

$$= \frac{A^2}{2}E[\sin(\omega\tau)]$$

$$= \frac{A^2}{2}\sin\omega\tau \quad \text{...(1)}$$

This is the cross correlation.

- Similarly we can obtain $R_{YX}(\tau)$ as follows :

$$R_{YX}(\tau) = E[Y(t)X(t+\tau)] = -\frac{A^2}{2}E[\sin(\omega\tau)]$$

$$= -\frac{A^2}{2}\sin(\omega\tau) \quad \text{...(2)}$$

4.7 Ergodicity :

SPPU : Dec. 06. May 09. May 12. May 13. May 15. May 16

University Questions

Q. 1 What are the conditions for a random process to be wide sense stationary. What is ergodicity ? **(Dec. 06, 6 Marks)**

Q. 2 Explain stationary random processes, non-stationary random processes, and wide sense stationary processes and Ergodic processes, with help of mathematical expression. **(May 09, 8 Marks)**

Q. 3 Explain in detail about stationary, non stationary, wide sense stationary and ergodic processes with suitable mathematical expressions and examples. **(May 12, 8 Marks)**

Q. 4 Explain stationary, non stationary, wide sense stationary and Ergodic processes with the help of mathematical expression. **(May 13, 8 Marks)**

Q. 5 Explain ergodic process if $X(t) = A \cos(2\pi f_c t + \phi)$ is random process with ϕ is a random variable uniformly distributed over $(0, 2\pi)$. Prove that $x(t)$ is ergodic in mean. **(May 15, 6 Marks)**

Q. 6 Explain in detail about stationary, wide sense stationary and ergodic process with suitable mathematical expressions. **(May 16, 6 Marks)**

- Earlier we have defined the ensemble average and time average of a random process.
- Consider the sample function x (t) of a wide-sense stationary process X (t). Let the observation interval be $-T \le t \le T$. Then the time average or dc value of the sample function x (t) is given by

  Time average :

$$m_X(T) = \frac{1}{2T}\int_{-T}^{T} x(t)\,dt \quad ...(4.7.1)$$

- The value of time average m_X (T) depends on the interval in which it is being observed and which sample function x (t) of the random process X (t) is being chosen.
- Therefore the time average m_X (T) itself is a random variable.
- The mean value of the time average m_x (T) is given by,

$$E[m_X(T)] = E\left[\frac{1}{2T}\int_{-T}^{T} x(t)\,dt\right] \quad ...(4.7.2)$$

- Interchanging the operation of expectation and integration we get,

$$E[m_X(T)] = \frac{1}{2T}\int_{-T}^{T} E[x(t)]\,dt \quad ...(4.7.3)$$

- The term E [x (t)] represents the mean value of x (t) and when we integrate it over the interval $-T \le t \le T$ and divide it by 2T we obtain the mean of the random process X (t) i.e. m_X or the ensemble mean.

$$\therefore \; E[m_X(T)] = m_X \quad ...(4.7.4)$$

Conditions for Ergodicity in mean :

The process X (t) is said to **ergodic in the mean** if the following conditions are satisfied :

1. The value of time average m_X (T) becomes equal to the ensemble average m_X when the observation interval T is tending to infinity.

   That means,

(E-823)

$$\underbrace{\lim_{T\to\infty}}_{\text{Observation interval tends to }\infty} \underbrace{m_X(T)}_{\text{Time average}} = \underbrace{m_X}_{\text{Ensemble average}} \quad ...(4.7.5)$$

2. The variance of m_X (T) which is treated as a random variable, will become zero with the observation interval T tending to infinity.

   That means

$$\lim_{T\to\infty} \operatorname{var}[m_X(T)] = 0 \quad ...(4.7.6)$$

Time average autocorrelation function :

- The other important time average is the autocorrelation function $R_x(\tau, T)$. It is defined for the sample function x (t) and over the observation interval $-T \le t \le T$ as follows :

$$R_X(\tau, T) = \frac{1}{2T}\int_{-T}^{T} x(t+\tau)\,x(t)\,dt \; ...(4.7.7)$$

- This time average also is a random variable. It has its own mean value and variance.
- Now we can define the ergodicity in terms of autocorrelation function, in a similar manner in which we defined the ergodicity in terms of mean.

Conditions for ergodicity in the autocorrelation function :

The process X (t) is said to be **ergodic in autocorrelation function**, if the following conditions are satisfied :

1. $\lim\limits_{T\to\infty} R_X(\tau, T) = R_X(\tau)$

   That means the time averaged autocorrelation function becomes equal to the autocorrelation function of the system with observation interval T tending to ∞.

2. $\lim\limits_{T\to\infty} \operatorname{Var}[R_X(\tau, T)] = 0$

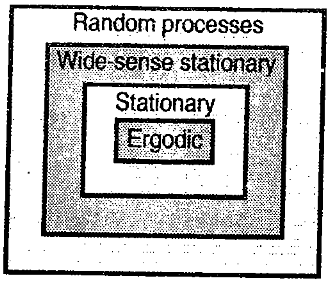
   That means the variance of $R_X(\tau, T)$ which is a random variable, becomes equal to zero with the observation interval T tending to ∞.

Conclusions on Ergodicity of a process :

- When all the ensemble averages become equal to the corresponding time averages the random process is called as an "ergodic" process. Thus for an ergodic process,

  Time average = Ensemble average

- Similarly if the time autocorrelation function is equal to the autocorrelation function then the system is ergodic.



(E-824) **Fig. 4.7.1 : Classification of random processes**

- The ensemble average and autocorrelation are just two of the many possible averages. For an ergodic process all the possible ensemble averages are equal to the corresponding time averages of one of its sample functions.
- Because the time average cannot be a function of time and time average is equal to the ensemble average, that means the ensemble averages also will be independent of time.
- Therefore the ergodic processes will be stationary processes, but the converse is not true. This is as shown in Fig. 4.7.1.

Ex. 4.7.1 : If $X(t) = A\cos(2\pi f_c t + \phi)$ is random process with ϕ is a random variable uniformly distributed over $(0, 2\pi)$. Prove that x(t) is ergodic in mean. **Dec. 10. May 13. Dec. 13, 8 Marks, May 15. 6 Marks**

Soln. :

Given : Random process $X(t) = A\cos(2\pi f_c t + \phi)$

$$f_\phi(\phi) = k \qquad (0, 2\pi)$$
$$= 0 \qquad \text{elsewhere}$$

The given random process will be ergodic in mean if

$$<x(t)> \; = m_x(t) \qquad \ldots(1)$$

Where

$$<x(t)> = \lim_{t\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} A\cos(2\pi f_c t + \phi)\,dt \qquad \ldots(2)$$

ϕ is a random variable uniformly distributed over $(0, 2\pi)$.

$$\therefore \; f_\phi(\phi) = \frac{1}{b-a} \qquad 0 \le \phi \le 2\pi = \frac{1}{2\pi - 0}$$

$$\therefore \; f_\phi(\phi) = \frac{1}{2\pi} \qquad \ldots(3)$$

Substituting this into Equation (2) we get,

Find < x (t) > :

$$<x(t)> = \lim_{t\to\infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} A\cos(2\pi f_c t + \phi)\,dt$$

$$\text{But } \frac{1}{T_0} = f_c$$

$$\therefore \; <x(t)> = \lim_{t\to\infty} \frac{1}{T_0}\left[\frac{A\sin(2\pi f_c t + \phi)}{2\pi f_c}\right]_{-T_0/2}^{T_0/2}$$

$$= \lim_{t\to\infty} \frac{1}{T_0}\,\frac{A}{2\pi f_c}$$

$$\left[\sin\frac{2\pi}{T_0}\cdot\frac{T_0}{2} - \sin\left(\frac{2\pi}{T_0}\times\frac{-T_0}{2}\right)\right]$$

$$\therefore \; <x(t)> = \lim_{t\to\infty} \frac{A}{2\pi}[\sin\pi + \sin\pi] = 0 \qquad \ldots(4)$$

Find $m_x(t)$:

$$m_x(t) = \int_0^{2\pi} A\cos(2\pi f_c t + \phi) \times f_\phi(\phi)\,d\phi$$

$$= \int_0^{2\pi} A\cos(2\pi f_c t + \phi) \times \frac{1}{2\pi}\,d\phi$$

$$= \frac{A}{2\pi}\left[\sin(2\pi f_c t + \phi)\right]_0^{2\pi} = 0 \qquad \ldots(5)$$

Since $<x(t)> = m_x(t)$ the given random system is ergodic in mean.

4.8 Transmission of a Random Process through a Linear Filter :

SPPU : May 06. May 09. Dec. 09. Dec. 10. May 11. Dec. 11

University Questions

Q. 1 Show that if a wide sense stationary process X(t) is passed through a LTI filter with impulse response h(t) then its output has constant mean square value. **(May 06, May 09, 8 Marks)**

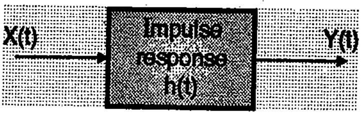
Q. 2 State properties of auto-correlation function. Show that when wide sense stationary process passed through a LTI filter with impulse response h(t) produces constant mean-square value. **(Dec. 09, 9 Marks)**

Q. 3 Find the mean square value of output random process when a WSS process is passed through an LTI filter. **(Dec. 10, 8 Marks)**

Q. 4 Show that the output of LTI system is WSS if the input applied to it is WSS. **(May 11, 6 Marks)**

Q. 5 Show that if the wide sense stationary process X(t) is passed through LTI system with impulse response h(t), then its output has constant mean square value. **(Dec. 11, 8 Marks)**

- Assume that a linear process X (t) is applied as input to a linear time invariant (LTI) filter. Let the impulse response of such filter be h (t).
- At the output of this filter let a new random process Y (t) be produced as shown in Fig. 4.8.1.



(E-825) **Fig. 4.8.1 : Transmission of random process through an LTI filter**

- Even if the distribution of X (t) is known to us, it is difficult to describe the probability of distribution of Y (t).
- Let us obtain the mean and autocorrelation function of the output process Y (t). We assume that the input process X (t) is a wide sense stationary process.

First mean of output random process Y (t) :

- The first mean of Y (t) is defined as,

$$m_Y(t) = E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(\tau_1)\, X(t-\tau_1)\, d\tau_1\right]$$

where τ_1 is a dummy variable.

- We can interchange the order of the expectation and integration with respect to τ_1 provided that [X (t)] is finite for all t and the system is stable, to write

$$m_Y(t) = \int_{-\infty}^{\infty} h(\tau_1)\, E[X(t-\tau_1)]\, d\tau_1$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, m_X(t-\tau_1)\, d\tau_1 \qquad ...(4.8.1)$$

- If the input random process X (t) is a wide sensed stationary process then m_X (t) will be a constant equal to m_X. So Equation (4.8.1) is simplified as follows :

$$m_Y = m_X \int_{-\infty}^{\infty} h(\tau_1)\, d\tau_1 \qquad ...(4.8.2)$$

$$\therefore \quad m_Y = m_X H(0) \qquad ...(4.8.3)$$

Where H (0) is the zero frequency (dc) response of the system.

- From Equation (4.8.3) we conclude that the mean of the random process Y (t) obtained at the output of LTI system in response to X (t) applied at the input is equal to the product of mean of X (t) and the dc response of the system.

Autocorrelation function of Y (t) :

- The autocorrelation function of Y (t) is given by,

$$R_Y(t, u) = E[Y(t)\, Y(u)]$$

where t and u are the two values of time at which the output process is observed.

- The autocorrelation function of Y (t) can be obtained by using the convolution integral as follows.

$$R_Y(t, u) = E\left[\int_{-\infty}^{\infty} h(\tau_1)\, X(t-\tau_1)\, d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)\, X(u-\tau_2)\, d\tau_2\right] \qquad ...(4.8.4)$$

- If the system X (t) is stable and if the mean square value E [X^2 (t)] is finite for all t, it is possible to interchange the order of expectation and integration

$$R_Y(t, u) = \int_{-\infty}^{\infty} h\, \tau_1\, h(\tau_1) \int_{-\infty}^{\infty} d\tau_2\, h(\tau_2)\; E[X(t-\tau_1)\, X(u-\tau_2)]$$

$$= \int_{-\infty}^{\infty} d\tau_1\, h(\tau_1) \int_{-\infty}^{\infty} d\tau_2\, h(\tau_2)\; R_X(t-\tau_1, u-\tau_2) \qquad ...(4.8.5)$$

- When the input process X (t) is a wide sense stationary random process, the autocorrelation function of X (t) will only be the function of the difference $(t-\tau_1)$ and $(u-\tau_2)$.
- Therefore we will substitute $\tau = (t - u)$ in Equation (4.8.5) we get

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\, h(\tau_2)\, R_X(\tau-\tau_1+\tau_2)\, d\tau_1\, d\tau_2 \qquad ...(4.8.6)$$

- If we look at the expressions of m_Y (t) and R_Y (τ), then we can conclude that if the input to a stable LTI filter X (t) is a wide sense stationary process, then the output of the filter Y (t) also is a wide sense stationary random process.

Mean square value of output process :

- Since $R_Y(0) = E[Y^2(t)]$ i.e. the mean square value of Y (t), we can obtain the expression for E [Y^2 (t)] by substituting $\tau = 0$ into Equation (4.8.6) as,

$$E[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\, h(\tau_2)\, R_X(\tau_2-\tau_1)\, d\tau_1\, d\tau_2 \qquad ...(4.8.7)$$

- The mean square value of Y(t) will be constant.

4.9 Power Spectral Density :

SPPU : Dec. 07. May 08. May 12

University Questions

Q. 1 What is power spectral density ? Derive the expression of PSD when a random process is transmitted through a LTI filter. **(Dec. 07, 8 Marks)**

Q. 2 Define the terms related to random processes : Power spectral density **(May 08, 2 Marks)**

Q. 3 Define the power spectral density and auto-correlation function of periodic signals. Show that both are related in frequency domain. **(May 12, 8 Marks)**

- So far we have considered the characteristics of wide-sense stationary random process in linear systems in the time domain.

- Let us now characterize the wide-sense stationary random process in the linear systems however in the frequency domain now.
- In this section we will obtain the mean squared value of the filter output in frequency domain, when a random process X (t) has been applied at its input.
- Let the impulse response of a LTI filter is equal to the IFT of the transfer function H (f).

$$\therefore \quad h(\tau_1) = \int_{-\infty}^{\infty} H(f)\cdot e^{j2\pi f\tau_1}\, df \quad ...(4.9.1)$$

- Substitute this expression into Equation (4.8.7) to get,

$$E[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} H(f)\, e^{j2\pi f\tau_1} df\right] h(\tau_2)\, R_X(\tau_2-\tau_1)\, d\tau_1\, d\tau_2$$

Rearranging we get,

$$E[Y^2(t)] = \int_{-\infty}^{\infty} df H(f) \int_{-\infty}^{\infty} d\tau_2\, h(\tau_2) \int_{-\infty}^{\infty} R_X(\tau_2-\tau_1)\, e^{j2\pi f\tau_1}\, d\tau_1 \quad ...(4.9.2)$$

- Let $\tau = (\tau_2 - \tau_1)$, $\therefore \; \tau_1 = \tau_2 - \tau$ and $d\tau_1 = -d\tau$

$\therefore$ Equation (4.9.2) will get modified as follows :

$$E[Y^2(t)] = \int_{-\infty}^{\infty} df\, H(f) \int_{-\infty}^{\infty} d\tau_2\, h(\tau_2) \int_{-\infty}^{\infty} R_X(\tau)\cdot e^{j2\pi f(\tau_2-\tau)}\, d\tau$$

$$= \int_{-\infty}^{\infty} df\, H(f) \int_{-\infty}^{\infty} d\tau_2\, h(\tau_2)\, e^{j2\pi f\tau_2} \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j2\pi f\tau}\, d\tau \quad ...(4.9.3)$$

- But $\int_{-\infty}^{\infty} h(\tau_2)\cdot e^{j2\pi f\tau_2}\, d\tau_2 = H^*(f)$

$$\therefore E[Y^2(t)] = \int_{-\infty}^{\infty} df\, H(f)\, H^*(f) \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

But $H(f)\, H^*(f) = |H(f)|^2$

$$\therefore E[Y^2(t)] = \int_{-\infty}^{\infty} df\, |H(f)|^2 \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j2\pi f\tau}\, d\tau \quad ...(4.9.4)$$

- In this expression the last term is the Fourier transform of the autocorrelation function $R_X(\tau)$ of the input random process X (t). That means,

$$\therefore \quad X(t) \overset{F}{\longleftrightarrow} \int_{-\infty}^{\infty} R_X(\tau)\cdot e^{-j2\pi f\tau}\, d\tau$$

- Let us introduce a new parameter for the R.H.S. of this expression as $S_X(f)$ as,

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j2\pi f\tau}\, d\tau \quad ...(4.9.5)$$

- The parameter $S_X(f)$ is known as the power spectral density or power spectrum of the wide sensed stationary process X (t).

$$\therefore E[Y^2(t)] = \int_{-\infty}^{\infty} |H(f)|^2\, S_X(f)\, df \quad ...(4.9.6)$$

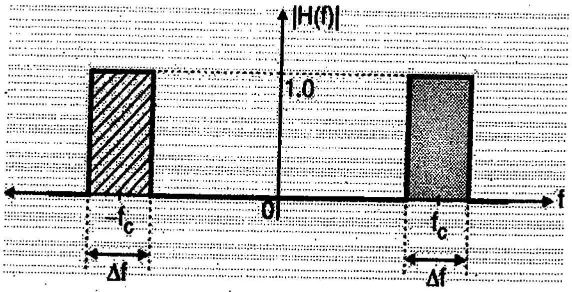
Conclusion :

Equation (4.9.6) tells us that the mean square value $E[Y^2(t)]$ of the output of a LTI filter for a wide sense stationary process X (t) applied at its input, is equal to the integration of psd and squared magnitude of transfer function in the frequency domain over the entire frequency range $-\infty$ to $+\infty$.

4.9.1 Physical Significance of PSD :

- Assume that the random process X (t) is passed through an ideal narrowband filter whose amplitude response is shown in Fig. 4.9.1 and expressed mathematically as,

$$|H(f)| = \begin{cases} 1 & |f \pm f_c| < \frac{1}{2}\Delta f \\ 0 & |f \pm f_c| > \frac{1}{2}\Delta f \end{cases} \quad ...(4.9.7)$$



(E-826) **Fig. 4.9.1 : Amplitude response of ideal narrowband filter**

Where Δf represents the bandwidth of the filter.

- Consider the expression for $E[Y^2(t)]$.

$$E[Y^2(t)] = \int_{-\infty}^{\infty} |H(f)|^2\, S_X(f)\, df$$

- In this expression we will assume that Δf is very small as compared to f_c and $S_X(f)$ is a continuous function of frequency. Hence,

$$E[Y^2(t)] \approx \int_{f_c - \frac{1}{2}\Delta f}^{f_c + \frac{1}{2}\Delta f} |H(f)|^2 S_X(f)\, df$$

$$\approx 2\,\Delta f\, S_X(f_c) \qquad ...(4.9.8)$$

- We know that the filter will pass only those frequency components of the input random process X (t) which reside inside the narrow passband of the filter which has a width Δf centered about the frequency $\pm f_c$.
- Therefore S_X (f) represents the frequency density of the average power in the random process X (t) calculated at $f = f_c$.
- The dimensions of psd are therefore Watts per Hz.

4.9.2 Properties of PSD :

In this section we will define and prove some of the important properties of PSD.

Einstein - Wiener - Khintchine relations :

- The psd S_X (f) and the autocorrelation R_X (τ) of a wide sense random process X (t) always form a Fourier - transform pair as follows :

$$\therefore\ R_X(\tau) \underset{\leftrightarrow}{F} S_X(f).$$

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \cdot e^{-j2\pi f\tau}\, d\tau \qquad ...(4.9.9)$$

and $$R_X(\tau) = \int_{-\infty}^{\infty} S_X(\tau) \cdot e^{j2\pi f\tau}\, df \qquad ...(4.9.10)$$

- These expressions are called as the Einstein - Wiener - Khintchine relations.
- These relations indicate that

$$S_X(f) \xleftrightarrow{F} R_X(\tau)$$

Hence if one of them is known, then we can obtain the value of the other.

Property 1 :

Statement : The value of psd of a wide sense stationary process at f = 0 is equal to the total area under the curve of autocorrelation function. That means,

$$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau)\, d\tau \qquad ...(4.9.11)$$

Proof :

We know that, $$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \cdot e^{-j2\pi f\tau}\, d\tau$$

Substitute f = 0 to get,

$$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau)\, e^0\, d\tau$$

$$\therefore\ S_X(0) = \int_{-\infty}^{\infty} R_X(\tau)\, d\tau \qquad \textbf{...Proved.}$$

Property 2 :

Statement : The mean square value of a wide-sense stationary random process is equal to the total area under the curve of power spectral density. That means,

$$E[X^2(t)] = \int_{-\infty}^{\infty} S_X(f)\, df \qquad ...(4.9.12)$$

Proof :

- We know that

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) \cdot e^{j2\pi f\tau}\, df$$

- Substitute τ = 0 into this expression to get,

$$R_X(0) = \int_{-\infty}^{\infty} S_X(f)\, e^0\, df$$

$$\therefore\ R_X(0) = \int_{-\infty}^{\infty} S_X(f)\, df \qquad ...(4.9.13)$$

- We also know that, $E[X^2(t)] = R_X(0)$
- Therefore,

$$E[X^2(t)] = \int_{-\infty}^{\infty} S_X(f)\, df \qquad \textbf{...Proved.}$$

Property 3 :

Statement : The psd of a wide sense random process will always be non-negative for all the values of " f ". That means,

$$S_X(f) \geq 0 \text{ for all f} \qquad ...(4.9.14)$$

Proof :

- The psd $S_X(f) = \dfrac{E[Y^2(t)]}{2\Delta f}$ ...from Equation (4.9.14)
- The mean square value $E[Y^2(t)]$ will always be non-negative. Δf will also be non-negative. Therefore S_X (f) also will be a non-negative function for all the values of f.

Property 4 :

Statement : The psd of a real valued random process is an even function of frequency. That means,

$$S_X(-f) = S_X(f) \qquad ...(4.9.15)$$

Proof : We know that,

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \cdot e^{-j2\pi f\tau} d\tau$$

Substitute $-f$ for f to get,

$$S_X(-f) = \int_{-\infty}^{\infty} R_X(\tau) \cdot e^{j2\pi f\tau} d\tau$$

Now substitute $-\tau$ for τ to get,

$$S_X(-f) = \int_{-\infty}^{\infty} R_X(-\tau) \cdot e^{-j2\pi f\tau} d\tau$$

But $R_X(-\tau) = R_X(\tau)$ because it is an even function.

$$\therefore \quad S_X(-f) = \int_{-\infty}^{\infty} R_X(\tau) \cdot e^{-j2\pi f\tau} d\tau$$

But the R.H.S. is nothing but $S_X(f)$.

$$\therefore \quad S_X(-f) = S_X(f) \qquad \textbf{...Proved.}$$

Property 5 :

If the psd is properly normalized, then it will have all the properties that are usually associated with a probability density function (pdf).

Explanation :

- The normalization of PSD is done with respect to the total area, covered by the graph of power spectral density. (This is same as the mean square value of the process).
- Consider the following function, which represents the normalized form of PSD.

$$P_x(f) = \frac{S_X(f)}{\int_{-\infty}^{\infty} S_X(f)\, df} \qquad ...(4.9.16)$$

- Note that the area under the function $P_x(f) = 1$ and using the properties 2 and 3 we note that $P_x(f) \geq 0$.
- Therefore the normalised form of power spectral density as defined by Equation (4.9.16) above will behave similar to a PDF.

Ex. 4.9.1 : The output of an oscillates is described by $x(t) = A \cos(\pi Ft + \theta)$

Where the amplitude A is constant and F and ϕ are independent random variables. The probability density function of θ is defined by

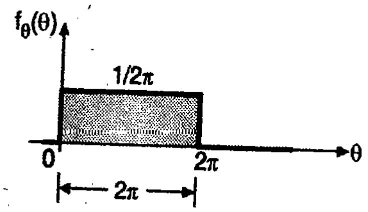
$$f_\phi(\theta) = \begin{cases} \frac{1}{2\pi}, 0 \leq \theta \, 2\pi \text{ otherwise} \\ 0 \end{cases}$$

Find the power spectral density of x(t) in terms of the probability density function of the frequency F. **Dec. 15. 8 Marks**

Soln. :

Step 1 : Draw the PDF :

The PDF is as shown in Fig. P. 4.9.1.



(E-1608) **Fig. P. 4.9.1 : PDF of θ**

Step 2 : Find the auto correlation $R_{XX}(t, t+\tau)$:

$$R_{XX}(t, t+\tau) = E[X(t) \cdot X(t+\tau)]$$

$$= \int_{-\infty}^{\infty} \{A\cos(\omega_c t+\theta) \cdot A\cos[\omega_c(t+\tau)+\theta]\} f_\theta(\theta)\, d\phi$$

$$= \frac{A^2}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2}\cos[\omega_c(t+\tau)+\theta-\omega_c t-\theta]$$

$$+ \frac{1}{2}\cos[\omega_c(t+\tau)+\theta+\omega_c t+\theta]\, d\theta$$

$$= \frac{A^2}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2}[\cos(\omega_c t) + \cos(2\omega_c t + 2\theta + \omega_c t)]\, d\theta$$

The integration of the second term in the equation above is zero.

$$\therefore \quad R_{XX}(t, t+\tau) = \frac{A^2}{2\pi} \times \frac{1}{2}[\cos\omega_c t]\,[\theta]_{-\pi}^{\pi}$$

$$= \frac{A^2}{2\pi} \times \frac{1}{2}\cos\omega_c t \times [2\pi]$$

$$= \frac{A^2}{2}\cos\omega_c t$$

Step 3 : Find PSD $S_{XX}(f)$:

We know that

$$R_{XX} \overset{F}{\longleftrightarrow} S_{XX}(f)$$

$$\therefore \text{ PSD } S_{XX}(f) = F[R_{XX}(t, t+\tau)]$$

$$= \frac{\pi A^2}{2} [\delta(f+f_c) + \delta(f-f_c)] \qquad \text{...Ans.}$$

4.10 Gaussian Process :

- In this section we will discuss an important family of random processes known as Gaussian processes.
- Let the observation interval in which we will observe a random process X(t) be t = 0 to t = T.
- Let us weight the random process X(t) by a function g(t), and obtain another random variable Y by integrating the product g(t) X(t) over the interval 0 to T.
- The random variable is given by,

$$Y = \int_0^T g(t)\, X(t)\, dt \qquad \text{...(4.10.1)}$$

- Y is the linear functional of X(t).

Definition :

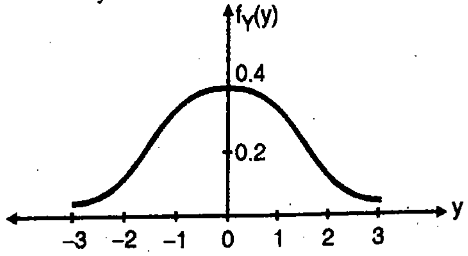
- If in the expression for Y, the weighing function g(t) is such that the mean square value of the random variable Y is a Gaussian distributed random variable for every g(t), then the process X(t) is called as a **Gaussian process.**
- In simple words the process X(t) is a Gaussian process if every linear functional of X(t) is a Gaussian random variable.
- A random variable is called as Gaussian random variable if its probability density function (PDF) is of the following nature.

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_Y} e^{-(y-m_y)^2/2\sigma_y^2} \qquad \text{...(4.10.2)}$$

where m_y = Mean of random variable Y.

σ_Y^2 = Variance

- The plot of Gaussian PDF is shown in Fig. 4.10.1. Note that the Gaussian random variable has been normalised to get the mean at $m_y = 0$ and variance $\sigma_y^2 = 1$.



(E-827) **Fig. 4.10.1 : Normalized PDF curve of a Gaussian variable**

- The normalized Gaussian PDF can be mathematically expressed by substituting $m_y = 0$ and $\sigma_y^2 = 1$ into Equation (4.10.2) as follows :

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-(y^2/2)} \qquad \text{...(4.10.3)}$$

- The normalized Gaussian PDF is denoted as N (0, 1).

Merits of Gaussian Process :

The Gaussian process has two main advantages :

1. We can obtain the analytical results for this process because Gaussian process exhibits a variety of properties.
2. The random processes produced by the physical phenomenon are generally such that we can use the Gaussian model to explain them properly.

4.10.1 Central Limit Theorem : SPPU : May 07

University Questions

Q. 1 State and explain central limit theorem. **(May 07, 4 Marks)**

- We can use the Gaussian process as a model to represent a large number of physical phenomenons.
- The central limit theorem is used for providing the mathematical justification for this fact.
- Let X_i be a set of random variables with i = 1, 2... N. Let all these random variables satisfy the following requirements.
  1. All the random variables are statistically independent.
  2. Let all of them have the same PDF with mean m_X and variance σ_X^2.
- Let the random variables X_i be normalized in the following way,

$$Y_i = \frac{1}{\sigma_X}(X_i - m_X), i = 1, 2, ..., N$$

so that the mean $E[Y_i] = 0$ and the variance $\text{Var}[Y_i] = 1$.

- We can then define a random variable V_N in the following manner,

$$V_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Y_i \qquad \text{...(4.10.4)}$$

Statement of Central Limit theorem :

The central limit theorem states that the probability distribution of the random variable V_N will become identical to the normalized Gaussian distribution N (0, 1) in the limit as N tends to infinity.

Note : Note that if N is finite, sometimes the approximation provided by the central limit theorem is of poor quality even though N is large.

4.10.2 Properties of a Gaussian Process :

The properties of a Gaussian process are as follows :

Property 1 :

If a Gaussian process X (t) is applied to a stable linear filter then the process Y (t) obtained at the filter output is also Gaussian.

Property 2 :

Consider the set of random variables or samples $X(t_1), X(t_2) \ldots X(t_n)$ obtained by observing a random process X (t) at the time instants $t_1, t_2, \ldots t_n$. If the process X (t) is Gaussian, then these random variables also are jointly Gaussian for any value of n. Their joint PDF is completely determined by specifying the set of means as follows :

$$m_X(t_i) = E[X(t_i)], \; i = 1, 2, \ldots n.$$

and the set of autocovariance functions, that means

$$C_X(t_k, t_i) = \left[E(X(t_k) - m_{X(t_k)})(X(t_i) - m_{X(t_i)}) \right] k,$$

$$i = 1, 2, \ldots n$$

Property 3 :

If a Gaussian process is wide sense stationary, then the process is also stationary in strict sense.

Property 4 :

If the random variables $X(t_1), X(t_2), \ldots, X(t_n)$ obtained by observing a Gaussian process X (t) at the time instants $t_1, t_2, \ldots, t_n$ are uncorrelated, that means if,

$$E[(X(t_k) - m_{X(t_k)})(X(t_i) - m_{X(t_i)})] = 0 \quad i \neq k$$

Then these random variables are statistically independent.

Ex. 4.10.1 : A voltage V(t) which is a Gaussian Ergodic random process with a mean of zero and a variance of 4 Volts^2, is measured by a dc meter, a true RMS meter and a meter which first squares V(t) and then reads its dc component. Find the output of each meter.

Dec. 14. 6 Marks

Soln. :

Given : Mean $m_x = 0$ and variance = 4 Volts^2.

1. **DC meter reading :**

The dc meter will indicate the average or mean value.

$\therefore$ DC meter reading = 0 V.

2. **True rms meter reading :**

$$\text{Variance} = E[x^2] - m_x^2$$

$$\text{But } m_x = 0$$

$$\therefore \text{Variance} = E[x^2] = \text{Mean square value}$$

$$\therefore \text{True rms meter reading} = \text{Rms value}$$

$$= \sqrt{\text{Mean square value}}$$

$$= \sqrt{\text{Variance}} = \sqrt{4} = 2 \text{ Volts}$$

3. The reading of a meter which first squares V(t) and then reads its dc component will be equal to $E[x^2] = 4V$.

4.11 Noise :

- Noise is an unwanted electrical disturbance which gives rise to audiable or visual disturbances in the communication systems and errors in the digital communication.
- It is possible to control the noise but it is not possible to completely eliminate it.
- There are many potential sources of noise. The noise may be generated internally by the devices and components or it may be generated by some external source.
- The external noise include atmospheric noise, galactic noise, man made noise etc.

Fundamental or internal sources of noise :

- The fundamental sources of noise are within the electronic equipment. They are called fundamental sources because they are the integral part of the physical nature of the material used for making electronic components.
- This type of noise follows certain rules. Therefore it can be eliminated by properly designing the electronic circuits and equipments.

4.11.1 Types of Noise :

- The fundamental noise sources produce different types of noise. They are as follows :
  1. Thermal noise
  2. Shot noise
  3. Partition noise
  4. Low frequency or flicker, noise.
  5. High frequency or transit time, noise.
  6. Avalanche noise.
  7. Burst noise.

  Let us know them one by one.
- The two most common types of them are thermal noise and shot noise.

4.11.2 Thermal Noise or Johnson Noise :

SPPU : May 16

University Questions

Q. 1 Write short notes on : Thermal noise or Johnson noise. **(May 16, 3 Marks)**

- The free electrons within a conductor are always in random motion.
- This random motion is due to the thermal energy received by them. The distribution of these free electrons within a conductor at a given instant of time is not uniform.
- It is possible that an excess number of electrons may appear at one end or the other of the conductor.

- The average voltage resulting from this non-uniform distribution is zero but the average power is not zero.
- As this power has appeared as a result of the thermal energy, it is called as the "thermal noise power".
- The average thermal noise power is given by,

$$P_n = k\,TB \text{ Watts} \qquad ...(4.11.1)$$

Where, k = Boltzmann's constant

= 1.38×10^{-23} Joules/Kelvin.

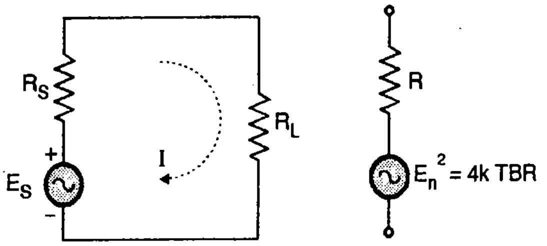
B = Bandwidth of the noise spectrum (Hz).

T = Temperature of the conductor, °Kelvin

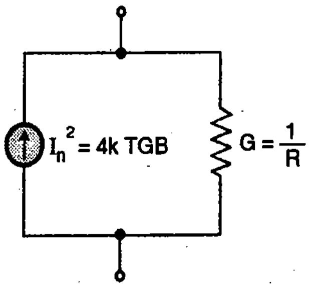
- Equation (4.11.1) indicates that a conductor operated at a finite temperature can work as a generator of electrical energy.
- The thermal noise power P_n is proportional to the noise BW and conductor temperature.

Equivalent circuit :

- As the conductor is generating an electrical noise energy, it must have an equivalent voltage or equivalent current generator circuit which will represent the noise source.
- The voltage generator and the current generator equivalent circuits are as shown in Figs. 4.11.1(a) and (b) respectively.



(a) A voltage generator for a noise source **(b) Voltage generator equivalent circuit**



(c) A current generator equivalent circuit for a noise source

(D-321) **Fig. 4.11.1**

Consider the circuit shown in Fig. 4.11.1(a).

- Assume that the rms source voltage is E_s. Then the maximum power is delivered to the load resistance R_L when $R_L = R_S$.

$$P_{L\,max} = \frac{E_s^2}{4\,R_S} \qquad ...(4.11.2)$$

- Applying the same logic to a conductor which is generating noise, we can write the expression for the noise power P_n as :

$$P_n = \frac{E_n^2}{4\,R} \qquad ...(4.11.3)$$

Where, R = Resistance of the conductor and

E_n = RMS noise voltage.

- But referring to Equation (4.11.1)

$$P_n = kTB$$

$$\therefore\ kTB = \frac{E_n^2}{4\,R}$$

$$\therefore\ E_n = \sqrt{4\,k\,TBR} \qquad ...(4.11.4)$$

- This is the expression for the rms value of the thermal noise voltage.
- Hence a conductor of resistance R which generates a noise voltage E_n can be represented as a voltage source E_n with internal resistance R as shown in Fig. 4.11.1(b).
- The current equivalent circuit for the thermal noise is as shown in Fig. 4.11.1(c). It can be obtained by using the same logic as that for the voltage equivalent circuit. The rms noise current is given by,

$$I_n = \sqrt{4\,GkTB} \qquad ...(4.11.5)$$

Where G = Conductance = $\frac{1}{R}$

4.11.3 Shot Noise :

- The shot noise is produced due to shot effect. Due to the shot effect, shot noise is produced in all the amplifying devices or for that matter in all the active devices.
- The shot noise is produced due to the random variations in the arrival of electrons (or holes) at the output electrode of an amplifying device.
- Therefore it appears as a randomly varying noise current superimposed on the output. The shot noise "sounds" like a shower of lead shots falling on a metal sheet if amplified and passed through a loud speaker.
- The shot noise has a uniform spectral density like thermal noise. The exact formula for the shot noise can be obtained only for diodes.
- For all other devices an approximate equation is stated. The mean square shot noise current for a diode is given as,

$$I_n^2 = 2\,I_{dc}\,q\,B \text{ Amperes}^2 \qquad ...(4.11.6)$$

Where, I_{dc} = Direct current across the junction (in Amp)

q = Electron charge = 1.6×10^{-19} C.

B = Effective noise bandwidth in Hz.

For the amplifying devices the shot noise is :

1. Inversely proportional to the transconductance of the device.

2. Directly proportional to the output current.

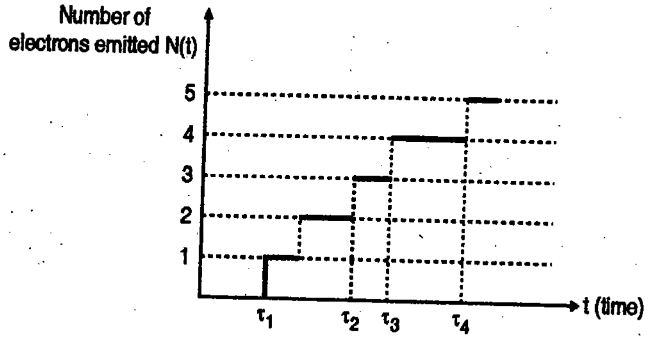
- Shot noise is present in the devices such as diodes, transistors, photo detectors etc. because of the discrete nature of current flow in these devices.
- Consider a photodetector circuit. Every time when the light is incident on it, the cathode emits an electron. This is generate a current pulse.
- These electrons are being randomly emitted at time instants denoted by τ_k where $-\infty < k < \infty$.
- If we assume that such a random emission is going on for a long time, then the photo current flowing through the photodetector can be imagined to be an infinite sum of current pulses.
- This can be expressed mathematically as follows :

$$X(t) = \sum_{k=-\infty}^{\infty} h(t-\tau_k) \quad ...(4.11.7)$$

$$.......\underbrace{h(t-\tau_{-1})}_{\text{Current pulse at } \tau_{-1}} + \underbrace{h(t-\tau_0)}_{\text{Current pulse at } \tau_0} + \underbrace{h(t-\tau_1)}_{\text{Current pulse at } \tau_1} +.......$$

(E-829)

- In Equation (4.11.7), $h(t-\tau_k)$ represents the current pulse generated at instant τ_k, and the process X(t) is called as **shot noise.**
- The number of electrons N (t), which are emitted during the time interval (0, t) gives rise to a discrete random process.
- The value of this process increases by 1 every time an electron is emitted. A sample function of such a random process is shown in Fig. 4.11.2.



(E-830) **Fig. 4.11.2 : Sample function**

- Let the number of electrons emitted during the time interval t and $(t + t_0)$ be v. Then the mean value of number of electrons is given by

$$\text{Mean} \quad E[v] = \lambda t_0 \quad ...(4.11.8)$$

- λ is a constant and called as the "rate" of the process.
- So the total number of electrons emitted during the interval t to $t + t_0$ i.e. v is given by,

$$v = N(t+t_0) - N(t) \quad ...(4.11.9)$$

- The variable "v" in the above expression will follow a Poisson distribution with a mean value of λt_0.
- The probability of emitting k number of electrons during the interval $(t, t + t_0)$ is given by the Poisson's distribution as,

$$P(v=k) = \frac{(\lambda t_0)^k}{k!} e^{-\lambda t_0} \quad k = 0, 1 \quad ...(4.11.10)$$

- After the detailed statistical analysis, the first two moments of the process X(t) are given by

1. The mean of $X(t) = m_X = \lambda \int_{-\infty}^{\infty} h(t)\, dt \quad ...(4.11.11)$

   where λ = rate and h (t) = waveform of the current pulse.

2. The autocovariance function of X(t) is given by

$$C_X(\tau) = \lambda \int_{-\infty}^{\infty} h(t)\, h(t+\tau)\, dt \quad ...(4.11.12)$$

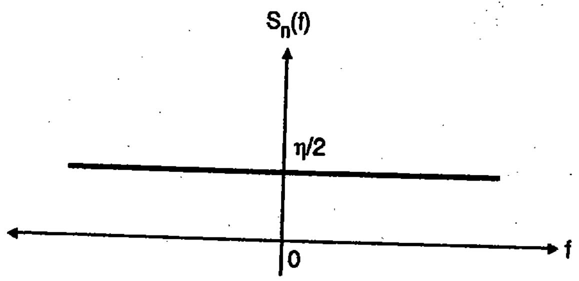
This expression is also called as **Cambell's theorem.**

4.12 White Gaussian Noise : SPPU : May 16

University Questions

Q. 1 Write short notes on : White Gaussian Noise. **(May 16, 6 Marks)**

- "White" noise is the noise whose power spectral density (PSD) is uniform over the entire frequency range of interest, as shown in Fig. 4.12.1. That means PSD is same for all the frequency components from 0 to ∞.



(D-1406) **Fig. 4.12.1 : Power spectral density of white noise**

Why is it called as white noise ?

- The white noise contains all the frequency components in equal proportion. This is analogous with white light which is a superposition of all visible spectral components.

Why is it called as Gaussian noise ?

- The white noise has a Gaussian distribution. That means the PDF of white noise has the shape of Gaussian PDF. Therefore it is called as Gaussian noise.

Power spectral density of white noise :

- As shown in Fig. 4.12.1 the power spectral density (PSD) of a white noise is given by,

$$S_n(f) = \frac{\eta}{2} \quad ...(4.12.1)$$

- This equation shows that the power spectral density of white noise is independent of frequency. As η is constant, the psd is uniform over the entire frequency range including the positive as well as the negative frequencies. η in Equation (4.12.1) is defined as :

$$\eta = KT_e \quad ...(4.12.2)$$

where, K = Boltzmann's constant and

T_e = Equivalent noise temperature of the system.

4.12.1 Auto Correlation Function of White Gaussian Noise :

Let us obtain expression for the auto correlation function of the white Gaussian noise. Refer to the following example for the same.

Ex. 4.12.1 : Derive and plot the auto correlation function of a white Gaussian noise which has a power spectral density of η / 2.

Soln. :

We know that the power spectral density and auto correlation function form a Fourier transform pair. That means,

$$R(\tau) \overset{F}{\leftrightarrow} S(f)$$

$$\text{i.e.} \quad FT\{R(\tau)\} = S(f)$$

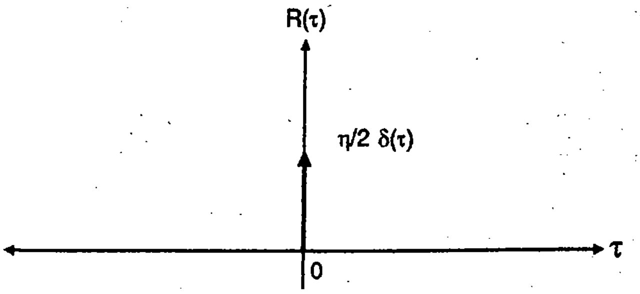
$$\text{OR} \quad R(\tau) = IFT\{S(f)\}$$

Therefore for the white noise the autocorrelation function R (τ) is obtained as follows :

$$R(\tau) = IFT\{S_n(f)\} = IFT\{\eta / 2\}$$

$$\therefore \quad R(\tau) = \frac{\eta}{2}\delta(t) \quad \text{...Ans.}$$

This is the expression for the auto correlation function of white noise. The auto correlation function can be plotted as shown in Fig. P. 4.12.1.



(D-1407) **Fig. P. 4.12.1 : Auto correlation function of white noise**

Note : As shown in Fig. P. 4.12.1 the auto correlation function of white noise is a delta function weighted by a factor (η / 2) at τ = 0. The meaning of this is that any two samples of white noise are totally uncorrelated.

- In this chapter, we are going to concentrate our focus on the noise which can be described as an ergodic random process.
- Further, we are going to assume that the probability density of the noise is Gaussian.
- In a lot of communication systems and most of the circumstances the assumption of Gaussian density proves to be perfectly correct.
- However the assumption of Gaussian density can be wrong under certain operating conditions.

4.13 Superposition of Noises :

- We can represent noise n(t) in the Fourier series form. That means noise is represented as "superposition" of individual noise components.
- However the frequency ranges of these components were not overlapping.
- What happens if the frequency ranges of the components are overlapping ? It can be proved that even though the frequency ranges are overlapping then also the total noise power is superposition of powers of individual spectral components. Let us prove it as follows :
- Suppose that we have two noise processes n_1 (t) and n_2 (t) whose spectral ranges overlap fully or partially.
- The total noise due to these two components is given by,

$$n(t) = n_1(t) + n_2(t) \text{ .... by superposition.}$$

- Then the normalized power P_{12} of the sum $n_1(t) + n_2(t)$ is given by,

$$P_{12} = \left[\overline{n(t)}\right]^2 = \overline{[n_1(t) + n_2(t)]^2}$$

$$\therefore P_{12} = \left[\overline{n_1(t)}\right]^2 + \left[\overline{n_2(t)}\right]^2 + 2\,\overline{n_1(t)\,n_2(t)} \quad ...(4.13.1)$$

$$\therefore P_{12} = P_1 + P_2 + 2\,\overline{n_1(t)\,n_2(t)} \quad ...(4.13.2)$$

- In the above expression P_1 and P_2 are the normalized powers of n_1 (t) and n_2 (t) respectively and the third term is the expected value of the product. The expected value of the product is nothing but "cross-correlation" of the processes n_1 (t) and n_2 (t). As these processes are uncorrelated the cross-correlation will be zero and the Equation (4.13.2) can be simplified as :

$$P_{12} = P_1 + P_2 \qquad \text{...(4.13.3)}$$

Conclusion :

From the above expression we can draw a conclusion that the total normalized noise power can be obtained by superposition of the powers of individual noise components. For k number of noise components the expression is,

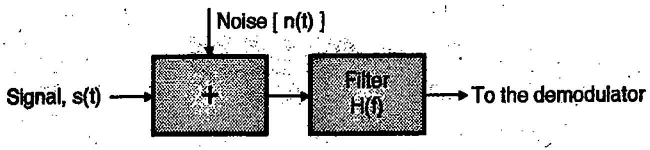
$$P = P_1 + P_2 + P_3 + \ldots + P_k$$

4.14 Effects of Linear Filtering on Noise :

- The thermal noise has a uniform power spectral density upto the frequency of the order of 10^{13} Hz.
- Shot noise also has a wide frequency range. Other noise sources also have very wide spectral ranges.
- We are going to consider the "white noise" for understanding the effect of filtering on noise.
- As discussed earlier, the white noise has a uniform power spectral density over the entire frequency range of interest.

Where to connect a filter ?

- Filters are connected in order to reduce the noise power.
- Generally these filters are narrowband filters which are designed to pass a specific range of frequencies.
- In order to minimize the noise power at the input of the demodulator of a receiver, we introduce a filter before the demodulator as shown in Fig. 4.14.1.



(D-361) Fig. 4.14.1 : A filter is connected before a demodulator to reduce the noise power input

Types of filter :

- In this section we are going to see the effect of following filters on noise :

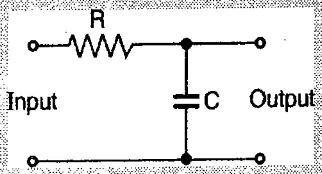
- R-C low pass filter.
- Ideal low pass filter.
- Ideal bandpass filter.

4.14.1 Effect of R-C Low Pass Filter :

SPPU : Dec. 12

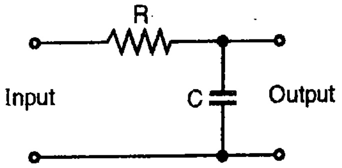
University Questions

Q. 1 A random telegraph signal x (t), characterized by the autocorrelation function Rx (τ) = exp (– 2 v | τ |) where v is constant is applied to the low-pass RC filter of figure. Determine the power spectral density and auto correlation function of the random process at the filter output. **(Dec. 12, 8 Marks)**

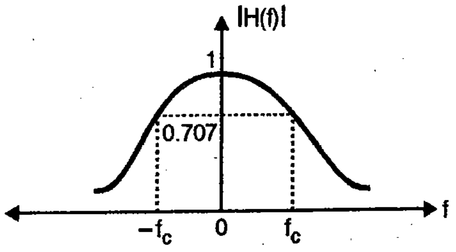


(E-1345) Fig. Q. 1

The R-C low pass filter and variation of its transfer function with frequency are as shown in Fig. 4.14.2 (a) and (b) respectively.



(a) R-C low pass filter



(b) Transfer function of R-C low pass filter

(E-834) Fig. 4.14.2

| H (f) | is the transfer function of the filter and f_c is its cut-off frequency. The transfer function of the R-C low pass filter is given by,

$$H(f) = \frac{1}{1 + j f / f_c} \qquad \text{...(4.14.1)}$$

Assumptions :

1. Let the input noise signal be n_i (t) and the output noise signal be n_o (t).
2. Let the psd of input noise signal be S_i (f) and the psd of the output noise signal be S_o (f).

   Let us now obtain the expression for the average noise power at the output of the R-C filter.

Average noise power at the filter output :

- We know that the relation between the psd of input and output is given by,

$$S_o(f) = |H(f)|^2 S_i(f) \qquad \text{...(4.14.2)}$$

- Assume that the input noise is AWGN (Additive White Gaussian Noise) then the psd at the filter input is given by,

$$S_i(f) = \frac{N_o}{2} \qquad ...(4.14.3)$$

- The magnitude of the transfer function of the filter can be obtained from Equation (4.14.1) as,

$$|H(f)|^2 = \frac{1}{1+(f/f_c)^2} \qquad ...(4.14.4)$$

- Substitute Equations (4.14.3) and (4.14.4) into Equation (4.14.2) to get,

$$S_o(f) = \frac{N_o}{2}\cdot\frac{1}{1+(f/f_c)^2} \qquad ...(4.14.5)$$

Thus we have obtained the power spectral density of the output noise signal.

- The next step is to obtain the average output noise power from the psd. It is done as follows :

$$P = \int_{-\infty}^{\infty} S(f)\,df \qquad ...(4.14.6)$$

- This is the general expression relating the average power and psd. Using this expression we can obtain the output noise power as :

$$P_{no} = \int_{-\infty}^{\infty} S_o(f)\,df$$

$$= \int_{-\infty}^{\infty} \frac{N_o}{2}\,\frac{1}{1+(f/f_c)^2}\,df$$

- Substitute $\frac{f}{f_c} = x$ in the above expression.

$$\therefore \quad f_c\,dx = df$$

- The limits of integration will remain unchanged.

$$\therefore \quad P_{no} = \frac{N_o}{2} f_c \int_{-\infty}^{\infty} \frac{1}{(1+x^2)}\,dx$$

But $\int \frac{1}{(1+x^2)}\,dx = \pi$

$$\therefore \quad P_{no} = \frac{N_o}{2} f_c \pi$$

- Thus the average noise power at the output of an R-C filter is given by,

$$P_{no} = \frac{\pi N_o f_c}{2} \quad \text{(R-C low pass filter)} \qquad ...(4.14.7)$$

- This equation shows that the output noise power can be reduced by reducing the value of filter cut-off frequency f_c.

Power Spectral Density :

- Let the psd of the output noise be denoted by $S_o(f)$. Let us obtain the expression for the same.
- Referring to Equation (4.14.5) we can write that the psd of output noise is

$$S_o(f) = \frac{N_o/2}{1+(f/f_c)^2}$$

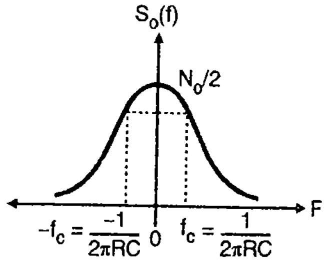
- But the cut-off frequency f_c of the RC low pass filter is given by

$$f_c = \frac{1}{2\pi RC}$$

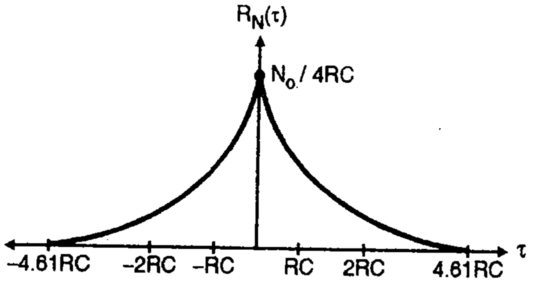
- Substituting this we get

$$S_o(f) = \frac{N_o/2}{1+(2\pi f RC)^2} \qquad ...(4.14.8)$$

- This is the required expression for the psd of output noise and it is plotted in Fig. 4.14.3(a).



(a) PSD of output noise



(b) Autocorrelation function of output noise

(E-835) Fig. 4.14.3

Autocorrelation function of output noise :

- The autocorrelation function of output noise is denoted by $R_N(\tau)$ and it can be obtained by taking the inverse Fourier transform of PSD $S_o(f)$.

$$\therefore R_N(\tau) = \text{IFT}[S_o(f)]$$

$$= \text{IFT}\left[\frac{N_o/2}{1+(2\pi f RC)^2}\right]$$

- Substitute $RC = 1/\alpha$ to get

$$R_N(\tau) = \text{IFT}\left[\frac{N_o/2}{1+\left(\frac{2\pi f}{\alpha}\right)^2}\right]$$

$$= \text{IFT}\left[\frac{\alpha^2 N_o/2}{\alpha^2+(2\pi f)^2}\right]$$

- Substitute $\alpha^2 = \alpha\times\alpha = \frac{2\alpha}{2\,RC}$ in the numerator to get

$$R_N(\tau) = \text{IFT}\frac{[(N_o/2)\,2\,\alpha/2\,RC]}{[\alpha^2+(2\pi f)^2]}$$

$$= \frac{N_o}{4\,RC}\,\text{IFT}\left[\frac{2\alpha}{\alpha^2+(2\pi f)^2}\right]$$

- We know that $e^{-\alpha|(\tau)|} \overset{F}{\longleftrightarrow} \frac{2\alpha}{\alpha^2+(2\pi f)^2}$

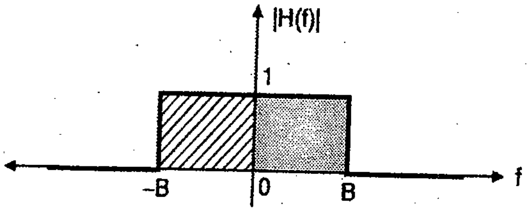
$$\therefore\quad R_N(\tau) = \frac{N_o}{4\,RC}\cdot e^{-\alpha|\tau|}$$

$$= \frac{N_o}{4\,RC}\,e^{-|\tau|/RC} \qquad ...(4.14.9)$$

- This is the required expression for autocorrelation function $R_N(\tau)$ and it is plotted in Fig. 4.14.3(b).

4.14.2 Rectangular (Ideal) Low Pass Filter :

The transfer function of an ideal (rectangular) low pass filter is given by :



(E-836) **Fig. 4.14.4 : Transfer function of an ideal low pass filter**

$$H(f) = 1, \qquad \text{for} -B \le f \le B$$
$$= 0, \qquad \text{elsewhere} \qquad ...(4.14.10)$$

Expression for the output noise power :

- The noise at the input is assumed to bc AWGN. Therefore the psd of the input noise is given by,

$$S_i(f) = \frac{N_o}{2} \qquad ...(4.14.11)$$

- We know that the relation between the input and output spectral densities is given by,

$$S_o(f) = |H(f)|^2 \times S_i(f)$$

Substituting Equation (4.14.11) into the above expression we get,

$$S_o(f) = |H(f)|^2\,\frac{N_o}{2} \qquad ...(4.14.12)$$

- The average noise power at the filter output is given by,

$$P_{no} = \int_{-\infty}^{\infty} S_o(f)\,df$$

$$= \int_{-\infty}^{\infty} |H(f)|^2 \times \frac{N_o}{2}\,df$$

$$= \int_{-B}^{B} \frac{N_o}{2} \times (1)^2\,df$$

$$P_{no} = N_o\,B$$

- Thus the average output noise power of an ideal low pass filter is given by,

$$P_{no} = N_o\,B \quad \text{Ideal low pass filter} \quad ...(4.14.13)$$

This equation shows that the output noise power can be reduced, by reducing the bandwidth of the ideal low pass filter.

Noise Power Spectral Density :

- Refer to Equation (4.14.12) to write the expression for output noise power spectral density as

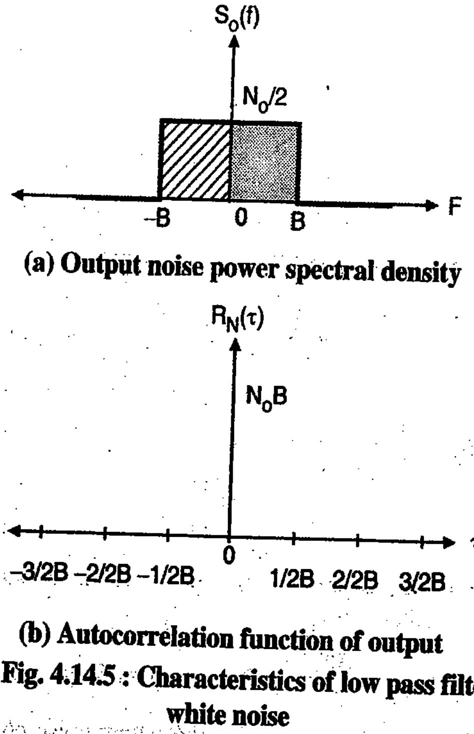
$$S_o(f) = |H(f)|^2\frac{N_o}{2}$$

- But $|H(f)| = 1$ for $-B \le f \le B$

$$\therefore\quad S_o(f) = \frac{N_o}{2} \quad ....-B \le f \le B$$

$$= 0 \quad .... \text{else where} \quad ...(4.14.14)$$

- The output noise power spectral density is graphically shown in Fig. 4.14.5(a).



(a) Output noise power spectral density

(b) Autocorrelation function of output

(E-837) **Fig. 4.14.5 : Characteristics of low pass filtered white noise**

Autocorrelation function of output noise :

- Let us now obtain the expression for the autocorrelation function of output noise of the ideal low pass filter.
- Autocorrelation of the output noise be denoted by $R_N(\tau)$. It can be obtained by taking the inverse Fourier transform of the power spectral density.

$$\therefore\quad R_N(\tau) = \text{IFT}[S_o(f)] = \text{IFT}[N_o/2]$$

$$= \int_{-B}^{B} \frac{N_o}{2} e^{j2\pi f\tau} df$$

$$= \frac{N_o}{j \times 2 \times 2\pi\tau} \left[e^{j2\pi f\tau} \right]_{-B}^{B}$$

$$= \frac{N_o}{2\pi\tau} \left[\frac{e^{j2\pi\tau B} - \epsilon^{-j2\pi\tau B}}{2j} \right]$$

$$\therefore \quad R_N(\tau) = \frac{N_o}{2\pi\tau} \sin(2\pi\tau B) \quad \text{...(4.14.15)}$$

- Multiply and divide by B to get

$$R_N(\tau) = \frac{N_o B}{2\pi\tau B} \sin(2\pi\tau B)$$

$$= N_o B \text{ sinc} (2\tau B) \quad \text{...(4.14.16)}$$

- This sinc function will have a maximum value of N_o B at f = 0 and it will pass through zero at $2\tau B = \pm 1, \pm 2 \ldots$
- So the $R_N(\tau)$ passes through zero at

$$\tau = \pm\frac{1}{2B}, \pm\frac{2}{2B}, \ldots\ldots \quad \text{...(4.14.17)}$$

- The autocorrelation is plotted in Fig. 4.14.5(b).

4.15 Noise Bandwidth :

- Assume that a white noise is present at the input of a receiver (filter). Let the filter have a transfer function H(f) as shown in Fig. 4.15.1.
- This filter is being used to reduce the noise power actually passed on to the receiver. Now draw the frequency response of an ideal (rectangular) filter as shown by the dotted plot in Fig. 4.15.1. The center frequency of this ideal filter also is f_0.
- Let the bandwidth "B_N" of the ideal filter be adjusted in such a way that the noise output power of the ideal filter is exactly equal to the noise output power of a real R-C filter.
- Then B_N is called as the noise bandwidth of the real filter.

Thus the noise bandwidth "B_N" is defined as the bandwidth of an ideal (rectangular) filter which passes the same noise power as passed by the real filter.

Noise bandwidth of an R-C low pass filter :

- Let us calculate the noise bandwidth B_N of an R-C low pass filter. The transfer function of an R-C low pass filter is given by,

$$H(f) = \frac{1}{1 + jf/f_c}$$

For this filter H(f) = 1 at f = 0.

- The noise output power of such a filter is given by Equation (4.14.7) as,

$$P_{no} = \frac{\pi N_o f_c}{2}$$

- In presence of the same AWGN noise, the noise output power of an ideal low pass filter with a bandwidth B is given by,

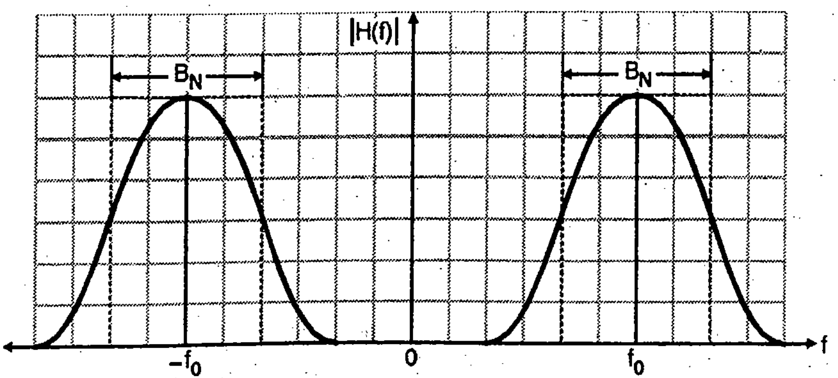
$$P_{no} = N_o B \text{ ...(from Equation (4.14.13))}$$

- According to the definition of noise bandwidth, if $B = B_N$ then output noise power of ideal and real low pass filters should be identical.

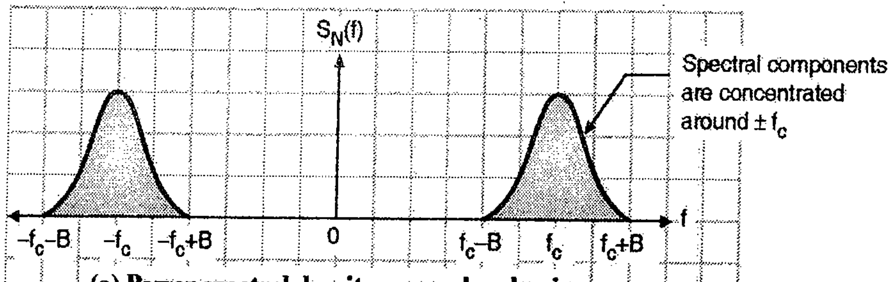
$$\therefore \frac{\pi N_o f_c}{2} = N_o B_N$$

$$\therefore \quad B_N = \frac{\pi}{2} f_c \quad \text{R-C low pass filter} \quad \text{...(4.15.1)}$$

- Thus the noise bandwidth of an R-C low pass filter is $\frac{\pi}{2}$ times or 1.57 times its 3-dB bandwidth f_0.

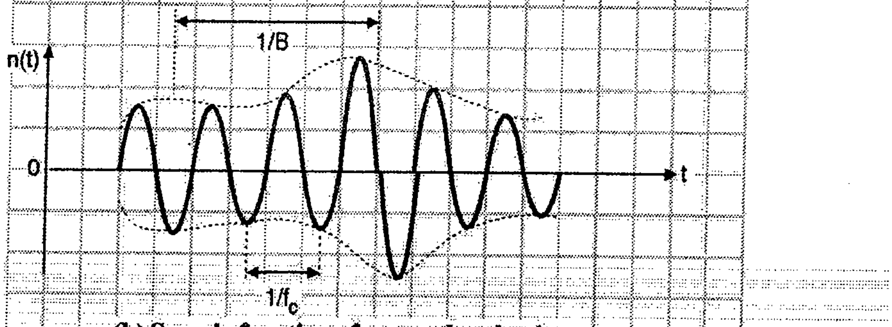


(D-327) **Fig. 4.15.1 : Noise bandwidth of a filter**



Spectral components are concentrated around ± f_c

(a) Power spectral density narrowband noise



(b) Sample function of narrowband noise

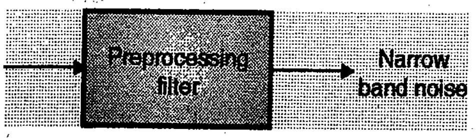
(E-840) **Fig. 4.16.2**

4.16 Narrow-Band Noise : SPPU : Dec. 12

University Questions

Q. 1 Explain narrowband noise and represent an narrowband noise in terms of inphase and quadrature components. **(Dec. 12, 8 Marks)**

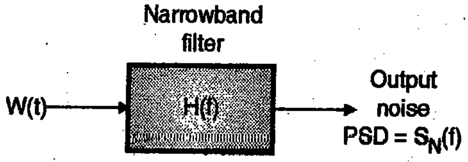
- Generally the preprocessing of received signal is performed in every communication receiver.
- A narrow band filter is generally employed for the purpose of preprocessing.
- Bandwidth of the narrow band filter is just large enough to pass the signals only but the bandwidth is not very large as to allow excessive noise to pass through.
- The noise at the output of such a filter is called as the narrowband noise.



(E-839) **Fig. 4.16.1 : Narrowband noise**

- Fig. 4.16.2(a) shows the power spectral density of the narrow band noise. It shows that the spectral components of narrow band noise are concentrated about some midband frequency ± f_c.
- Narrowband noise is a random process, the sample function of which is denoted by n (t) and shown in Fig. 4.16.2(b).
- Fig. 4.16.2(b) shows that the sample function n (t) of a narrowband noise is similar to a sinewave. Its frequency is f_c and it undergoes slow variations in both amplitude and phase.

Power spectral density S_N (f) :



(E-841) **Fig. 4.16.3**

- Refer Fig. 4.16.3. Let the noise produced at the output of a narrowband filter be n (t) due to application of a white Gaussian noise having a sample function W (t).
- The white Gaussian noise applied at the filter input has a zero mean and unit power spectral density.
- W (t) is the sample function of the input noise process W (t) and n (t) is the sample function of the output noise process N (t).
- Let the transfer function of the filter be H (f). So the PSD of n (t) is given by

$$S_N(f) = |H(f)|^2 \times \text{PSD of } W(t)$$

$$\therefore \quad S_N(f) = |H(f)|^2 \times 1 = |H(f)|^2 \qquad ...(4.16.1)$$

Conclusion :

The shape of S_N (f) shown in Fig. 4.16.2(a) is obtained by squaring the shape of transfer function H (f) of the narrow band filter.

4.16.1 Representing the Narrowband Noise in Terms of in Phase and Quadrature Components :

SPPU : May 06, Dec. 09, Dec. 12, Dec. 13

University Questions

Q. 1 Show that a random process X(t) can be represented in terms of its quadrature components **(May 06, 8 Marks)**

Q. 2 State the properties of in-phase and quadrature phase components of narrow band noise and explain the process of generation with PSD. **(Dec. 09, 9 Marks)**

Q. 3 Explain narrowband noise and represent an narrowband noise in terms of inphase and quadrature components. **(Dec. 12, 8 Marks)**

Q. 4 Show that a narrowband random process X(t) can be completely represented in terms of its in phase and Quadrature components. **(Dec. 13, 8 Marks)**

- Let $n_+(t)$ denote the pre-envelope and $\tilde{n}(t)$ denote the complex envelope of the narrowband noise n (t).
- Let the power spectrum of n (t) be centered about the frequency f_c. Then we can write that

$$n_+(t) = n(t) + j\,\hat{n}(t) \qquad \text{...(4.16.2)}$$

where $\hat{n}(t)$ = Hilbert transform of n(t).

$$\text{And } \tilde{n}(t) = n_+(t)\,e^{-j2\pi f_c t} \qquad \text{...(4.16.3)}$$

when $\tilde{n}(t)$ is the complex envelope.

- The complex envelope can be expressed in terms of the inphase component $n_I(t)$ and the quadrature component $n_Q(t)$ of the narrowband noise as follows :

$$\tilde{n}(t) = n_I(t) + j\,n_Q(t) \qquad \text{...(4.16.4)}$$

- Combining the Equation (4.16.2) through (4.16.4), we get the expressions for the inphase and quadrature components as follows :

$$n_I(t) = n(t)\cos(2\pi f_c t) + \hat{n}(t)\sin(2\pi f_c t) \text{ ...(4.16.5)}$$

$$\text{and } n_Q(t) = \hat{n}(t)\cos(2\pi f_c t) - n(t)\sin(2\pi f_c t)\text{...(4.16.6)}$$

- From Equations (4.16.5) and (4.16.6) by eliminating $\hat{n}(t)$, we can express n (t) in terms of the in phase and quadrature components as follows :

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \text{ ...(4.16.7)}$$

4.16.2 Properties of Quadrature Components of Noise :

- Using Equations (4.16.5) to (4.16.7) we can derive the following important properties of the quadrature components $n_I(t)$ and $n_Q(t)$.

Property 1 :

The in-phase and quadrature components $n_I(t)$ and $n_Q(t)$ have zero mean.

Property 2 :

If the narrowband noise n (t) is Gaussian then its in phase component $n_I(t)$ and the quadrature component $n_Q(t)$ are jointly Gaussian.

Property 3 :

If the narrowband noise n (t) is wide sense stationary, then the in phase component $n_I(t)$ and the quadrature component $n_Q(t)$ are jointly wide-sense stationary.

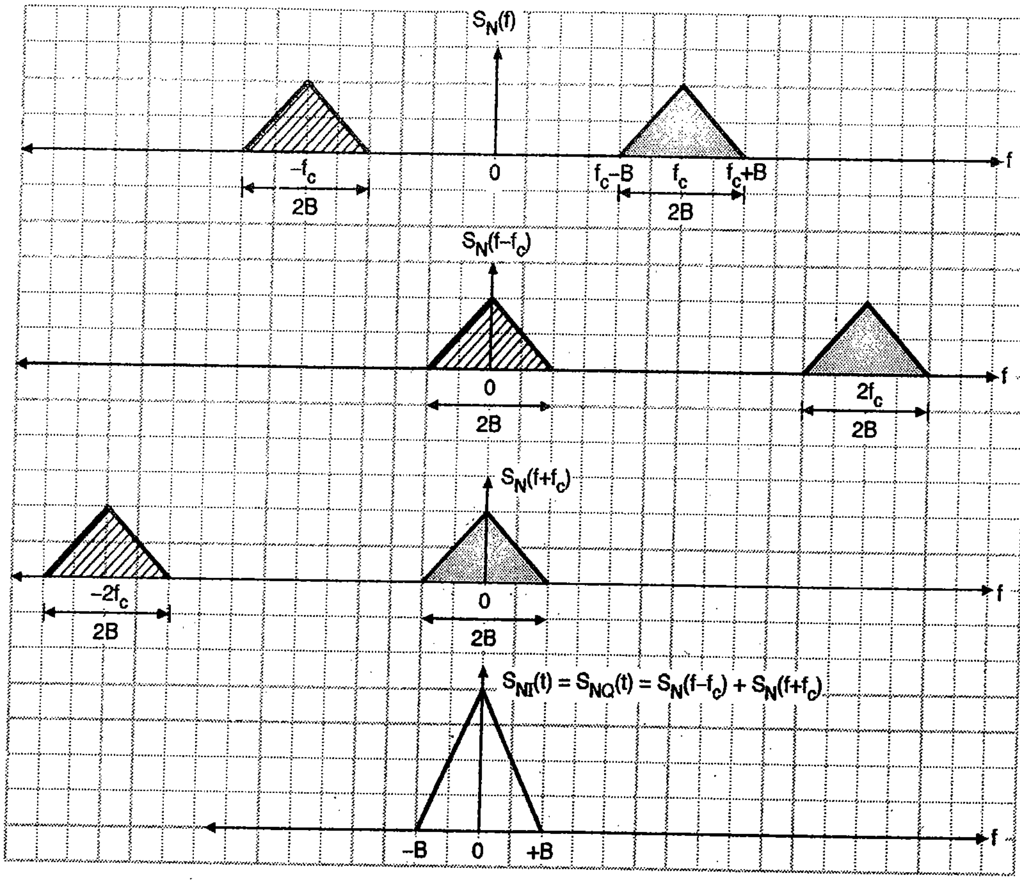
Property 4 :

- The power spectral density of $n_I(t)$ and $n_Q(t)$ is same and it is related to the psd of n (t) as follows :

$$S_{NI}(f) = S_{NQ}(f) = \begin{cases} S_N(f-f_c) + S_N(f+f_c) & \ldots -B \le f \le B \\ 0 & \ldots\text{elsewhere} \end{cases} \text{ ...(4.16.8)}$$

where we assume that $S_N(f)$ occupies the spectrum $(f_c - B)$ to $(f_c + B)$ and that $f_c > B$.

- This property is illustrated graphically in Fig. 4.16.4.



(E-842) **Fig. 4.16.4 : PSD of quadrature components**

Property 5 :

The quadrature components of n (t) have the same variance as that of n (t).

Property 6 :

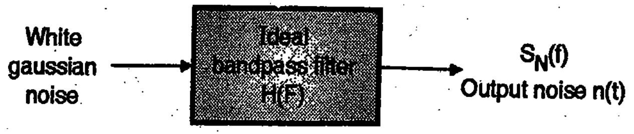
The cross-spectral densities of the quadrature components n_I (t) and n_Q (t) of the narrowband noise n (t) are purely imaginary.

Property 7 :

The in phase component n_I (t) and the quadrature component n_Q (t) are statistically independent if the narrowband noise n (t) is Gaussian and has a zero mean with a power spectral density S_N (f) which is symmetrical about the mid frequency $\pm f_c$.

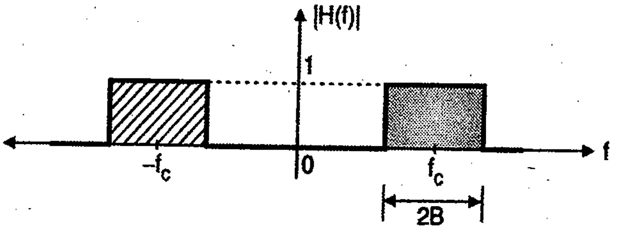
4.16.3 White Noise Passed Through an Ideal Bandpass Filter :

- Let the input to an ideal bandpass filter be a white Gaussian noise which has a zero mean and power spectral density of $N_o/2$.



(E-843) **Fig. 4.16.5**

- The transfer function of an ideal bandpass filter is as shown in Fig. 4.16.6.



(E-844) **Fig. 4.16.6 : Transfer function of an ideal bandpass filter**

- The transfer function of the ideal bandpass filter is given by,

$$H(f) = 1, \quad \text{for } |f_c| - B \text{ to } |f_c| + B$$
$$= 0, \quad \text{elsewhere} \quad ...(4.16.9)$$

- The midband frequency of the filter is f_c and bandwidth is 2B.

Power spectral density of filtered noise :

- Assuming that the noise at filter input is AWGN, the psd of the input noise is given by,

$$S_i(f) = \frac{N_o}{2} \quad ...(4.16.10)$$

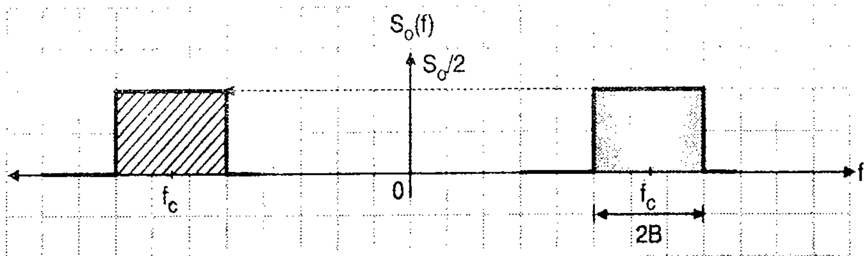
- The psd of the output noise is given by,

$$S_o(f) = |H(f)|^2 \cdot S_i(f)$$

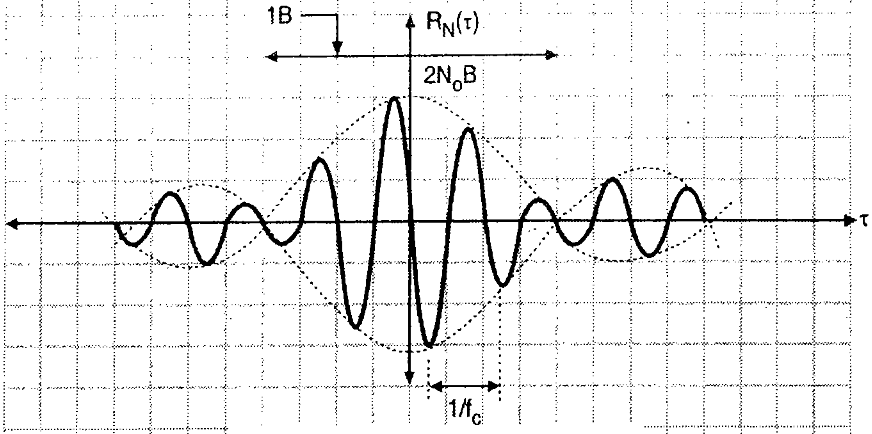
- But H (f) = 1 corresponding to the passband of the filter.

$$\therefore \; S_o(f) = S_i(f)$$
$$= \frac{N_o}{2} \text{ for } |f_c| - B \text{ to } |f_c| + B \quad ...(4.16.11)$$

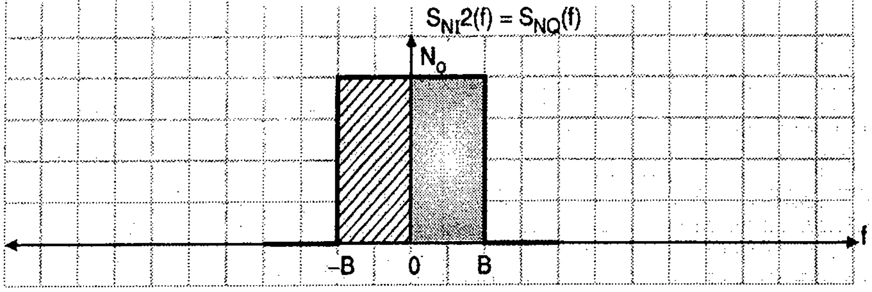
- The psd of output noise is graphically shown in Fig. 4.16.7(a).



(a) PSD of output noise of ideal BPF



(b) Autocorrelation of output noise



(c) PSD of in phase amount of phase components

(E-845) Fig. 4.16.7

Autocorrelation function of output noise :

- Let the autocorrelation function of the output noise be denoted by $R_N(\tau)$. It is equal to the inverse Fourier transform of psd $S_o(f)$.

$$\therefore R_N(\tau) = \text{IFT}[S_o(f)]$$

$$= \int_{-f_c-B}^{-f_c+B} \frac{N_o}{2} e^{j2\pi f\tau} df + \int_{f_c-B}^{f_c+B} \frac{N_o}{2} e^{j2\pi f\tau} df$$

$$R_N(\tau) = \frac{N_o}{2} \times \frac{1}{j2\pi\tau}\left[e^{j2\pi f\tau}\right]_{-f_c-B}^{-f_c+B} + \frac{N_o}{2} \times \frac{1}{j2\pi\tau}\left[e^{j2\pi f\tau}\right]_{f_c-B}^{f_c+B}$$

$$= \frac{N_o}{2\pi\tau}\left[\frac{e^{-j2\pi f_c\tau} . e^{j2\pi B\tau} - e^{-j2\pi f_c\tau} . e^{-j2\pi B\tau}}{2j}\right] + \frac{N_o}{2\pi\tau}\left[\frac{e^{j2\pi f_c\tau} . e^{j2\pi B\tau} - e^{j2\pi f_c\tau} . e^{-j2\pi B\tau}}{2j}\right]$$

$$= \frac{N_o}{2\pi\tau}\left[e^{j2\pi f_c\tau}\frac{(e^{j2\pi B\tau} - e^{-j2\pi B\tau})}{2j} + e^{-j2\pi f_c\tau}\frac{(e^{j2\pi B\tau} - e^{-j2\pi B\tau})}{2j}\right]$$

$$= \frac{N_o}{2\pi\tau}\left[e^{j2\pi f_c\tau}\sin(2\pi B\tau) + e^{-j2\pi f_c\tau}\sin(2\pi B\tau)\right]$$

$$= N_o\frac{\sin(2\pi B\tau)}{2\pi\tau}\left[e^{j2\pi f_c\tau} + e^{-j2\pi f_c\tau}\right]$$

$$= 2N_oB\frac{\sin(2\pi B\tau)}{2\pi\tau}\left[\frac{e^{j2\pi f_c\tau} + e^{-j2\pi f_c\tau}}{2}\right]$$

$$\therefore R_N(\tau) = 2N_oB\,\text{sinc}(2B\tau)\cos(2\pi f_c\tau) \qquad ...(4.16.12)$$

- This is the required expression for $R_N(\tau)$ and it is plotted in Fig. 4.16.7(b).

Power spectral density of quadrature components of noise :

- We know that according to property 4,

$$S_{NI}(f) = S_{NQ}(f)$$
$$= S_N(f - f_c) + S_N(f + f_c) \quad ... -B \le f \le B$$

- Hence $S_{NI}(f) = S_{NQ}(f) = \frac{N_o}{2} + \frac{N_o}{2} = N_o$
- $\ldots -B \leq f \leq B$
- The PSD of n_I (t) and n_Q (t) is shown in Fig. 4.16.7(c).

Autocorrelation of quadrature components :

- The autocorrelation of the quadrature components is denoted by $R_{NI}(\tau)$ and $R_{NQ}(\tau)$.

$$\therefore \quad R_{NI}(\tau) = R_{NQ}(\tau) = \int_{-B}^{B} N_o e^{j2\pi f\tau} df$$

$$= \frac{N_o}{j2\pi\tau} \left[e^{j2\pi f\tau}\right]_{-B}^{B}$$

$$\therefore \quad R_{NI}(\tau) = R_{NQ}(\tau) = \frac{N_o}{\pi\tau}\left[\frac{e^{j2\pi B\tau} - e^{-j2\pi B\tau}}{2j}\right]$$

$$= \frac{N_o}{\pi\tau} \sin(2\pi B\tau)$$

$$= 2BN_o \frac{\sin(2\pi B\tau)}{2\pi B\tau}$$

$$\therefore \quad R_{NI}(\tau) = R_{NQ}(\tau)$$

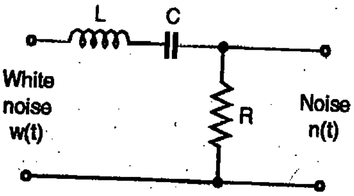
$$= 2BN_o \operatorname{sinc}(2B\tau) \qquad \ldots(4.16.13)$$

This is the required expression.

4.17 Transmission of White Noise Through a High Q Tuned Filter :

- Fig. 4.17.1 shows the bandpass LCR filter.
- The transfer function of this filter is given by,

$$H(f) = \frac{R}{R + j2\pi fL + (1/j2\pi f_c)} \qquad \ldots(4.17.1)$$



(E-846) Fig. 4.17.1 : High Q tuned filter

Power spectral density of output noise :

- Let the psd of output noise of the filter be $S_N(f)$.
- The resonant frequency of the filter is given by

$$f_r = \frac{1}{2\pi\sqrt{LC}} \qquad \ldots(4.17.2)$$

and its Q factor is defined as,

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} \qquad \ldots(4.17.3)$$

- Equation (4.17.1) for H(f) can be rearranged as follows :

$$H(f) = \frac{1}{1 + j2\pi f\frac{L}{R} + \frac{1}{j2\pi fCR}} \qquad \ldots(4.17.4)$$

Now $Q\, f_r = \frac{1}{R}\sqrt{\frac{L}{C}} \times \frac{1}{2\pi\sqrt{LC}}$

$$= \frac{1}{2\pi RC} \qquad \ldots(4.17.5)$$

and $\frac{Q}{f_r} = \frac{1}{R}\sqrt{\frac{L}{C}} \times 2\pi\sqrt{LC} = 2\pi\frac{L}{R} \qquad \ldots(4.17.6)$

- Substituting Equations (4.17.5) and (4.17.6) into Equation (4.17.4) we get,

$$H(f) = \frac{1}{1 + j\frac{Qf}{f_r} + \frac{1Qf_r}{jf}}$$

$$= \frac{1}{1 + j\frac{Qf}{f_r} - j\frac{Qf_r}{f}}$$

$$\therefore \quad H(f) = \frac{1}{1 + jQ\left[\left(\frac{f}{f_r}\right) - \left(\frac{f_r}{f}\right)\right]} \qquad \ldots(4.17.7)$$

- If the Q factor is larger than 1, then the above expression can be approximated as follows :

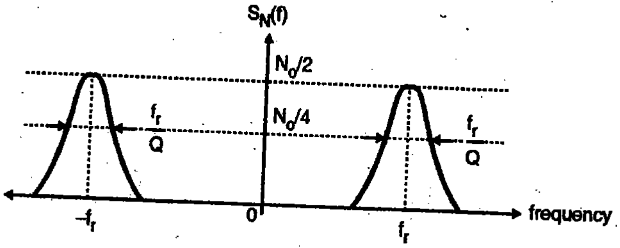
$$H(f) = \begin{cases} \frac{1}{1 + j2Q(f - f_r)/f_r} & \ldots\ f > 0 \\ \frac{1}{1 + j2Q(f + f_r)f_r} & \ldots\ f < 0 \end{cases} \qquad \ldots(4.17.8)$$

- Let the white Gaussian noise of zero mean and psd $N_o/2$ is applied at the input of this filter.
- Then the psd of the output noise is given by

$$S_N(f) = |H(f)|^2 \cdot \frac{N_o}{2}$$

$$\therefore S_N(f) = \begin{cases} = \frac{N_o/2}{1 + 4Q^2(f - f_r)^2/f_r^2} & \ldots\ f > 0 \\ = \frac{N_o/2}{1 + 4Q^2(f + f_r)^2/f_r^2} & \ldots\ f < 0 \end{cases} \qquad \ldots(4.17.9)$$

- The PSD of noise output is shown in Fig. 4.17.2.



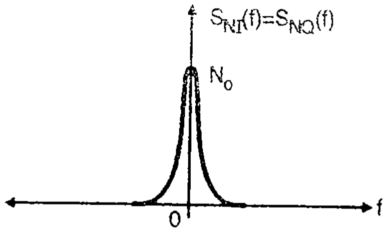
(E-847) Fig. 4.17.2 : Power spectral density of n (t)

PSD of in phase and quadrature components :

- The psd of the in phase and quadrature components are denoted by $S_{NI}(f)$ and $S_{NQ}(f)$ respectively and are given by

$$S_{NI}(f) = S_{NQ}(f) = \frac{N_o}{1 + (2Qf/f_r)^2} \qquad \ldots(4.17.10)$$

- This psd is plotted in Fig. 4.17.3.



(E-848) **Fig. 4.17.3 : PSD of in phase and quadrature components of noise output**

Conclusion :

- Compare the psd of the in phase and quadrature components of high Q filter and RC low pass filter of section 4.14.1. You will find that the psd in both these cases is almost the same.
- Thus the in phase and quadrature components of noise have approximately the same characteristics when the white Gaussian noise is passed through a low pass RC filter or an LC filter.

4.17.1 Representation of Narrow Band Noise in Terms of Envelope and Phase Components :

- So far we have represented the narrow band noise n (t) in terms of n_I (t) and n_Q (t).
- Now we may represent the noise n (t) in terms of its envelope and phase components as follows :

$$n(t) = \underbrace{r(t)}_{\text{Envelope of n(t)}} \cos[2\pi f_c t + \underbrace{\psi(t)}_{\text{Phase of n(t)}}] \quad \text{(E-1476)} \quad ...(4.17.11)$$

where $$r(t) = \left[n_I^2(t) + n_Q^2(t)\right]^{1/2}$$

and $$\psi(t) = \tan^{-1}\left[\frac{n_Q(t)}{n_I(t)}\right]$$

- The function r (t) is called as the envelope of n (t) and ψ (t) is called as the phase of n (t).

PDF of r (t) and ψ (t) :

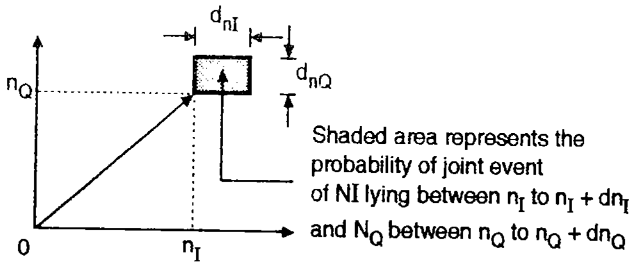
- The PDF of r (t) and ψ (t) can be obtained from the PDFs of n_I (t) and n_Q (t).
- Let N_I and N_Q represent the random variables which are obtained by observing the random processes represented by the sample functions n_I (t) and n_Q (t).
- N_I and N_Q are independent Gaussian random variables having zero mean and variance σ^2.
- Hence we can express their joint PDF with the following expression :

$$f_{N_I N_Q}(n_I\, n_Q) = \frac{1}{2\pi\sigma^2}\, e^{\frac{-n_I^2 + n_Q^2}{\sigma^2}} \quad ...(4.17.12)$$

- That means, probability of joint event of N_I lying between n_I and $(n_I + dn_I)$ and N_Q lying between n_Q and $(n_Q + dn_Q)$ is given by,

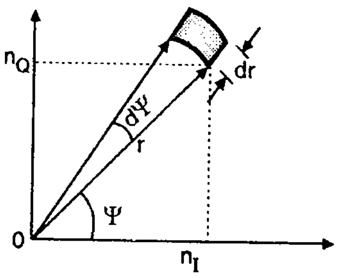
$$f_{N_I N_Q}(n_I, n_Q)\, dn_I\, dn_Q = \frac{1}{2\pi\sigma^2}\, e^{\frac{(-n_I^2 + n_Q^2)}{2\sigma^2}}\, dn_I\, dn_Q \quad ...(4.17.13)$$

- It is graphically shown in Fig. 4.17.4(a) as the shaded area.



Shaded area represents the probability of joint event of NI lying between n_I to $n_I + dn_I$ and N_Q between n_Q to $n_Q + dn_Q$

(a) In terms of in phase and quadrature components



(b) In terms of envelope and phase

(E-849) **Fig. 4.17.4 : Representation of narrowband noise**

- Let us define the relation between n_I, n_Q with r and ψ by referring to Fig. 4.17.4(a) as follows :

$$n_I = r\cos\psi \quad ...(4.17.14)$$

$$n_Q = r\sin\psi \quad ...(4.17.15)$$

- The two shaded regions in Fig. 4.17.4 (a) and (b) can be equated to write

$$\text{Area } dn_I\, dn_Q = \text{Area } r\, dr\, d\psi \quad ...(4.17.16)$$

- Let R and ψ denote the random variables obtained by observing the random processes represented by r (t) and ψ (t).
- So by substituting Equations (4.17.14) to (4.17.16) into Equation (4.17.13) we get the probability of the RVSR and Y lying jointly inside the shaded area of Fig. 4.17.4 (b) as,

$$P = \frac{r}{2\pi\sigma^2}\, e^{(-r^2/2\sigma^2)}\, dr\, d\psi \quad ...(4.17.17)$$

- That means the joint PDF of R and ψ is

$$f_{R,\psi}(r, \psi) = \frac{r}{2\pi\sigma^2} \cdot e^{-(r^2/2\sigma^2)} \quad ...(4.17.18)$$

- This PDF is independent of angle ψ, which means the random variables R and ψ are statistically independent.

- Therefore we can express $f_{R,\psi}(r, \psi)$ as the product of $f_R(r)$ and $f_\psi(\psi)$. The R.V. ψ which represents phase is uniformly distributed in the range 0 to 2π.

$$\therefore \quad f_\psi(\psi) = \begin{cases} \frac{1}{2\pi} & 0 \le \psi \le 2\pi \\ 0 & \text{elsewhere} \end{cases} \qquad ...(4.17.19)$$

$$f_{R,\psi}(r, \psi) = f_R(t) \times f_\psi(\psi)$$

$$= \frac{1}{2\pi} \times f_R(r) = \frac{r}{2\pi\sigma^2} e^{-(r^2/2\sigma^2)}$$

$$\therefore \quad f_R(r) = \begin{cases} \frac{r}{\sigma^2} e^{-(r^2/2\sigma^2)} & ...r \ge 0 \\ 0 & ... \text{elsewhere} \end{cases} \qquad ...(4.17.20)$$

which represents the PDF of R. The expression for $f_R(r)$ shows that R.V. R is having Rayleigh distribution.

- In order to represent $f_R(r)$ graphically let

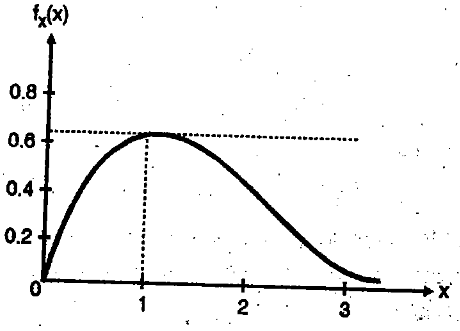
$$x = r/\sigma$$

$$\therefore \quad f_X(x) = \sigma f_R(r) \qquad ...(4.17.21)$$

- We can rewrite the Rayleigh distribution of Equation (4.17.21) in the normalized form as,

$$f_X(x) = \begin{cases} x e^{-(x^2/2)} & ... \quad x \ge 0 \\ 0 & ... \quad \text{elsewhere} \end{cases} \qquad ...(4.17.22)$$

- This expression is plotted in Fig. 4.17.5. The maximum value of $f_X(x)$ corresponds to x = 1 and it is equal to 0.607.
- The PDF $f_X(x)$ is 0 for the negative values of x.



(E-850) **Fig. 4.17.5 : $f_X(x)$ has a Rayleigh distribution**

4.18 Solved Examples :

Ex. 4.18.1 : A random process is expressed as, $X(t) = A \cos(\omega t + \theta)$

where ω and θ are constants and A is a random variable. Determine whether X (t) is wide sensed stationary (WSS) process.

Soln. :

To decide whether the given process is WSS or not, we have to calculate the values of mean $m_x(t)$ and autocorrelation $R_{XX}(t, t+\tau)$.

1. $m_X(t) = E[X(t)] = E[A \cos(\omega t + \theta)]$

$$= \int_{-\infty}^{\infty} A \cos(\omega t + \theta) f_\theta(\theta) \, d\theta$$

Since θ is constant.

$$m_X(t) = \cos(\omega t + \theta) E[A]. \qquad ...(1)$$

This equation indicates that $m_X(t)$ is not constant unless $E[A] = 0$.

2. $R_{XX}(t, t+\tau) = E[X(t) X(t+\tau)]$

$$= \int_{-\infty}^{\infty} A^2 \cos(\omega t + \theta) \cos[\omega(t+\tau) + \theta] f_\theta(\theta) \, d\theta$$

$$= \int_{-\infty}^{\infty} \frac{A^2}{2} [\cos \omega\tau + \cos(2\omega t + 2\theta + \omega\tau)] f_\theta(\theta) \, d\theta$$

Since θ is constant

$$R_{XX}(t, t+\tau) = \frac{1}{2} [\cos \omega\tau + \cos(2\omega t + 2\theta + \omega\tau)] E[A^2]. \qquad ...(2)$$

This equation indicates that the autocorrelation is not a function of time difference τ only.

Conclusion :

Since $m_X(t)$ is not constant and since the autocorrelation is not a function of time difference τ only, the given process is not WSS.

$$\therefore R_{XX}(t, t+\tau) = \frac{1}{2} [\cos(2\omega_o t + \tau) + \cos \omega_o \tau] E[M^2]$$

$$+ \frac{1}{2} [\sin(2\omega_o t + \tau) + \sin \omega_o \tau] E(MN)$$

$$+ \frac{1}{2} [\sin(2\omega_o t + \tau) - \sin(\omega_o \tau)] E(MN)$$

$$+ \frac{1}{2} [\cos(\omega_o \tau) - \cos(2\omega_o t + \tau)] E(N^2)$$

But $E[MN] = 0$ and $E[M^2] = E[N^2] = \sigma^2$

$$\therefore R_{XX}(t, t+\tau) = \frac{1}{2} [\cos(2\omega_o t + \tau) + \cos(\omega_o \tau)] \sigma^2$$

$$+ 0 + 0 + \frac{1}{2} [\cos(\omega_o t) - \cos(2\omega_o t + \tau)] \sigma^2$$

$$\therefore R_{XX}(t, t+\tau) = \sigma^2 \cos(\omega_o \tau) \qquad ...(3)$$

- But RHS of Equation (2) is nothing but $R_{XX}(\tau)$

$$\therefore \quad R_{XX}(t, t+\tau) = \sigma^2 \cos(\omega_o \tau) = R_{XX}(\tau)$$

- Since $R_{XX}(t, t+\tau)$ is a function of only τ, the process X (t) is WSS.
- Thus we have proved that X (t) is WSS if

$$E[MN] = 0 \text{ and } E[M^2] = E[N^2] = \sigma^2$$

Ex. 4.18.2 : A random process X (t) is expressed as

$X(t) = M \cos \omega_0 t + N \sin \omega_0 t$

where ω_0 is constant while M and N are random variables.

(a) Prove that the necessary condition for X (t) to be stationary is,

$E[M] = E[N] = 0$

(b) Prove that X (t) is wide sense stationary if M and N only if and are uncorrelated and have equal variance.

i.e. $E[MN] = 0$

and $E[M^2] = E[N^2] = \sigma^2$

Dec. 16, 6 Marks

Soln. :

Part (a) :

$$m_X(t) = E[X(t)] = E[M \cos \omega_0 t + N \sin \omega_0 t]$$

$$\therefore \quad m_X(t) = \cos(\omega_0 t)\, E[M] + \sin \omega_0 t\, E[N] \qquad ...(1)$$

- For X (t) to be stationary it is necessary that $m_X(t)$ should be independent of t.
- Equation (1) shows that this possible if and only if E[M] and E[N] are equal to zero.
- Thus we have proved that the necessary condition for X (t) to be stationary is

  $E[M] = E[N] = 0.$

Part (b) :

Given :

$$E[MN] = 0 \text{ and } E[M^2] = E[N^2] = \sigma^2$$

$$R_{XX}(t, t+\tau) = E[X(t)\, X(t+\tau)]$$

$$= E[(M \cos \omega_0 t + N \sin \omega_0 t)$$

$$(M \cos \omega_0 (t+\tau) + N \sin \omega_0 (t+\tau))]$$

$$= E[M^2 \cos \omega_0 t \cos \omega_0 (t+\tau)$$

$$+ MN \cos \omega_0 t \sin \omega_0 (t+\tau)$$

$$+ MN \sin \omega_0 t \cos \omega_0 (t+\tau)$$

$$+ N^2 \sin \omega_0 t \sin \omega_0 (t+\tau)]$$

Ex. 4.18.3 : Consider a random process X (t) given by,

$X(t) = A \cos(\omega t + \theta)$

where A and ω are constants and θ is a random variable over [– π, π]. Show that X (t) is ergodic in both the mean and autocorrelation.

Dec. 08, 4 Marks

Soln. :

Step 1 : Obtain the time average :

- The time average of a sample function x (t) of the random process X (t) is defined as,

$$< X(t) > = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} A \cos(\omega t + \theta)\, dt$$

$$= \frac{A}{T_0} \int_{-T_0/2}^{T_0/2} \cos(\omega t + \theta)\, dt$$

where $T_0 = 2\pi / \omega$. since $\omega T_0 = 2\pi$

$$\therefore \quad < x(t) > = 0 \qquad ...(1)$$

Step 2 : Obtain the time averaged autocorrelation :

- The time averaged autocorrelation of the sample function is defined as,

$$\bar{R}_{XX}(\tau) = < X(t)\, X(t+\tau) >$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x(t+\tau)\, dt$$

$$= \frac{A^2}{T_0} \int_{-T_0/2}^{T_0/2} \cos(\omega t + \theta) \cos[\omega(t+\tau) + \theta]\, dt$$

$$= \frac{A^2}{2 T_0} \int_{-T_0/2}^{T_0/2} [\cos(2\omega t + \omega \tau + 2\theta) + \cos(\omega \tau)]\, dt$$

$$= \frac{A^2}{2 T_0} \int_{-T_0/2}^{T_0/2} \cos[2\omega t + \omega t + 2\theta]\, dt$$

$$+ \frac{A^2}{2 T_0} \int_{-T_0/2}^{T_0/2} \cos(\omega \tau)\, dt$$

$$\therefore \quad \bar{R}_{XX}(\tau) = \frac{A^2}{2 T_0} \times 0 + \frac{A^2}{2 T_0} \cos(\omega \tau)\, [t]_{-T_0/2}^{T_0/2}$$

$$= \frac{A^2}{2 T_0} \cos(\omega \tau) \times T_0$$

$$\therefore \quad \bar{R}_{XX}(\tau) = \frac{A^2}{2} \cos(\omega \tau) \qquad ...(2)$$

Conclusion :

- In Ex. 4.18.2 we have proved that

  $$m_X(t) = 0 \text{ and } R_{XX}(t, t+\tau) = \frac{A^2}{2} \cos(\omega \tau).$$

- Comparing these results with the Equations (1) and (2) of this example we conclude that,

1. $m_X(t) = < x(t) > = 0$

   i.e. Ensemble mean = Time averaged mean.

2. $R_{XX}(\tau) = \bar{R}_{XX}(\tau)$

- Hence the process X (t) is an ergodic process.

Ex. 4.18.4 : If the random process is WSS then prove that,

$$R_{XX}(-\tau) = R_{XX}(\tau).$$

Soln. :

- We know that

$$R_{XX}(-\tau) = E[X(t)\,X(t+\tau)]$$

- Substitute $(t+\tau) = t'$ and $\therefore$ $t = (t'-\tau)$ we get,

$$R_{XX}(-\tau) = E[X(t'-\tau)\,X(t')]$$

$$= E[X(t')\,X(t'-\tau)]$$

- But the RHS represents the autocorrelation $R_{XX}(\tau)$.

$$\therefore \quad R_{XX}(-\tau) = R_{XX}(\tau). \qquad \textbf{... Proved.}$$

Ex. 4.18.5 : If the random process X (t) is WSS then prove that, $|R_{XX}(\tau)| \le R_{XX}(0)$

Soln. :

- We know that,

$$E\{[X(t) \pm X(t+\tau)]^2\} \ge 0$$

because the mean square value will always be non-negative.

$$\therefore \quad E = [X^2(t)] \pm E[2X(t)X(t+\tau)] + E[X^2(t+\tau)] \ge 0$$

$$\therefore \quad R_{XX}(0) \pm 2R_{XX}(\tau) + R_{XX}(0) \ge 0$$

$$\therefore \quad 2R_{XX}(0) \ge |2R_{XX}(\tau)|$$

$$\therefore \quad R_{XX}(0) \ge |R_{XX}(\tau)| \qquad \textbf{...Proved.}$$

Ex. 4.18.6 : If a random process X (t) is real, then prove that its power spectrum is real. Also verify that $S_{XX}(-\omega) = S_{XX}(\omega)$

Soln. :

Part I : To prove that $S_{XX}(\omega)$ is real :

- Power spectrum is same as power spectral density and the power spectral density of X (t) is equal to the Fourier transform of its auto correlation $R_{XX}(\tau)$.

$$\therefore \text{ Power spectrum } S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\cdot e^{-j\omega\tau\, d\tau}$$

- But $e^{-j\omega\tau} = \cos\omega\tau - j\sin\omega\tau$

$$\therefore S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\cdot[\cos(\omega\tau) - j\sin(\omega\tau)]\,d\tau$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau)\cos\omega\tau\,d\tau - j\int_{-\infty}^{\infty} R_{XX}(\tau)\sin\omega\tau\,d\tau$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau)\cos\omega\tau\,d\tau - j\left[\int_{-\infty}^{0} R_{XX}(-\tau)\sin\omega\tau\,d\tau + \int_{0}^{\infty} R_{XX}(\tau)\sin\omega\tau\,d\tau\right]$$

- Since $R_{XX}(-\tau) = R_{XX}(\tau)$ and $\sin(\omega\tau)$ is an odd function, the imaginary part of the above expression will be zero.

$$\therefore \quad S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\cos(\omega\tau)\,d\tau \qquad ...(1)$$

- This equation indicates that $S_{XX}(\omega)$ is real.

Part II : To prove that $S_{XX}(-\omega) = S_{XX}(\omega)$:

- Substitute $\omega = -\omega$ into Equation (1) to get,

$$S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\cos(-\omega\tau)\,d\tau$$

- But $\cos(-\omega\tau) = \cos(\omega\tau)$

$$\therefore \quad S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\cos(\omega\tau)\,d\tau$$

$$\therefore \quad S_{XX}(-\omega\tau) = S_{XX}(\omega) \qquad \textbf{...Proved.}$$

Ex. 4.18.7 : X (t) and Y (t) are the two WSS random processes with zero mean. If the random process Z(t) is such that,

$Z(t) = X(t) + Y(t)$.

Then obtain the autocorrelation of Z (t) if X (t) and Y (t) are jointly WSS.

Soln. :

Part I : Autocorrelation of Z (t) :

The autocorrelation of the random process Z (t) is given by,

$$R_{ZZ}(t_1, t_2) = E[Z(t_1)\,Z(t_2)]$$

$$= E[[X(t_1) + Y(t_1)]]\;[[X(t_2) + Y(t_2)]]$$

$$= E[X(t_1)X(t_2)] + E[X(t_1)Y(t_2)] + E[Y(t_1)X(t_2)] + E[Y(t_1)Y(t_2)]$$

$$\therefore R_{ZZ}(t_1, t_2) = R_{XX}(t_1, t_2) + R_{XY}(t_1, t_2) + R_{YX}(t_1, t_2) + R_{YY}(t_1, t_2) \qquad ...(1)$$

- If X (t) and Y (t) are jointly WSS, then

$R_{XX}(t_1, t_2) = R_{XX}(\tau)$ where $\tau = (t_2 - t_1)$

$R_{XY}(t_1, t_2) = R_{XY}(\tau)$, $R_{YX}(t_1, t_2) = R_{YX}(\tau)$

and $R_{YY}(t_1, t_2) = R_{YY}(\tau)$

- Substituting in Equation (1) we get,

$$\therefore \quad R_{ZZ}(\tau) = R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau) \qquad ...(2)$$

This is the required result.

Ex. 4.18.8 : For the same processes in the previous example find the autocorrelation of Z (t) if X (t) and Y (t) are orthogonal.

Soln. :

- If X (t) and Y (t) are orthogonal then

$$R_{XY}(\tau) = R_{YX}(\tau) = 0$$

- Substituting this into Equation (2) of the previous example we get,

$$R_{ZZ}(\tau) = R_{XX}(\tau) + R_{YY}(\tau) \qquad \text{...Ans.}$$

Ex. 4.18.9 : Random processes X (t) and Y (t) are WSS with zero mean. The random process Z (t) is such that,

Z (t) = X (t) + Y (t)

If X (t) and Y (t) are orthogonal, then prove that the mean square of Z (t) is equal to the sum of mean squares of X (t) and Y (t).

Soln. :

- In the previous example we have proved that if X (t) and Y (t) are orthogonal, then

$$R_{ZZ}(\tau) = R_{XX}(\tau) + R_{YY}(\tau)$$

- Substituting $\tau = 0$ we get,

$$R_{ZZ}(0) = R_{XX}(0) + R_{YY}(0)$$

- That means

$$E[Z(t) \cdot Z(t)] = E[X(t)\,X(t)] + E[Y(t)\,Y(t)]$$

$$\therefore \quad E[Z^2(t)] = E[X^2(t)] + E[Y^2(t)] \qquad ...(1)$$

- Equation (1) shows that the mean square of Z (t) is equal to mean squares of X (t) and Y (t).

Ex. 4.18.10 : If X (t) and Y (t) are given by :

X (t) = M cos ωt + N sin ωt

Y (t) = N cos ωt – M sin ωt

where ω is constant, M and N are independent random variables having zero mean and variance σ^2. Find the cross correlation of X (t) and Y (t).

Soln. :

- The cross correlation of X (t) and Y (t) is given by,

$$R_{XX}(t_1, t_2) = E[X(t_1)\,Y(t_2)]$$

$$= E[(M\cos\omega t_1 + N\sin\omega t_1)(N\cos\omega t_2 - M\sin\omega t_2)]$$

$$= E[(MN)](\cos\omega t_1 \cos\omega t_2 - \sin\omega t_1 \sin\omega t_2)$$

$$- E[M^2]\cos\omega t_1 \sin\omega t_2 + E[N^2]\sin\omega t_1 \cos\omega t_2$$

- But $E[MN] = E[M]\,E[N] = 0$

and $E[M^2] = E[N^2] = \sigma^2$

$$\therefore \quad R_{XY}(t_1, t_2) = -\sigma^2 \cos\omega t_1 \sin\omega t_2 + \sigma^2 \sin\omega t_1 \cos\omega t_2$$

$$= \sigma^2 [\sin\omega t_1 \cos\omega t_2 - \cos\omega t_1 \sin\omega t_2]$$

$$= \sigma^2 [\sin(\omega t_1 - \omega t_2)]$$

$$\therefore \quad R_{XY}(t_1, t_2) = \sigma^2 [\sin\omega(t_1 - t_2)]$$

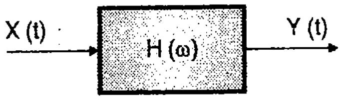
$$\therefore \quad R_{XY}(\tau) = R_{XX}(t_2 - t_1) = \sigma^2 [-\sin\omega(t_2 - t_1)]$$

$$\therefore \quad R_{XY}(\tau) = -\sigma^2 \sin\omega\tau \qquad \text{...Ans.}$$

This is the required expression for cross correlation.

Ex. 4.18.11 : The random process X (t) is WSS. It is applied to an LTI system having an impulse response $4\,e^{-3t}\,u(t)$. Find the mean value of output Y (t) of the system if E [X (t)] = 3.

Soln. :



(E-851) **Fig. P. 4.18.11**

- Refer to the LTI system shown in Fig. P. 4.18.11.
- The frequency response of the LTI system is given by,

$$H(\omega) = \text{F.T. of } h(t) = \int_{-\infty}^{\infty} h(t) \cdot e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} 4e^{-3t}\,u(t)\,e^{-j\omega t}\,dt = 4\int_{0}^{\infty} e^{-3t}\,e^{-j\omega t}\,dt$$

$$\therefore \quad H(\omega) = 4\int_{0}^{\infty} e^{-(3+j\omega)t}\,dt = \frac{4}{-(3+j\omega)}\left[e^{-(3+j\omega)t}\right]_0^{\infty}$$

$$= \frac{-4}{(3+j\omega)}\left[e^{-\infty} - e^{0}\right]$$

$$\therefore \quad H(\omega) = \frac{4}{3+j\omega} \qquad ...(1)$$

- The mean value of Y (t) is given by,

$$E[Y(t)] = m_X(t)\,H(0) = E[X(t)]\,H(0)$$

$$= 3 \times \frac{4}{3} = 4 \qquad \text{...Ans.}$$

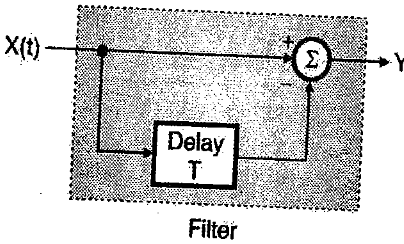
Ex. 4.18.12 : A wide sense stationary random processes X(t) is applied to input of an LTI system with impulse response $h(t) = 3e^{-2t}\,u(t)$. Find the mean value of output Y(t) of system if E[X(t)] = 2. **Dec. 08. 4 Marks**

Ans. : Similar to Ex. 4.18.11.

Ex. 4.18.13 : A WSS random process X (t) is applied to the input of a LTI system with impulse response h (t) = a exp (– at) u (t). Find the mean value of the output Y (t) of the system if E [X (t)] = 6 and a = 2. **May 11, 6 Marks**

Ans. : Similar to Ex. 4.18.11.

Ex. 4.18.14 : The WSS random process X (t) is applied to the input of filter shown in Fig. P. 4.18.14. Find the expression of the power spectrum of output process Y (t) if the power spectral density of X (t) is $S_{XX}(\omega)$.



(E-852) **Fig. P. 4.18.14**

Soln. :

- From Fig. P. 4.18.14 the expression for Y (t) is given by,

$$Y(t) = X(t) - X(t-T) \qquad \ldots(1)$$

- The impulse response of this filter can be obtained by applying the impulse δ (t) at the input of the filter and noting down the filter response.

$$\therefore\ h(t) = \delta(t) - \delta(t-T)$$

- Hence the frequency response of the filter is given by,

$$H(\omega) = F[h(t)] = F[\delta(t) - \delta(t-T)]$$

$$\therefore\ H(\omega) = 1 - e^{-j\omega T} \qquad \ldots(2)$$

- So the output power spectrum is given by,

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega) = |1 - e^{-j\omega T}|^2 S_{XX}(\omega)$$

$$= |1 - (\cos\omega T - j\sin\omega T)|^2 S_{XX}(\omega)$$

$$= |(1-\cos\omega T) - j\sin\omega T|^2 S_{XX}(\omega)$$

$$= \left[\sqrt{(1-\cos\omega T)^2 + \sin(\sin\omega T)^2}\right]^2 S_{XX}(\omega)$$

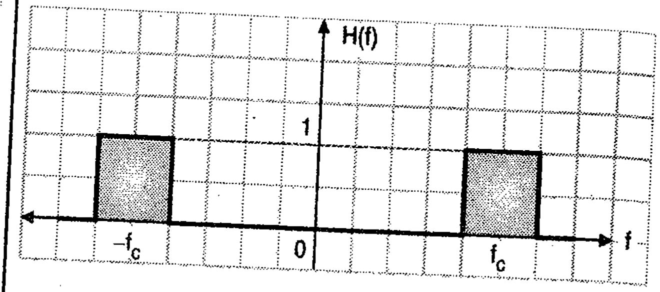
$$= [(1-\cos\omega T)^2 + \sin(\omega T)^2]\, S_{XX}(\omega)$$

$$\therefore\ S_{YY}(\omega) = [1 - 2\cos\omega T + \cos^2\omega T + \sin^2\omega T]\, S_{XX}(\omega)$$

$$= [1 + 1 - 2\cos\omega T]\, S_{XX}(\omega)$$

$$= [2(1-\cos\omega T)]\, S_{XX}(\omega) \qquad \ldots\textbf{Ans.}$$

Ex. 4.18.15 : The input X (t) to an ideal bandpass filter having a frequency response characteristics shown in Fig. P. 4.18.15 is a white noise process. Determine the total noise power of the filter.



(E-853) **Fig. P. 4.18.15**

Ans. : Refer section 4.16.3 for answer.

Ex. 4.18.16 : If the process X (t) = A cos (2π f_c t + φ), where φ is a random variable uniformly distributed in the range (0, 2π) is passed through a filter with H (f) = j 2πf. Find the output PSD. **May 11, 6 Marks**

Soln. :

Step 1 : Find the PSD of the given process :

1. Autocorrelation :

The autocorrelation of the given process can be obtained as follows :

$$\text{Process} : X(t) = A\cos(2\pi f_c\, t) + \phi$$

$$\text{PDF} : P_\phi(\phi) = \frac{1}{2\pi} \quad : 0 \le \phi \le 2\pi$$

$$= 0 \quad : \text{elsewhere}$$

Therefore the autocorrelation is given by,

$$R_X(t_1, t_2) = E[X(t_1)\, X(t_2)]$$

$$R_X(t_1, t_2) = E[A\cos(2\pi f_c t_1 + \phi)\, A\cos(2\pi f_c\, t_2 + \phi)]$$

$$= \frac{A^2}{2} E[\cos 2\pi f_c (t_1 - t_2) + \cos 2\pi f_c (t_1 + t_2) + 2\phi]$$

$$= \frac{A^2}{2}\cos[2\pi f_c (t_1 - t_2)] \qquad \ldots(1)$$

$$\therefore\ R_X(\tau) = \frac{A^2}{2}\cos 2\pi f_c\, \tau$$

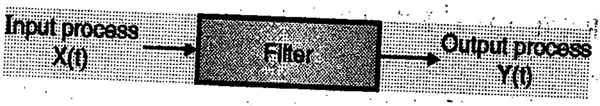
2. PSD :

PSD and autocorrelation form a Fourier transform pair.

$$\therefore\ \text{PSD } S_X(f) = F(R_x(\tau))$$

$$= F\left[\frac{A^2}{2}\cos 2\pi f_c\, \tau\right]$$

$$\therefore\ S_X(f) = \frac{A^2}{4}[\delta(f - f_c) + \delta(f + f_c)] \quad \ldots(2)$$



(E-1155) **Fig. P. 4.18.16(a)**

This PSD is as shown in Fig. P. 4.18.16(b).

The PSD of the filter output is given by,

$$S_Y(f) = |H(f)|^2 S_X(f) \quad \ldots(3)$$

$$\text{But } H(f) = j\,2\pi f \quad \therefore \; |H(f)|^2 = 4\pi^2 f^2$$

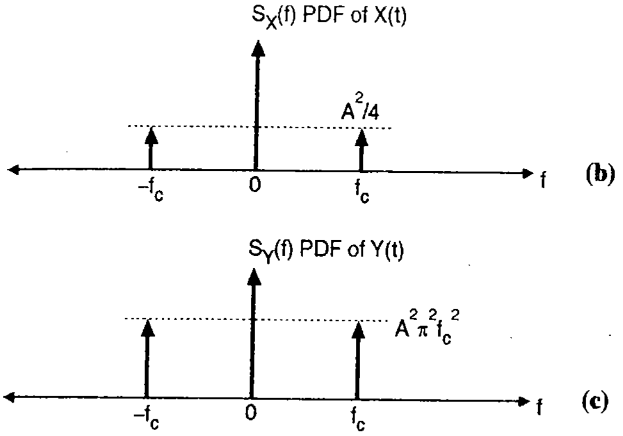
$$\therefore \; S_Y(f) = 4\pi^2 f^2 \times \frac{A^2}{4} [\,\delta(f-f_c) + \delta(f+f_c)\,] \quad \ldots(4)$$

$$\text{But } x(t)\,\delta(t-t_d) = x(t_d)\,\delta(t-t_d)$$

$$\therefore \; S_Y(f) = A^2 \pi^2 f_c^2 [\,\delta(f-f_c) + \delta(f+f_c)\,]$$

$$\therefore \; S_Y(f) = A^2 \pi^2 f_c^2[\delta(f-f_c) + A^2 \pi^2 f_c^2\, \delta(f+f_c)] \quad \ldots(5)$$

The output PSD is plotted in Fig. P. 4.18.16(c).



(E-1156) Fig. P. 4.18.16 : PSD of process X (t) and the filter output process Y (t)

Ex. 4.18.17 : The random variable X has a uniform distribution over a $0 \le x \le 2$. Find mean and mean square value for the random process $V(t) = 6e^{Xt}$. **May 11, 4 Marks**

Soln. :

Given : $V(t) = 6\,e^{Xt}$ ...Given process

$$f_X(x) = \frac{1}{2} \quad \ldots 0 \le x \le 2 \qquad \text{Uniform distribution of X over 0 to 2}$$
$$= 0 \quad \ldots\text{Elsewhere}$$

∴ Mean of the process is given by,

$$m_v = E[V(t)] = E[\,6\,e^{Xt}\,]$$

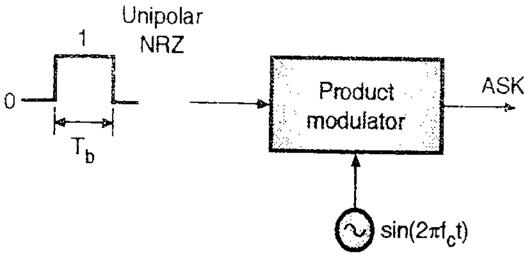
$$= \int_0^2 6\,e^{Xt} \frac{1}{2} dx = \frac{3}{t}[e^{Xt}]_0^2 = \frac{3}{t}[e^{2t} - e^0]$$

$$= \frac{3}{t}[e^{2t} - 1] \quad \ldots\textbf{Ans.}$$

Ex. 4.18.18 : Derive an equation of PSD for ON-OFF signaling. **May 11, 6 Marks**

Soln. :

- ON-OFF signalling means BASK. The ASK signal is obtained by multiplying the NRZ unipolar signal, with sin $\omega_c t$ as shown in Fig. P. 4.18.18(a).

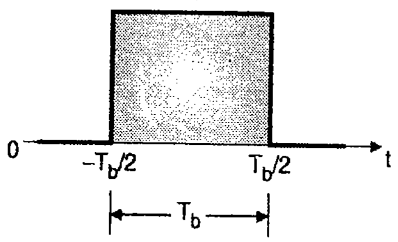


(E-1158) Fig. P. 4.18.18(a) : Generation of ASK signal

- The NRZ unipolar signal is as shown in Fig. P. 4.18.18(b). The Fourier transform of this signal is given by,

$$X(f) = T_b \sin c\,(f\,T_b)$$

$$= T_b \frac{\sin(\pi f\,T_b)}{\pi f\,T_b} \quad \ldots(1)$$



(E-1159) Fig. P. 4.18.18(b) : Unipolar NRZ signal

The PSD of the NRZ signal is given by,

$$S(f) = \frac{|\overline{X(f)}|^2}{T_b}$$

Where $\overline{X(f)}$ represents the average value of X(f) due to large number of pulses in the unipolar NRZ input signal.

Substituting the value of X(f) we get,

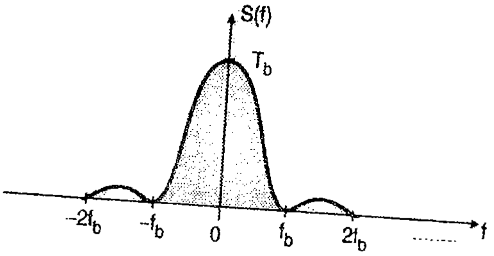
$$\text{PSD, } S(f) = T_b \left[\frac{\sin(\pi\,f\,T_b)}{\pi\,f T_b}\right]^2 \quad \ldots(2)$$

This is the PSD of the unipolar NRZ signal. This spectrum is as shown in Fig. P. 4.18.18(c).

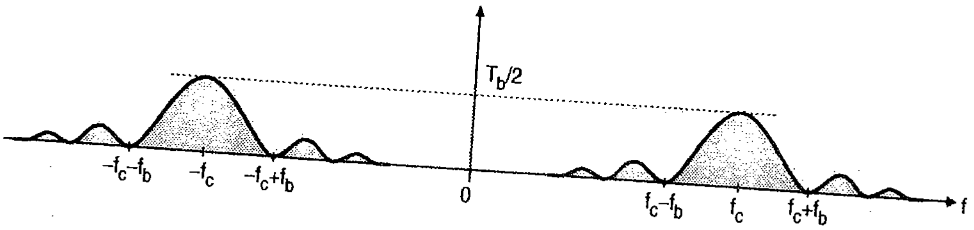
In order to obtain the BASK signal this unipolar NRZ signal is multiplied by the carrier signal sin ($2\pi\,f_c t$). Due to modulation the spectrum of the baseband NRZ signal gets translated about + f_c and − f_c and its amplitude reduces to half of the baseband spectrum. Hence the PSD of BASK is given by,

$$S_{(BASK)}(f) = \frac{T_b}{2}\left\{\left[\frac{\sin\pi\,(f_c - f)\,T_b}{\pi\,(f_c - f)\,T_b}\right]^2 + \left[\frac{\sin\pi\,(f_c + f)\,T_b}{\pi\,(f_c + f)\,T_b}\right]^2\right\}$$

The PSD of BASK is plotted in Fig. 4.18.18(d).



(c) PSD of the unipolar NRZ signal



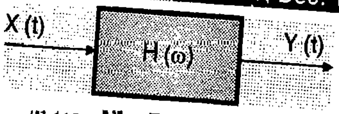
(d) PSD of BASK

(E-1160) Fig. P. 4.18.18

Ex. 4.18.19 : What are the different properties of random process ? A wide sense stationary random process X(t) is applied to the input of LTI system with impulse response $h(t) = 3e^{-2t} u(t)$. Find the mean value of output Y(t) of the system if E[X(t)] = 2.

Dec. 11, Dec. 13, 8 Marks

Soln. :



(E-1157) Fig. P. 4.18.19

- Refer to the LTI system shown in Fig. P. 4.18.19.
- The frequency response of the LTI system is given by,

$$H(\omega) = \text{F.T. of } h(t) = \int_{-\infty}^{\infty} h(t)\cdot e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} 3e^{-2t}\,u(t)\,e^{-j\omega t}\,dt = 3\int_{0}^{\infty} e^{-2t}\,e^{-j\omega t}\,dt$$

$$\therefore H(\omega) = 3\int_{0}^{\infty} e^{-(2+j\omega)t}\,dt = \frac{3}{-(2+j\omega)}\,[e^{-(2+j\omega)t}]_{0}^{\infty}$$

$$= \frac{-3}{(2+j\omega)}\,[e^{-\infty}-e^{0}]$$

$$\therefore H(\omega) = \frac{3}{2+j\omega} \qquad \text{...(1)}$$

- The mean value of Y (t) is given by,

$$E[Y(t)] = m_X(t)\,H(0) = E[X(t)]\,H(0)$$

$$= 2\times\frac{3}{2} = 3 \qquad \text{...Ans.}$$

Ex. 4.18.20 : Let X(t) be a zero-mean, stationary, Gaussian process with autocorrelation function $R_X(t)$. This process is applied to a square law device, which is defined by the input-output relation $Y(t) = X^2(t)$ Where Y(t) is the output. Show that the mean of Y(t) is $R_X(0)$.

May 13, 8 Marks

Soln. :

The autocorrelation function is also defined as,

$$R_X(\tau) = E[X(t+\tau)\,X(t)]$$

Substituting τ = 0 in this expression we get

$$R_X(0) = E[X(t)\,X(t)] = E[X^2(t)]$$

In this example it is given that

$$Y(t) = X^2(t)$$

$$\therefore R_X(0) = E[Y(t)]$$

The RHS of this expression represents the mean value of output Y(t)

$$\therefore R_X(0) = \text{mean of } Y(t) \qquad \text{...Proved.}$$

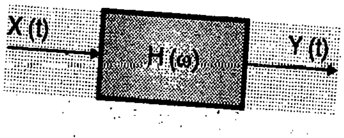
Ex. 4.18.21 : When a WSS random processes X(t) is applied to input of an LTI system with impulse response $h(t) = 3e^{-2t}$ u(t) find the mean value of system if E[X(t)] = 2.

May 15, 8 Marks

Soln. :

Refer to the LTI system shown in Fig. P. 4.18.21. The frequency response of the LTI system is given by,

$$H(\omega)=\text{F.T. of } h(t) = \int_{-\infty}^{\infty} h(t)\cdot e^{-j\omega t}\,dt$$



(E-851) Fig. P. 4.18.21

$$= \int_{-\infty}^{\infty} 3e^{-2t} u(t) e^{-j\omega t} dt = 3 \int_{0}^{\infty} e^{-2t} e^{-j\omega t} dt$$

$$\therefore \quad H(\omega) = 3 \int_{0}^{\infty} e^{-(2+j\omega)t} dt = \frac{3}{-(2+j\omega)} [e^{-(2+j\omega)t}]_{0}^{\infty}$$

$$= \frac{-3}{(2+j\omega)} [e^{-\infty} - e^{0}]$$

$$\therefore \quad H(\omega) = \frac{3}{2+j\omega} \qquad \text{...(1)}$$

The mean value of Y (t) is given by,

$$E[Y(t)] = m_X(t) H(0) = E[X(t)] H(0) = 2 \times \frac{3}{2} = 3$$

Review Questions

Q. 1 Define a random process.

Q. 2 What is sample function and ensemble ?

Q. 3 Explain the relationship between random variables and random processes.

Q. 4 Define ensemble mean and time average.

Q. 5 Define autocorrelation function of a random process, derive its expression and explain its significance.

Q. 6 State the types of random processes.

Q. 7 Define the stationary and nonstationary random processes.

Q. 8 What is a strictly stationary process ?

Q. 9 Define a jointly stationary processes.

Q. 10 What is a wide-sensed stationary process ?

Q. 11 Write a short note on : Ergodic process.

Q. 12 Derive the expression for the power spectral density (PSD) of a wide-sensed process.

Q. 13 State and explain the properties of PSD.

□□□

CHAPTER 5

Baseband Receivers

Unit IV

Syllabus :

Signal space representation, Geometric representation of signal, Conversion of continuous AWGN channel to vector channel, Likelihood functions, Coherent detection of binary signals in presence of noise, Optimum filter, Matched filter, Probability of error of matched filter, Correlation receiver.

5.1 Detection and Estimation :

- In this chapter we are going to discuss two basic issues in digital communication namely detection and estimation.
- **Detection** : The detection theory is used for designing a decision making device which looks at the input signals and makes a decision about whether 0 or 1 was transmitted.
- **Estimation** : The estimation theory has been developed in order to design a processor which uses the information in the received signal to extract the estimates of some physical parameters or waveforms.
- There is always a possibility of error getting introduced into the detection as well as estimation.

5.1.1 Digital Communication System :

The simplified digital communication system is shown in Fig. 5.1.1.

Transmitter :

- The message source produces one symbol per T seconds. There are M such symbols namely $m_1, m_2, ..., m_M$ or in general m_i, produced by the source.
- All these M symbols are equally likely hence the probability of each symbol is given by,

$$p_i = P(m_i \text{ transmitted}) = \frac{1}{M} \quad ...(5.1.1)$$

- The message source output is applied to a **vector transmitter** which produces a vector of real numbers. For every message signal m_i the vector transmitter produces an output signal s_i.
- For example when the message source output $m = m_i$ then the output of the vector transmitter is

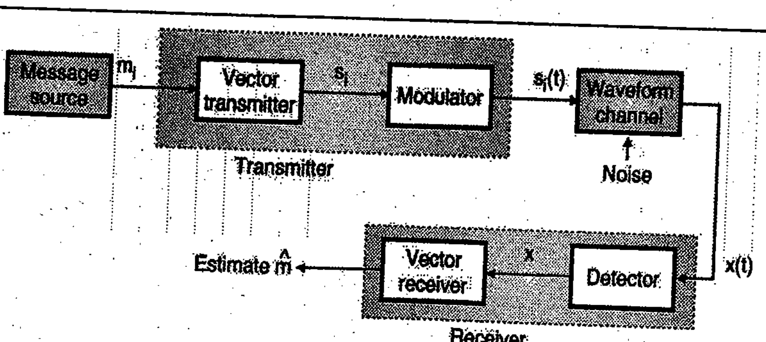
$$s_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ \vdots \\ s_{iN} \end{bmatrix} \text{ with } i = 1, 2, 3, ... M \quad ...(5.1.2)$$

where $N \le M$.

- This vector is applied to a modulator which produces a distinct signal $s_i(t)$ of duration T seconds. This signal is an energy signal.
- One such signal is transmitted every T seconds.

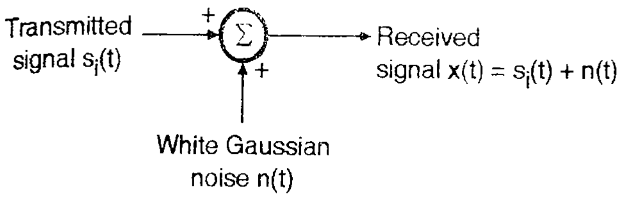
Channel :

- A channel in digital communication system should have the following two characteristics.
  1. The channel should be linear. It should have a sufficient bandwidth which ensures an undistorted transmission of $s_i(t)$.
  2. The signal $s_i(t)$ is contaminated by White Gaussian noise.



(E-867) Fig. 5.1.1 : Digital communication system

- Such a channel is called as Additive White Gaussian noise (AWGN) channel. The model of AWGN channel is shown in Fig. 5.1.2.



(E-868) **Fig. 5.1.2 : Model of an additive white Gaussian noise**

Receiver :

- The received signal is a random process that is mathematically expressed as :

$$X(t) = s_i(t) + n(t) \quad \ldots 0 \le t \le T$$

$$i = 1, 2, \ldots M \quad \ldots(5.1.3)$$

- The receiver observes the received signal x (t) over a duration of T seconds and makes the best estimate of the transmitted signal $s_i(t)$ or m_i. Any such estimate is a guess made by the receiver about the transmitted signal.
- This is achieved by using a detector and a vector receiver as shown in the block diagram of Fig. 5.1.2.
- In the detection and estimation process there is always a possibility of incorrect decision being made. This could introduce errors.
- The receiver is called as an **optimum receiver** if the error probability is successfully minimised. The error probability is defined as,

$$P_e = P(\hat{m} = m_i) \quad \ldots(5.1.4)$$

where $\hat{m}$ is the estimate obtained at the receiver output and m_i is the transmitted message. Minimum error probability ensures correct estimation of the transmitted message.

Types of receivers :

The receivers can be of two types :

1. Coherent receivers
2. Noncoherent receivers

Coherent receivers :

If the receiver is phase synchronized to the transmitter, then it is called as a coherent receiver.

Non-coherent receivers :

If there is no phase synchronization between the transmitter and receiver, then the receiver is called as non-coherent receiver.

5.2 Geometric Representation of Signals :

SPPU - Dec. 16

University Questions

Q. 1 Explain geometrical representation of signal. **(Dec. 16, 4 Marks)**

- We have seen that once we choose a convenient set of orthonormal basis functions $\{ \phi_j, (t) \}$, with j = 1, 2, ..., N then it is possible to expand each signal in the set $\{ s_i(t) \}$ with i = 1, 2, ..., M as follows :

$$s_i(t) = \sum_{j=1}^{N} s_{ij} \phi_j(t) \qquad 0 \le t \le T \qquad \ldots(5.2.1)$$

$$i = 1, 2, \ldots, M$$

- That means the set of signals $\{ s_i(t) \}$ for different values of i are as follows :

$$s_1(t) = \sum_{j=1}^{N} s_{ij} \phi_j(t)$$

$$\therefore \quad s_1(t) = s_{11} \phi_1(t) + s_{12} \phi_2(t) + \ldots.. + s_{1N} \phi_N(t) \qquad \ldots(5.2.2)$$

Similarly $s_2(t) = s_{21} \phi_1(t) + s_{22} \phi_2(t) + \ldots.. + s_{2N} \phi_N(t) \qquad \ldots(5.2.3)$

and so on.

- Here s_{ij} are called as the coefficients for expansion and they are defined as follows :

$$s_{ij} = \int_0^T s_i(t) \cdot \phi_j(t)\, dt, \; i = 1, 2, \ldots M$$

$$j = 1, 2, \ldots N \qquad \ldots(5.2.4)$$

That means $s_{11} = \int_0^T s_1(t) \cdot \phi_1(t)\, dt$

$$s_{12} = \int_0^T s_1(t) \cdot \phi_2(t)\, dt$$

and so on.

- Hence each signal in the set $\{s_i(t)\}$ can be completely determined by the vector of its coefficients as shown below.

$$\underbrace{s_1}_{\text{First signal in the set } \{s_i(t)\}} = \underbrace{\begin{bmatrix} s_{11} \\ s_{12} \\ \vdots \\ s_{1N} \end{bmatrix}}_{\text{Vector of coefficient}}$$

(E-871)

So in general we can express the signal s_i as follows :

(E-872)

$$\underbrace{s_i}_{\text{Signal}} = \underbrace{\begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix}}_{\text{Vector of coefficient}} \quad i = 1, 2, \ldots\ldots\ldots M \qquad \ldots(5.2.5)$$

- The signal s_i is also called as the signal vector.

5.3 Signal Space Representation :

- The signal space is an N-dimensional Euclidean space which is characterized by N axes that are mutually perpendicular and denoted as $\phi_1, \phi_2, \ldots \phi_N$.
- The set of signal vectors $\{ s_i \}$ with $i = 1, 2, \ldots M$ can be plotted in the signal space.
- Fig. 5.3.1 illustrates the concept of a two-dimensional signal space with three signals that means here value of N is 2 and that of M is 3.
- Since N = 2 we have used two mutually perpendicular axes labelled ϕ_1 and ϕ_2.

5.3.1 N-dimensional Signal Space :

- In an N-dimensional Euclidean space, we have to use two parameters namely length of the vectors and angles between the vectors.
- The length of norm of a signal vector s_i is denoted by the symbol $\| s_i \|$.
- And the squared length of any vector s_i is defined as the dot product of s_i with itself. It is expressed mathematically as follows :

$$\text{The squared length,} \quad \| s_i \|^2 = (s_i , s_i) = \sum_{j=1}^{N} s_{ij}^2 \qquad \ldots(5.3.1)$$

Where s_{ij} are the elements of s_i.

- The cosine of angle between any two vectors is equal to the ratio of their dot product and the product of their individual norms (lengths).
- So cosine of angle between the vectors s_i and s_j is given by,

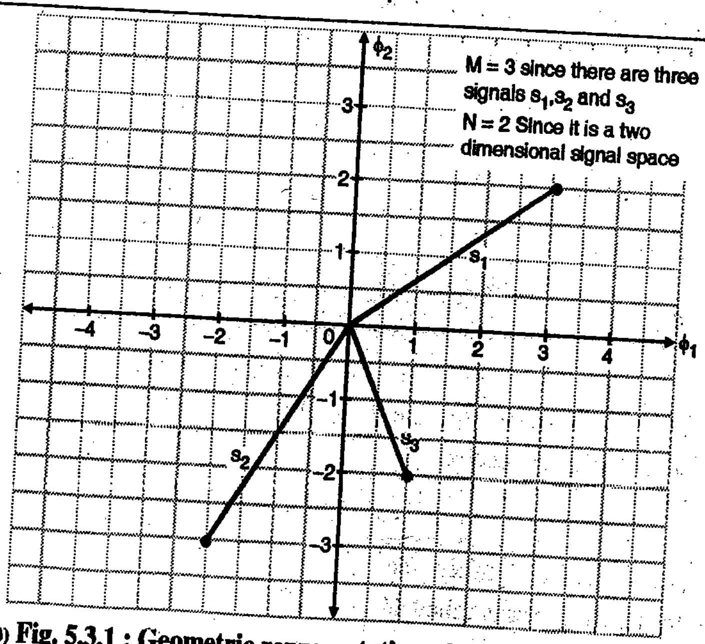
$$\cos \theta = \frac{\text{Dot product of } s_i \text{ and } s_j}{\text{Product of lengths of } s_i \text{ and } s_j} = \frac{(s_i , s_j)}{\| s_i \| \, \| s_j \|} \qquad \ldots(5.3.2)$$

Condition for orthogonality :

- The two vectors s_i and s_j are said to be orthogonal if the angle θ between them is 90° or if their dot product is equal to zero.

Relation between energy and vector representation :

- The energy of a signal $s_i (t)$ of duration T is given by,



(E-873) Fig. 5.3.1 : Geometric representation of signals for N = 2 and M = 3

$$E_i = \int_0^T s_i^2(t)\,dt \qquad \text{...(5.3.3)}$$

- But $s_i(t) = \sum_{j=1}^{N} s_{ij}\,\phi_j(t)$

or $$s_i(t) = \sum_{k=1}^{N} s_{ik}\,\phi_k(t)$$

- Substituting these into the expression for E_i we get,

$$E_i = \int_0^T \left[\sum_{j=1}^{N} s_{ij}\,\phi_j(t)\right]\left[\sum_{k=1}^{N} s_{ik}\,\phi_k(t)\right] \qquad \text{...(5.3.4)}$$

- Interchanging the order of summation and integration we get,

$$E_i = \sum_{j=1}^{N}\sum_{k=1}^{N} s_{ij}\,s_{ik}\int_0^T \phi_j(t)\,\phi_k(t)\,dt \qquad \text{...(5.3.5)}$$

- But the functions $\phi_j(t)$ and $\phi_k(t)$ are orthonormal if the following condition is satisfied.

$$\int_0^T \phi_j(t)\,\phi_k(t)\,dt = \begin{cases} 1 & \text{... if } j = k \\ 0 & \text{... if } j \neq k \end{cases} \qquad \text{...(5.3.6)}$$

- Applying this condition to the Equation (5.3.5) for E_i we get,

$$E_i = \sum_{j=1}^{N}\sum_{k=1}^{N} s_{ij}\,s_{ik} \quad \text{... with } j = k$$

$$\therefore \quad E_i = \sum_{j=1}^{N} s_{ij}\,s_{ij} = \sum_{j=1}^{N} s_{ij}^2 \qquad \text{...(5.3.7)}$$

Thus the energy of a signal s_i (t) is equal to the squared length of the signal vector s_i representing it.

5.4 Detection of Known Signals in Noise :

- Let us assume that a source is capable of transmitting M different signals S_1 (t), S_2 (t), ..., S_M (t) etc.
- Let all these signals are equally likely and each one has a probability of (1 / M). Let one of these M possible signals be transmitted in each time slot of duration T seconds.
- Then for an AWGN channel, a possible realization x (t) of the received random process X (t) is described as,

$$x(t) = S_i(t) + W(t) \quad ...0 \leq t \leq T \qquad \text{...(5.4.1)}$$

$$i = 1, 2, ..., M$$

Where W (t) is a sample function of the white noise process W (t). It has a zero mean value and power spectral density $N_0 / 2$.

- When the receiver receives the signal x (t), which is the sum of received signal and noise.
- The receiver has to observe the signal x (t) and make an "estimate" of the transmitted signal S_i (t) to decide whether a 0 or 1 was transmitted.

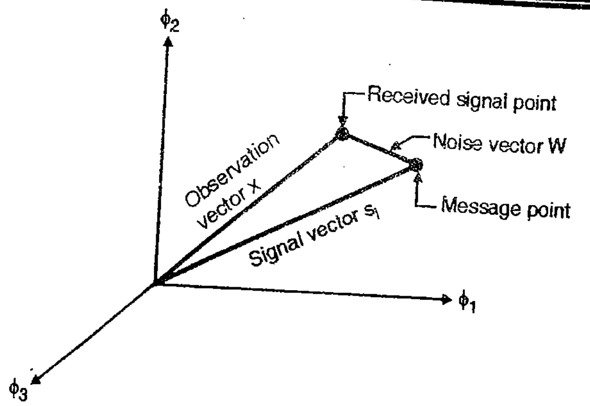
Signal Constellation :

- The transmitted signal S_i (t) is applied to a bank of correlators. An appropriate set of N orthonormal basis functions is also applied.
- The outputs of such a correlator define the signal vector S_i. These vectors give us all the information about the transmitted signal S_i (t).
- Hence we can represent S_i (t) by a point in the Euclidean space of dimension $N \leq M$. Such a point is called as the **transmitted signal point** or **message point.**
- The collection of such message points in the N-dimensional Euclidean space is called as the **signal constellation.**
- The representation of the transmitted signal is simple but that of the received signal is not so simple because the additive noise W (t), is also present alongwith the transmitted signal.
- When the signal x (t) is applied to the bank of N correlators the new vector x produced by the correlator outputs is called as the **observation or received vector.**
- This vector is denoted by x and given by,

$$x = S_i + W$$

$$i = 1, 2, ...M \qquad \text{...(5.4.2)}$$

- With the help of the observation vector x, we can represent the received signal x (t) by a point in the same Euclidean space in which we have represented the transmitted signal.
- This point is called as **received signal point.** Fig. 5.4.1 illustrates relationship between the observation vector x, signal vector S_i and noise vector W for N = 3.



(E-874) **Fig. 5.4.1 : Effect of noise on the position of received signal point**

5.5 Conversion of Continuous AWGN Channel into a Vector Channel :

- Consider the bank of N product integrators or correlators as shown in Fig. 5.5.1(a). Let its input signal be the received signal x(t) defined as per the idealized Additive White Gaussian Noise (AWGN) channel of Fig. 5.5.1(b).

$$\therefore \; x(t) = s_i(t) + \omega(t) \quad \ldots \begin{cases} 0 \le t \le T \\ i = 1, 2 \ldots M \end{cases} \qquad \ldots(5.5.1)$$

- In Equation (5.5.1) ω(t) represents the sample function of a White Gaussian noise process W(t) of zero mean and power spectral density $N_0/2$.
- Hence the output of correlator j in Fig. 5.5.1(a) will be the sample value of a random variable X_j and is given by,

$$x_j = \int_0^T x(t)\, \phi_j(t)\, dt$$

$$\therefore \; \underbrace{x_j}_{\text{Sample output of correlator } j} = \underbrace{s_{ij}}_{\text{Quantity contributed by transmitted signal } s_i(t)} + \underbrace{\omega_j}_{\text{Sample value of AWGN process } W(t)} \qquad j = 1, 2 \cdots\cdots N$$

(E-1392) …(5.5.2)

- In the above expression s_{ij} represents a deterministic quantity contributed by the transmitted signal $s_i(t)$ and is given by,

$$s_{ij} = \int_0^T s_i(t)\, \phi_j(t)\, dt \qquad \ldots(5.5.3)$$

- The second term ω_j in Equation (5.5.2) is the sample value of a random variable W_j which is due to the presence of channel noise ω(t). It is given by,

$$\omega_j = \int_0^T \omega(t) \cdot \phi_j(t)\, dt \qquad \ldots(5.5.4)$$

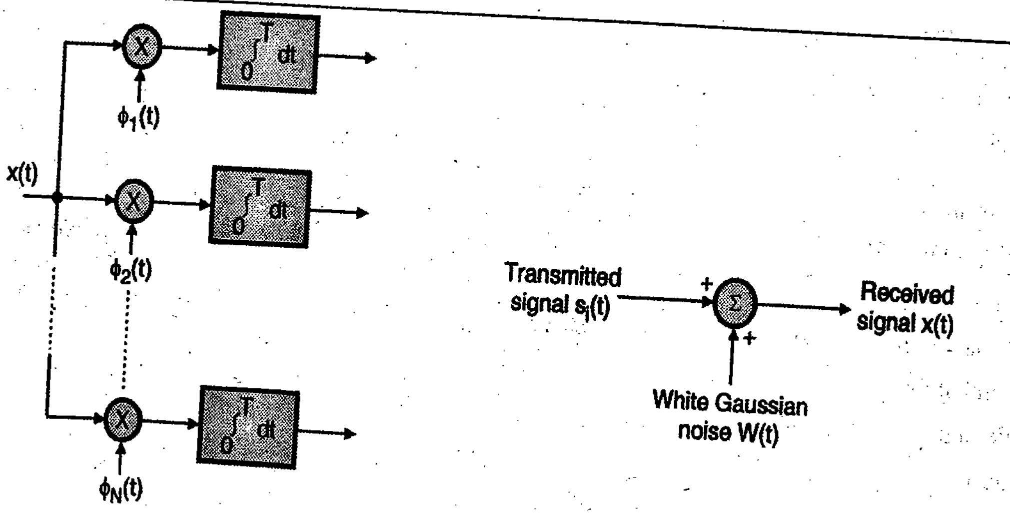
- Consider another random process X′(t). Its sample function is x′(t) and the relation between x′(t) and received signal x(t) is as follows :

$$x'(t) = x(t) - \sum_{j=1}^{N} x_j\, \phi_j(t) \qquad \ldots(5.5.5)$$

- Substituting Equations (5.5.1) and (5.5.2) into Equation (5.5.5) and then using the following expansion equation,

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\, \phi_j(t) \quad \ldots \begin{cases} 0 \le t \le T \\ i = 1, 2 \ldots M \end{cases}$$

We get,



(a) Bank of correlators

(b) AWGN model of a channel

(E-1391) **Fig. 5.5.1**

$$x'(t) = s_i(t) + \omega(t) - \sum_{j=1}^{N} (s_{ij} + \omega_j)\, \phi_j(t)$$

$$= \omega(t) - \sum_{j=1}^{N} \omega_j\, \phi_j(t)$$

$$\therefore\ x'(t) = \omega'(t) \qquad \ldots(5.5.6)$$

The sample function x'(t) thus depends only on the channel noise ω(t). Using Equations (5.5.5) and (5.5.6) we can express the received signal x (t) as,

$$x(t) = \sum_{j=1}^{N} x_j\, \phi_j(t) + x'(t)$$

$$\therefore\ x(t) = \sum_{j=1}^{N} x_j\, \phi_j(t) + \omega'(t) \qquad \ldots(5.5.7)$$

- Therefore ω'(t) can be viewed as a **remainder term** which must be included on the RHS in order to preserve the equality in Equation (5.5.7).
- Compare the equations of transmitted signal $s_i(t)$ and received signal x(t).

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\, \phi_j(t) \begin{cases} 0 \le t \le T \\ i = 1, 2 \ldots\ldots\ldots, M \end{cases}$$

$$x(t) = \sum_{j=1}^{N} x_j\, \phi_j(t) + \omega'(t)$$

- Here $s_i(t)$ is a deterministic signal whereas x(t) is a random (stochastic).

5.5.1 Statistical Characterization of the Correlator Outputs :

- Let us now develope the statistical characterization of the N outputs in the correlator bank of Fig. 5.5.1(a).
- Let X(t) is a random process. The received signal x(t) represents the sample function of X(t).
- Let X_j represent the random variable. Its sample value is represented by the correlator output x_j with j = 1, 2, ...N.
- As per the AWGN model of Fig. 5.5.1(b), the random process X(t) is a Gaussian process. Therefore X_j is a Gaussian random variable for all values of j, and it is completely characterized by its mean and variance. We will determine the mean and variance of X_j.
- Let W_j be the random variable the sample value of which is ω_j produced by the j^{th} correlator in response to the White Gaussian noise component ω(t).
- Since the noise process W(t) represented by ω(t) in the AWGN channel model has a zero mean, the corresponding random variable W_j also has a zero mean.
- Therefore, the mean value of X_j will depend only on s_{ij}.

$$\therefore\ \text{Mean value of } X_j = m_{xj} = E[X_j] = E[s_{ij} + W_j]$$

$$= s_{ij} + E[W_j]$$

$$\therefore\ m_{xj} = s_{ij} \qquad \ldots(5.5.8)$$

- Now let us find the variance of X_j.

$$\text{Variance of } X_j = \sigma_{xj}^2 = \text{Var}[X_j] = E[(X_j - s_{ij})^2]$$

$$= E[W_j^2] \qquad \ldots(5.5.9)$$

- As per Equation (5.5.4), the random variable W_j is defined as follows,

$$W_j = \int_0^T W(t)\, \phi_j(t)\, dt$$

- Hence the variance of X_j can be obtained as follows :

$$\sigma_{xj}^2 = E[(X_j - s_{ij})^2] = E[W_j^2]$$

$$= E\left[\int_0^T W(t)\, \phi_j(t)\, dt \int_0^T W(u)\, \phi_j(u)\, du\right]$$

$$= E\left[\int_0^T \int_0^T \phi_j(t)\, \phi_j(u)\, W(t)\, W(u)\, dt\, du\right] \qquad \ldots(5.5.10)$$

- Interchanging the integration and expectation we get,

$$\sigma_{xj}^2 = \int_0^T \int_0^T \phi_j(t)\, \phi_j(u)\, E[W(t)\, W(u)]\, dt\, du.$$

$$= \int_0^T \int_0^T \phi_j(t)\, \phi j(u)\, R_W(t, u)\, dt\, du \qquad \ldots(5.5.11)$$

- Where $R_W(t, u)$ is the autocorrelation function of the noise process W(t). This noise is a stationary process, therefore $R_W(t, u)$ depends only on the time difference (t − u).
- Also note that the noise W(t) is white noise having a constant PSD of $N_o/2$. Hence $R_W(t, u)$ can be expressed as follows,

$$R_W(t, u) = \frac{N_o}{2}\, \delta(t - u) \qquad \ldots(5.5.12)$$

- Substituting Equation (5.5.11) into Equation (5.5.12) and using sifting property of delta function we get.

$$\sigma_{xj}^2 = \frac{N_o}{2} \int_0^T \int_0^T \phi_j(t)\, \phi_j(u)\, \delta(t-u)\, dt\, du$$

$$= \frac{N_o}{2} \int_0^T \phi_j^2(t)\, dt \qquad \ldots(5.5.13)$$

- But $\int_0^T \phi_j^2(t)\, dt = \text{Energy of } \phi_j(t) = 1$

$$\therefore \quad \sigma_{xj}^2 = \frac{N_o}{2} \quad \text{for all } j \qquad \ldots(5.5.14)$$

- This result shows that the outputs of all correlators X_j with $j = 1$ .....N have a variance equal to the PSD $N_o/2$ of noise process W(t).
- Also, since the $\phi_j(t)$ form an orthogonal set, the correlator outputs X_j are mutually uncorrelated (not correlated to each other). This is proved as follows,

$$\therefore \ \text{cov}[X_j X_k] = E[(X_j - m_{X_j})(X_k - m_{X_k})]$$

$$= E[(X_j - s_{ij})(X_k - s_{ik})]$$

$$= E[W_i W_k]$$

$$= E\left[\int_0^T W(t)\, \phi_j(t)\, dt \int_0^T W(u)\, \phi_k(u)\, du\right]$$

$$= \int_0^T \int_0^T \phi_j(t)\, \phi_k(u)\, R_W(t, u)\, dt\, du \qquad \ldots(5.5.15)$$

$$= \frac{N_o}{2} \int_0^T \int_0^T \phi_j(t)\, \phi_k(u)\, \delta(t-u)\, dt\, du$$

$$= \frac{N_o}{2} \int_0^T \phi_j(t)\, \phi_k(t)\, dt$$

$$= 0, \quad j \neq k$$

- The X_j are Gaussian random variables. Therefore Equation (5.5.15) implies that they are statistically independent as well. (as per property 4 of a Guassian process).
- The vector of N random variables is defined as,

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} \qquad \ldots(5.5.16)$$

- The elements of this vector are independent Gaussian random variables with a mean value of s_{ij} and variance of $N_o/2$.
- As the elements $X_1, X_2, \ldots\ldots, X_N$ are statistically independent we can express the conditional PDF of vector X provided the signal $S_i(t)$ or symbol m_i was transmitted as follows :

$$f_X(x/m_i) = \prod_{j=1}^{N} f_{X_j}(x_j/m_i), \qquad i = 1, 2, \ldots\ldots, M$$

$$\ldots(5.5.17)$$

- Here the vector x and scalar x_j are sample values of the random vector X and random variable X_j respectively.
- The vector x is called as the **observation vector** and x_j is called as an **observable element**. Any channel that satisfies Equation (5.5.17) is called a **memoryless channel**.
- Output of each correlator X_j is a Gaussian R.V. It has a mean value of s_{ij} and variance equal to $N_o/2$. Therefore, we have,

$$f_{Xj}(x_j/m_i) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}(x_j - s_{ij})^2\right] \begin{array}{l} j = 1, \ldots, N \\ i = 1, \ldots, M \end{array}$$

$$\ldots(5.5.18)$$

- Substitute Equation (5.5.18) into Equation (5.5.17) to get,

$$f_X(x/m_i) = (\pi N_0)^{-N/2} \exp\left[-\frac{1}{N_0} \sum_{j=1}^{N} (x_j - s_{ij})^2\right]$$

$$i = 1, \ldots, M \qquad \ldots(5.5.19)$$

- Equation (5.5.7) states that,

$$x(t) = \sum_{j=1}^{N} x_j\, \phi_j(t) + \omega'(t)$$

- It has been proved that the elements of random vector X (i.e. $X_1, X_2, \ldots\ldots, X_N$) completely characterize the first term in Equation (5.5.7). But the second term $\omega'(t)$ in the expression depends only on the channel noise $\omega(t)$.
- The noise process W(t) represented by $\omega(t)$ is Gaussian and has a zero mean. Therefore, the noise process $W'(t)$ represented by $W'(t)$ will also be a Gaussian process with zero mean.
- Therefore any random variable $W'(t_k)$ derived from the noise process $W'(t)$ by sampling it at $t = t_k$ will be statistically independent of random variables $\{X_j\}$. This is mathematically expressed as follows :

$$E[X_j W'(t_k)] = 0 \begin{cases} j = 1, 2, \ldots\ldots, N \\ 0 \leq t_k \leq T \end{cases} \qquad \ldots(5.5.20)$$

- Thus any random variable obtained by sampling the remainder noise process $W'(t)$ is independent of the set of random variables $\{X_j\}$ and the set of transmitted signals $\{s_i(t)\}$, Equation (5.5.20) indicates that R.V. $W'(t_k)$ is **not relevant** in deciding about which particular signal was actually transmitted.

- That means, the correlator outputs obtained from signal x (t) are the only data which is useful for the decision making process and therefore represents **sufficient statistics** for the problem at hand.
- The sufficient statistics summarizes all the relevant information supplied by an observation vector.

Theorem of irrelavance :

- The results obtained in this section can be summarized by formulating the theorem of irrelevance which is as follows :

Statement :

As far as signal detection in AWGN noise is concerned, only the projections of noise onto the basis function of the signal set $\{s_i(t)\}_{i=1}^{M}$ affects the sufficient statistics of the detection problem. The remainder noise is not important and should not be considered in the decision making process.

Corollary :

- The corollary of this theorem may be stated as the AWGN channel of Fig. 5.5.1(b) is equivalent to an N-dimensional **vector channel** which is described by the following observation vector :

$$x = s_i + W \quad ... i = 1, 2, ..., M \qquad ...(5.5.21)$$

- The number of basis functions involved in formulating the signal vector s_i in Equation (5.5.21) will be equal to N (dimensions).

5.5.2 Likelihood Functions :

- The conditional probability density functions $f_X(X/m_i)$ i = 1, 2 ...., M are the characterizing functions of an AWGN channel. That means the AWGN channels can be completely characterised by these conditional PDFs.
- If we derive these conditional PDFs then we will get the functional dependence on the observation vector X, provided the transmitted message symbol is m_i.
- But the situation at the receiver is exactly the opposite. The observation vector X is given to us and we need to estimate the message symbol m_i, which is responsible for the generation of X.
- In order to do this, the likelihood function which is denoted by L (m_i) is introduced. It is defined as :

$$L(m_i) = f_X(X/m_i), i = 1, 2, ...M \quad ...(5.5.22)$$

- It is important to note that even though the mathematical form of L (m_i) and $f_X(X/m_i)$ is exactly the same, their individual meaning are completely different.

- Practically it is more convenient to use the **log-likelihood function** which is denoted by l (m_i) and defined as follows :

$$l(m_i) = \log_{10} L(m_i), i = 1, 2 ....M \quad ...(5.5.23)$$

- The log-likelihood function for AWGN channel is given by,

$$l(m_i) = \frac{-1}{N_0} \sum_{j=1}^{N} (x_j - s_{ij})^2, i = 1, 2 ......, M \quad ...(5.5.24)$$

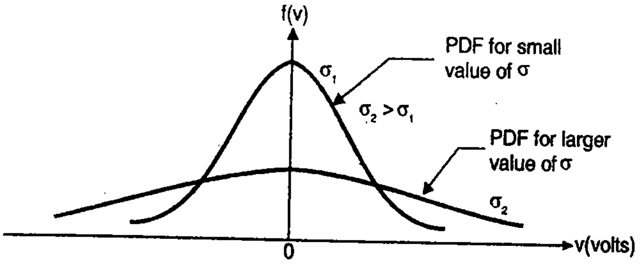
- This function is used for the basic receiver design problem.

5.6 Coherent Detection of Binary Signals in Presence of Noise :

- Noise entering into or contributed by a system results in errors in the detected signal. The number of errors or error rate is dependent on both the signal and noise amplitudes, and the characteristics of the noise.
- Large amplitude noise spikes will introduce errors in the detected signal. The noise getting added to the signal is assumed to be Gaussian noise.

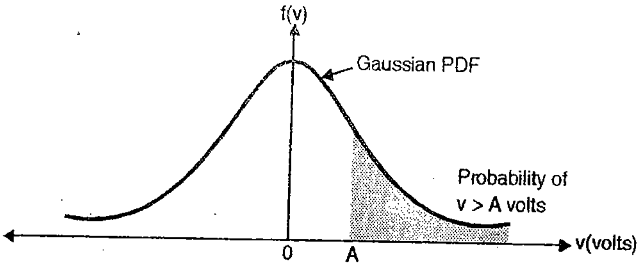
5.6.1 Gaussian Noise and its PDF :

- In many application the noise is considered to be Gaussian eventhough in practice it is not so. A noise is said to be Gaussian noise if it exhibits the Gaussian distribution.
- The familiar example of Gaussian noise is the thermal noise generated in a resistor. The probability density function (PDF) of Gaussian noise is as shown in Fig. 5.6.1(a).

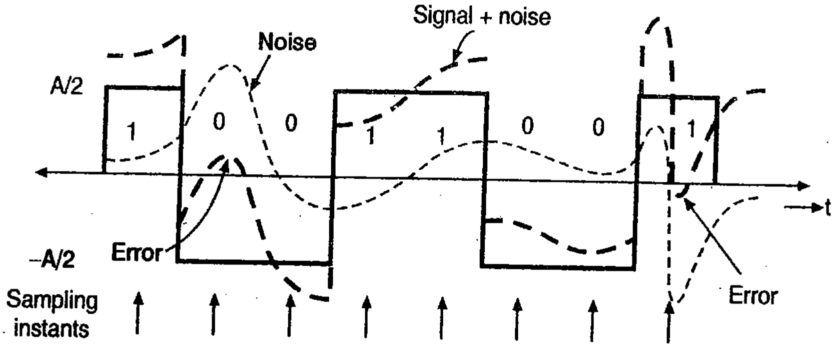


(E-875) **Fig. 5.6.1(a) : PDF of Gaussian noise**

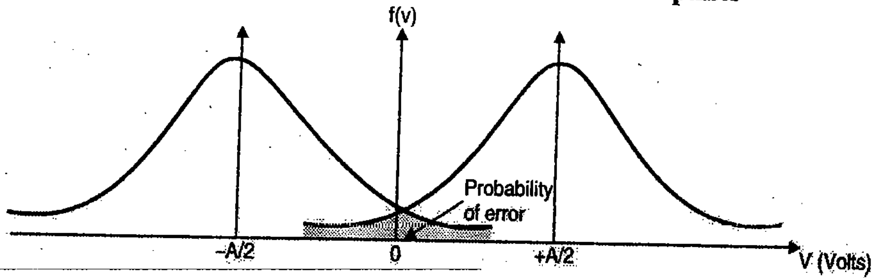
- The Gaussian noise PDF is of Bell shape as shown in Fig. 5.6.1(a) and the area under the curve represents a total probability of 1 or 100% existence.
- The peak value is centered around zero value of "v" and the curve is symmetrical about the Y- axis. This shows that the mean value or average value of the Gaussian noisé is zero.



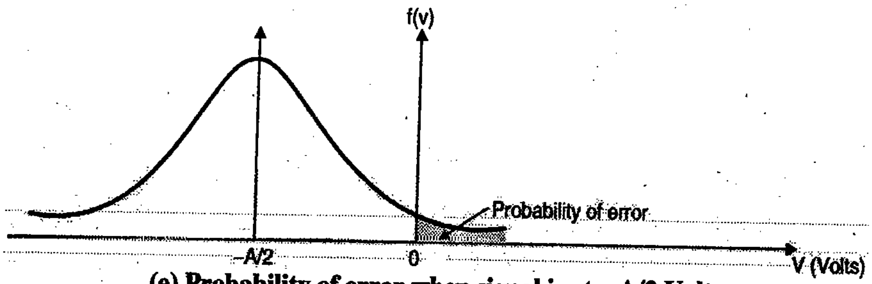
(E-876) **Fig. 5.6.1(b) : Shaded area represents the probability that v > A**



(E-877) **Fig. 5.6.1(c) : Effect of noise on two level pulses**



(d) Gaussian noise on a two level system



(e) Probability of error when signal is at – A/2 Volts

(E-878) **Fig. 5.6.1**

- The PDF shows that the larger values of "v" have less chance of occurrence, than the smaller values. This is because the PDF decreases with increase in the values of "v".
- The voltage near zero value has a greater probability of occurrence than the large positive or negative voltages.

5.6.2 Variance and Standard Deviation :

SPPU : May 13

University Questions

Q. 1 Define mean, correlation, standard deviation and variance of random process. **(May 13, 8 Marks)**

- To analyze probable signals we have to use the terms variance and standard deviation frequently. Out of them, the standard deviation "σ" is a measure of spread of the observed values.
- The square of standard deviation σ^2 is called as variance. The larger the variance, the larger is the spread. This is as shown in Fig. 5.6.1(a) where $\sigma_2 > \sigma_1$ therefore spread of curve of σ_2 is more.

5.6.3 Mathematical Representation of a Gaussian PDF :

The Gaussian PDF curves of Fig. 5.6.1 (a) are expressed mathematically as :

$$f(v) = \frac{1}{\sqrt{2\pi\sigma}} e^{-v^2/2\sigma^2} \quad \text{... (5.6.1)}$$

Where f(v) = Probability density of voltage V (PDF)

σ = Standard deviation and σ^2 = Variance

5.6.4 How to use the PDF Curves ?

- To find the probability that the voltage "v" is greater than "A" volts, the shaded area of Fig. 5.6.1(b) must be computed. The area under the entire PDF curve is 1 representing 100% probability. To get the shaded area, Equation (5.6.1) should be integrated for $v > A$.

5.6.5 Effect of Gaussian Noise on Bipolar Signals :

- Consider a bipolar or two level signal with two levels + A/2 or – A/2 as shown in Fig. 5.6.1(c).
- The decision threshold at the receiver is 0 volts. That means the decoder decides that a positive pulse is present if the instantaneous signal is higher than 0 volts, and a negative pulse is present if the signal level is below 0 Volts at the sampling instants. Both the levels are equiprobable i.e. they have the same probability of 0.5.
- In Fig. 5.6.1(c) the decoder responds to the amplitude of signal plus noise at the sampling instants shown by arrows. Two errors have occured in this pulse train in the second and the last digit due to large amplitude of the noise.

5.6.6 To Determine the Error Rate of Pulse System :

- Now let us find out the error rate of the pulsed system we have discussed so far. To do so, assume that the Gaussian noise is superimposed on the two levels, – A/2 and + A/2 as shown in Fig. 5.6.1(d).
- Let us find now the probability of error in the decision making at the decoder. When will the error occur ? One situation is that the transmitted signal is – A/2 Volts but due to noise the received voltage exceeds the decision threshold level of 0 Volts. The probability of error is represented by the shaded area in Fig. 5.6.1(e).
- The other situation in which an error can occur is when the transmitted voltage is + A/2 Volts but the noise makes the received voltage less than 0 Volts. (below the decision threshold). The probability of error is represented by the shaded area in Fig. 5.6.1(f).

5.6.7 Error Probability :

- Both the signals i.e. – A/2 and + A/2 representing binary 0 and 1 respectively are equally likely i.e. they have equal probabilities equal to 1/2 and the Gaussian distribution also is symmetrical. Therefore the shaded areas shown in Fig. 5.6.1(e) and Fig. 5.6.1(f) are equal. The probability of error is given by :

Error probability = [(Probability of signal at – A/2) × (Probability of signal in the shaded area for $v \geq 0$)] + [(probability of signal at + A/2) × (Probability of signal in the shaded area for $v \leq 0$)]

$\therefore$ Error probability =

$$\left[\frac{1}{2} \times \text{Probability of signal in shaded area for } v \geq 0\right]$$

$$+ \left[\frac{1}{2} \times \text{Probability of signal in shaded area for } v \leq 0\right]$$

$$= 2 \times \frac{1}{2} \times \text{Probability of signal in shaded area} \quad \text{... (5.6.2)}$$

Error probability = Probability of signal in shaded area.

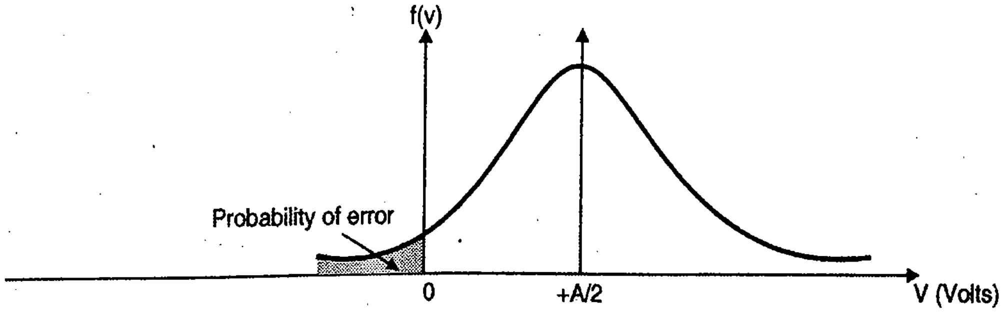
- Thus error probability of a digital transmission can be obtained by calculating the shaded area. The shaded area can be obtained by means of integration.

5.6.8 Maximum Likelihood Detector :

SPPU : Dec. 16

University Questions

Q. 1 Explain the principle of Maximum likelihood receiver with the help of various methods of detection of signal. **(Dec. 16, 8 Marks)**



(f) Probability of error when signal is at +A/2 Volts

(E-879) **Fig. 5.6.1**

- Let us now tackle the detection problem.
- If the observation vector x is given then we can perform a **mapping** from x to get an estimate $\hat{m}$ of the transmitted symbol m_i, in a certain way so as to minimize the average probability of symbol error in the decision.
- This is known as the maximum likelihood detection.

Maximum likelihood detection :

- Assume that if the observation vector has a value x, then the receiver will make the decision of $\hat{m} = m_i$.

The average probability of symbol error in such a decision is given by,

$$P_e(m_i, x) = P(m_i \text{ not sent} / x)$$

$$= P(m_i \text{ sent} / x) \quad ...(5.6.3)$$

Where P (m_i sent / x) is the conditional probability that m_i was sent provided x is received.

Optimum decision rule :

- We are interested in minimizing the probability of error in mapping each observation vector into a decision. Hence we have to deduce an optimum decision rule from Equation (5.6.3).
- The optimum decision rule is as follows :

The estimate $\hat{m}_i = m_i$ if

$$P(m_i \text{ sent} / x) \geq P(m_k \text{ sent} / x) \text{ for all } k \neq i$$

$$\text{where } k = 1, 2, ...M \quad ...(5.6.4)$$

- This rule is called as the maximum a posteriori probability.
- Applying Bayes' rule to Equation (5.6.4) we can restate the decision rule as follows :

The estimate $\hat{m}_i = m_i$ if

$$\frac{p_k f_X(x / m_k)}{f_X(x)} \text{ is maximum for } k = i \quad ...(5.6.5)$$

Where p_k is the a priori probability of occurrence of symbol m_k, $f_X(x / m_k)$ is the likelihood function corresponding to transmission of symbol m_k, and $f_X(x)$ is the unconditional joint pdf of random variable X.

- From Equation (5.6.5) we may note the following points :
  1. The denominator term $f_X(x)$ is independent of the transmitted signal.
  2. The a priori probability $p_k = p_i$ because all the transmitted signals are equally likely.
- So the decision rule may be further simplified as follows :

The estimate $\hat{m}_i = m_i$, if

$$f_X(x / m_k) \text{ is maximum for } k = i \quad ...(5.6.6)$$

- Generally it is more convenient to work with the natural logarithm of the likelihood function than the likelihood function itself.
- Such a natural logarithm of the likelihood function is called as **meric.**
- The likelihood function $f_X(x / m_k)$ is always non negative. Also if $X > Y > 0$ then $\log_e X > \log_e Y$. Hence the decision rule of Equation (5.6.6) can be restated as follows :

The estimate $\hat{m}_i = m_i$, if

$$\log_e [f_X(x / m_k)] \text{ is maximum for } k = i \quad ...(5.6.7)$$

- The decision rule stated above is called as the maximum likelihood and device used for its implementation is called as maximum likelihood detector.
- Looking at Equation (5.6.7) we conclude that the maximum likelihood detector computes the metric of each transmitted message, compares them and then makes a decision based on the maximum of them.

5.6.9 Detection Techniques : SPPU : Dec. 16

University Questions

Q. 1 Explain the principle of Maximum likelihood receiver with the help of various methods of detection of signal. **(Dec. 16, 8 Marks)**

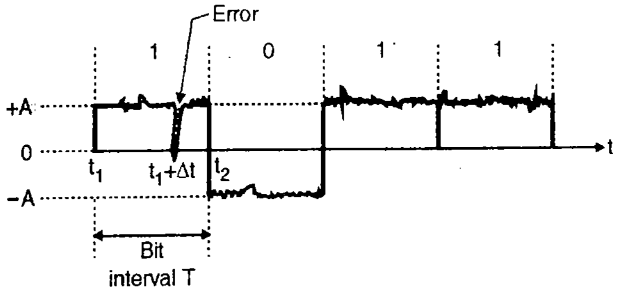
The important detection techniques (baseband receivers) used in digital communication are as follows :

1. Integrate and Dump receiver (filter).
2. Optimum filter.
3. Matched filter.
4. Correlator or coherent receiver.

5.7 Effect of Noise On the Transmitted Signal :

- The digital transmission systems are used for the transmission of a sequence of binary digits, 0s and 1s.
- These digits can be represented by different patterns such as unipolar RZ and NRZ, bipolar RZ and NRZ, AMI and Manchester.
- In general the binary digits are encoded in such a way that a "1" is represented by $x_1(t)$ and a "0" is represented by signal $x_2(t)$, where $x_1(t)$ and $x_2(t)$ each have a duration T.
- The resulting signal may be transmitted directly or used for modulating a carrier.
- The received signal is corrupted by noise. Therefore there is a probability that the receiver will make an error in deciding whether a 1 or a 0 was transmitted.
- In this chapter we are going to make calculations of such error probabilities and discuss methods to minimize them.

- Consider a binary sequence 1 0 1 1 being transmitted. While travelling from the transmitter to receiver, noise gets added to it.
- Thus the signal received by a receiver is corrupted by noise as shown in Fig. 5.7.1. The first transmitted bit is represented by voltage + A volts which extends over the time t_1 to t_2, i.e. over one bit interval.
- Noise has been superimposed on this signal. In order to make a judgement of whether a 1 or a 0 is received, the receiver, samples the received signal once in every bit interval.
- In the first bit interval of Fig. 5.7.1 if the sampling happens to take place at instant ($t_1 + \Delta t$), then the receiver will decide that a 0 has been received thus introducing an error.



(E-442) **Fig. 5.7.1 : Signal corrupted by noise**

- In order to reduce the probability of error the sampling instant in each interval should be selected in such a way that the signal amplitude is maximum at the instant of sampling.
- In the following sections we are going to discuss various types of detection methods to detect the baseband signals. These detection techniques must satisfy the following requirements.

5.7.1 Requirements of a Detection Technique :

1. A detection technique must have minimum probability of error.
2. It should sample (check) the received signal in every bit interval at the instant when the signal has maximum possible amplitude.
3. The detection method must maximize the signal to noise ratio, by amplifying the signal and attenuating the noise.
4. One of the detection techniques used is integrator and dump circuit.

5.8 Integrate and Dump Receiver (Filter) :

SPPU : May 08, Dec. 14, Dec. 16

University Questions

Q. 1 Write a note on : Integrator and dump circuit. **(May 08, 5 Marks)**

Q. 2 Derive the expression of SNR for integrator and dump filter and explain working of integrator and dump filter. **(Dec. 14, 8 Marks)**

Q. 3 Explain the principle of Maximum likelihood receiver with the help of various methods of detection of signal. **(Dec. 16, 8 Marks)**

The block diagram of an integrate and dump receiver is as shown in Fig. 5.8.1.

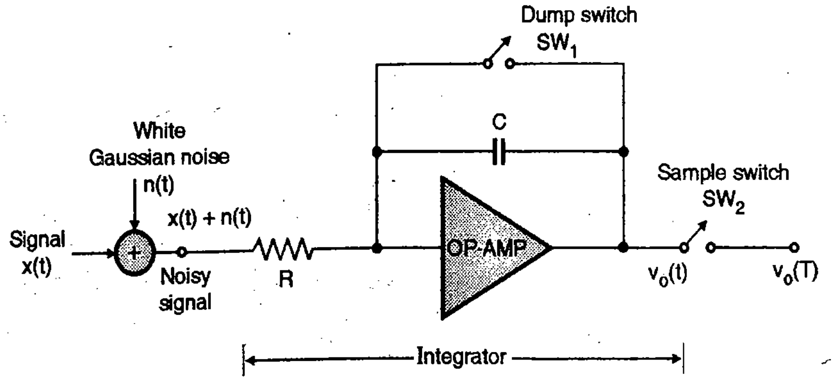
5.8.1 Operation of the Circuit :

SPPU : Dec. 14, Dec. 16

University Questions

Q. 1 Derive the expression of SNR for integrator and dump filter and explain working of integrator and dump filter. **(Dec. 14, 8 Marks)**

Q. 2 Explain the principle of Maximum likelihood receiver with the help of various methods of detection of signal. **(Dec. 16, 8 Marks)**



(E-443) **Fig. 5.8.1 : Integrate and dump receiver (filter)**

- The transmitted signal x (t) gets corrupted due to noise n (t) while travelling from the transmitter to receiver.
- This noise is assumed to Additive White Gaussian Noise (AWGN). Therefore an adder is shown in Fig. 5.8.1.
- At the input of the integrator we get a noisy signal x (t) + n (t).
- The integrator is supposed to integrate the noisy signal only over the bit interval T. Therefore just at the beginning of each bit interval, the dump switch SW_1 is closed momentarily to discharge the capacitor C completely.
- Thus voltage on C at the beginning of each bit interval is zero. This will ensure that voltage on C will not be dependent on the previous bit intervals.
- The dump switch SW_1 is then left open for the entire bit duration T. The capacitor then charges linearly as shown in Fig. 5.8.2(a).
- At the end of each bit interval, the sample switch SW_2 is closed momentarily to sample the integrator output. Note that integrator output is maximum at the end of each bit interval hence the second requirement mentioned in the preceding section is satisfied.
- The sampled output is then used for making a decision of whether a 0 is received or 1 is received.
- At the beginning of next pulse duration, switch SW_2 is open circuited and SW_1 is closed to discharge the capacitor. The integrator output is given by,

$$v_o(t) = \frac{1}{RC}\int_0^T [x(t) + n(t)]$$

$$dt = \frac{1}{RC}\int_0^T x(t)\,dt + \int_0^T n(t)\,dt = x_o(t) + n_o(t)$$

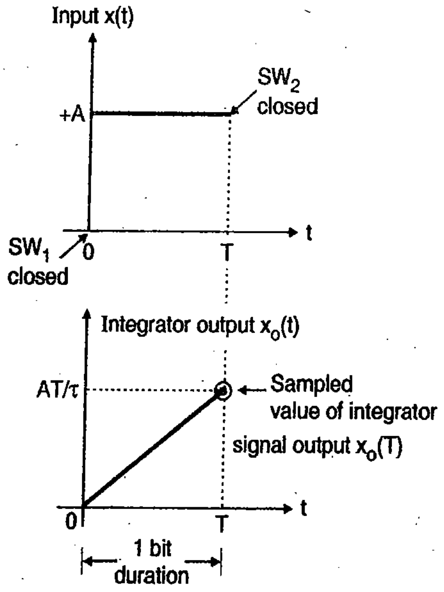
- Assuming x (t) = + A we get,

$$v_o(t) = \frac{1}{RC}\int_0^T A\,dt + n_o(t)$$

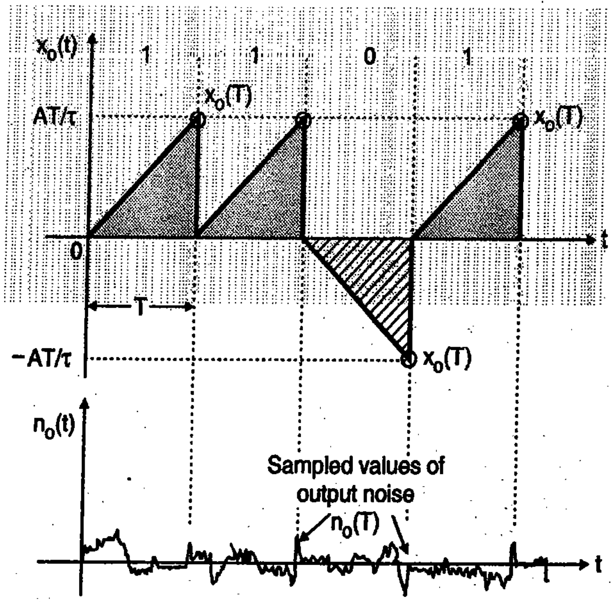
$$\therefore \quad v_o(t) = \frac{A}{RC} \times T + n_o(T) = \frac{AT}{\tau} + n_o(T),$$

$$\because \quad RC = \tau \qquad ...(5.8.1)$$

- The signal output voltage is shown in Fig. 5.8.2(a) and noise output has been shown in Fig. 5.8.2(b).
- x_o (t) and n_o (t) represent the instantaneous signal and noise voltages at the integrator output whereas x_o (T) and n_o (T) represent the sampled values of output voltage and noise voltage respectively at the end of each bit interval T.
- This receiver is called integrate and dump receiver because it uses an integrator and a dump switch SW_1 across the capacitor to abruptly discharge the capacitor voltage after each sampling.
- The sample switch SW_2 of Fig. 5.8.1 is momentarily closed at the end of each bit interval in order to make the integrator output available to the user.



(a) Input and output voltage over one bit duration T



(b) Output signal voltage $x_o(t)$ and output noise voltage $n_o(t)$

(E-444) **Fig. 5.8.2 : Waveforms of integrate and dump receiver**

5.8.2 Signal to Noise Ratio of Integrate and Dump Receiver (Filter) :

SPPU : May 07, Dec. 07, Dec. 11, Dec. 14, May 16, Dec. 16

University Questions

Q. 1 Derive expression for signal to noise ratio of integrator and dump filter. **(May 07, 8 Marks)**

Q. 2 Derive the expression of $\left(\frac{S}{N}\right)$ of integrator and dump filter, explain its operation. **(Dec. 07, 6 Marks)**

Q. 3 Derive the expression for signal to noise ratio of integrate and dump receiver. **(Dec. 11, May 16, Dec. 16, 8 Marks)**

Q. 4 Derive the expression of SNR for integrator and dump filter and explain working of integrator and dump filter. **(Dec. 14, 8 Marks)**

The steps to be followed for this derivation are as follows :

Step 1 : Obtain the normalized signal output power.

Step 2 : Obtain the output noise power.

Step 3 : Obtain the expression for signal to noise ratio.

Step 1 : To obtain the normalized signal output power :

1. The output of the integrator can be expressed as :

$$v_0(t) = \frac{1}{RC}\int_0^T [x(t) + n(t)]$$

$$dt = \frac{1}{RC}\left[\int_0^T x(t)\,dt + \int_0^T n(t)\,dt\right]$$

$$\therefore \quad v_0(t) = x_0(t) + n_0(t) \qquad ...(5.8.2)$$

Where $x_0(t)$ = Output signal voltage

$n_0(t)$ = Output noise voltage.

2. Consider the output signal voltage given by,

$$x_0(t) = \frac{1}{RC}\int_0^T x(t)\,dt$$

But $x(t) = +A$ for the interval 0 to T in Fig. 5.8.2(a).

$$\therefore \quad x_0(t) = \frac{1}{RC}\int_0^T A\,dt = \frac{A}{RC}[t]_0^T = \frac{AT}{RC}$$

But RC = τ = Time constant of the integrator

$$\therefore \quad x_0(t) = AT/\tau \qquad ...(5.8.3)$$

3. The normalized output signal power,

$$x_0^2(T) = A^2T^2/\tau^2 \qquad ...(5.8.4)$$

Step 2 : To obtain the output noise power :

4. As discussed earlier, average normalized power of a signal is given by :

$$P = \int_{-\infty}^{\infty} S(f)\,df \qquad ...(5.8.5)$$

Where S(f) = Power spectral density (psd) of the signal.

Applying the same concept to the output noise $n_0(t)$, we can write that;

$$\text{Average normalized noise power} = \int_{-\infty}^{\infty} S_{no}(f)\,df \qquad ...(5.8.6)$$

Where $S_{no}(f)$ = psd of output noise.

5. Average normalized noise = $\overline{n_0^2(t)}$ = Mean square value of $n_0(t)$. This is because the noise is assumed to be Gaussian with a zero mean value. Therefore Equation (5.8.6) can be written as

$$\overline{n_0^2(t)} = \int_{-\infty}^{\infty} S_{no}(f)\,df \qquad ...(5.8.7)$$

6. But $S_{no}(f) = |H(f)|^2 S_{ni}(f)$ ... standard result ...(5.8.8)

Where H(f) = Transfer function of the integrator and

$S_{ni}(f)$ = psd of input noise n(t)

7. The squared transfer function of an integrator (LPF) is given by,

$$|H(f)|^2 = \frac{\sin^2(\pi fT)}{(\pi f\tau)^2} \qquad ...(5.8.9)$$

And assuming that the input noise n(t) is a white noise its psd is given by,

$$S_{ni}(f) = \frac{N_0}{2} \qquad ...(5.8.10)$$

Substituting these values into Equation (5.8.8) we get,

$$S_{no}(f) = \frac{N_0}{2}\frac{\sin^2(\pi fT)}{(\pi f\tau)^2}$$

$$\text{and} \quad \overline{n_0^2(t)} = \int_{-\infty}^{\infty} \frac{N_0}{2}\frac{\sin^2(\pi fT)}{(\pi f\tau)^2}\,df \qquad ...(5.8.11)$$

8. Substitute $\pi f \tau = y$ $\qquad \therefore \pi f T = \frac{yT}{\tau}$

$\therefore \pi \tau \, df = dy$ $\qquad \therefore dy = \frac{df}{\pi \tau}$

Substituting these values into Equation (5.8.11) we get,

$$\overline{n_o^2(t)} = \frac{N_o}{2} \int_{-\infty}^{\infty} \frac{\sin^2[yT/\tau]}{y^2} \cdot \frac{dy}{\pi\tau}$$

$$= \frac{N_o}{2} \int_{-\infty}^{\infty} \frac{\sin^2[yT/\tau]}{[yT/\tau]^2 [\tau/T]^2} \cdot \frac{dy}{\pi\tau}$$

$$= \frac{N_o}{2} \int_{-\infty}^{\infty} \frac{\sin^2[yT/\tau]}{[yT/\tau]^2} \times \left[\frac{T}{\tau}\right]^2 \cdot \frac{1}{\pi\tau} dy$$

$$\therefore \overline{n_o^2(t)} = \frac{N_o}{2} \cdot \frac{T^2}{\pi\tau^3} \int_{-\infty}^{\infty} \frac{\sin^2[yT/\tau]}{[yT/\tau]^2} dy \qquad \text{...(5.8.12)}$$

9. Substitute $\frac{yT}{\tau} = x$ $\qquad \therefore \frac{T}{\tau} dy = dx$

$\therefore dy = \frac{\tau}{T} dx$

Substituting these values into Equation (5.8.12) we get,

$$\overline{n_o^2(t)} = \frac{N_o}{2} \cdot \frac{T^2}{\pi\tau^3} \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} \cdot \frac{\tau}{T} dx$$

$$= \frac{N_o T}{2\pi\tau^2} \int_{-\infty}^{\infty} \left[\frac{\sin x}{x}\right]^2 dx \qquad \text{...(5.8.13)}$$

10. As the function inside the integration in Equation (5.8.13) is a squared sine function, we can change the integration limits as follows :

$$\overline{n_o^2(t)} = \frac{N_o T}{2\pi\tau^2} \cdot 2 \int_{0}^{\infty} \left[\frac{\sin x}{x}\right]^2 dx \qquad \text{...(5.8.14)}$$

But $\int_{0}^{\infty} \left[\frac{\sin x}{x}\right]^2 dx = \frac{\pi}{2}$ ......standard result

$$\therefore \overline{n_o^2(t)} = \frac{N_o T}{2\pi\tau^2} \cdot 2 \cdot \frac{\pi}{2} = \frac{N_o T}{2\tau^2} \qquad \text{...(5.8.15)}$$

This is the expression for output noise power.

11. The output signal to noise power ratio is given by,

$$\frac{[x_o(T)]^2}{[n_o(T)]^2} = \frac{A^2 T^2/\tau^2}{N_o T/2\tau^2} = \frac{2A^2T^2}{N_o T}$$

$$\therefore \quad SNR_o = \frac{2A^2T}{N_o} \qquad \text{...(5.8.16)}$$

This is the required expression for the output signal to noise ratio.

Conclusions :

1. This is the required expression for the signal to noise ratio at the output of the integrate and dump receiver. It indicates that the signal to noise ratio can be improved by increasing the bit duration T and it is dependent on $A^2 T$ which is the normalized energy of the signal. The signal to noise ratio can be improved by increasing the normalized signal energy.
2. From Equations (5.8.3) and (5.8.15) it is evident that an integrator filters the signal and noise in such a way that the signal voltage increases linearly with time, but the rms noise voltage (square root of $\overline{n_o^2(T)}$ increases more slowly as $\sqrt{T}$). Thus an integrator enhances the signal as compared to noise and such an enhancement increases with time.

5.8.3 Probability of Error for Integrate and Dump Receiver :

- The function of a receiver in digital communication is to distinguish between bits 1 and 0, in presence of noise.
- Therefore one of its most important characteristics is the error probability P_e. Let us now obtain the expression for the error probability of integrate-and-dump receiver.

Assumptions :

1. Let the binary 0 be represented by – A Volts and a binary 1 be represented by + A Volts that means :

$x(t) = +A$ ... for binary 1.

$= -A$ ... for binary 0.

2. The corresponding output of the integrate-and-dump receiver is given by :

$x_o(T) = AT/\tau$ ... for $x(t) = +A$

and $= -AT/\tau$ ... for $x(t) = -A$

3. The input noise n (t) is a white noise having Gaussian distribution.

The steps to be followed for the derivation of error probability are :

Steps to be followed :

Step 1 : Obtain the integrator output.

Step 2 : Calculate the noise voltage required to introduce error.

Step 3 : Obtain the expression for the standard deviation of output noise.

Step 4 : Write the PDF of noise n_0 (t).

Step 5 : Calculate the error probability.

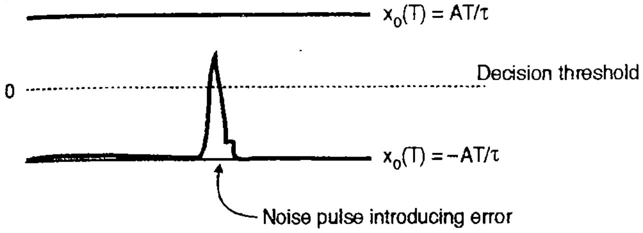
Step 1 : Integrator output :

Integrator output is given by $v_0(T) = \frac{AT}{\tau} + n_0(T)$

...for x (t) = + A

and $v_0(T) = \frac{-AT}{\tau} + n_0(T)$ ... for x (t) = − A

The decision threshold at the receiver is exactly at the center of AT / τ and − AT / τ as shown in Fig. 5.8.3.



(E-445) **Fig. 5.8.3 : Decision threshold and error introduction for integrate-and-dump receiver**

Step 2 : Noise voltage to introduce error :

- As shown in Fig. 5.8.3, if x_0 (T) is above the decision threshold then the receiver will consider it to be logic 1 and if x_0 (T) is below the decision threshold then it will be considered as logic 0.
- Suppose if x (t) = − A, then the correct output is − AT / τ but an error in judgement is introduced if the output noise n_0 (T) > AT / τ as shown in Fig. 5.8.3.
- Similarly for x (t) = + A, error in judgement is introduced if the output noise n_0 (T) > − AT /τ.
- Therefore the error probability P_e is nothing but the probability of n_0 (T) > AT / τ or n_0 (T) > − AT / τ.
- To obtain these probabilities we have to consider the PDF of n_0 (T), shown in Fig. 5.8.4. The noise is assumed to have a Gaussian distribution.
- The PDF of a random variable X having Gaussian distribution is given by,

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2} \quad ...(5.8.17)$$

where σ = Standard deviation of random variable X.

m = Mean value of random variable X.

- In this case the random variable is output noise n_0 (t). As the noise is assumed to have Gaussian distribution, its mean value m = 0. Substituting x = n_0 (t) and m = 0 in Equation (5.8.17) we get,

$$\text{PDF of } n_0(T),\ f_X[n_0(T)] = \frac{1}{\sigma\sqrt{2\pi}} e^{-[n_0(t)]^2/2\sigma^2} \quad ...(5.8.18)$$

Step 3 : Standard deviation σ :

The standard deviation of output noise n_0 (t) is given by,

σ = [Mean square value of n_0 (T) − Square of mean value of n_0 (t)$]^{1/2}$

$$= \left[\overline{n_0^2(t)} - 0\right]^{1/2} \quad \text{... because mean value of } n_0(t) = 0$$

$$\therefore \quad \sigma = \left[\overline{n_0^2(t)}\right]^{1/2} \quad ...(5.8.19)$$

Substitute the value of $\overline{n_0^2(t)}$ from Equation (5.8.15) to write,

$$\sigma = [N_0 T / 2\tau^2]^{1/2} = \sqrt{N_0 T / 2\tau^2} \quad ...(5.8.20)$$

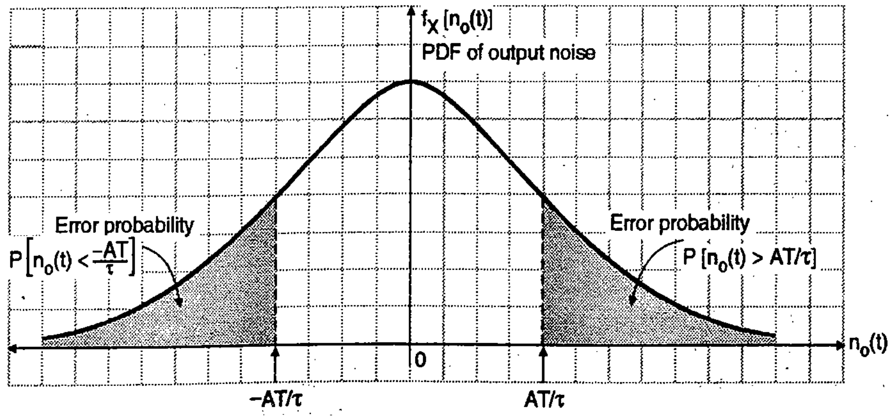
Step 4 : PDF of n_0 (t) :

Substitute the value of σ into Equation (5.8.18) to write expression for PDF of n_0 (t) as,

$$f_X[n_0(t)] = \frac{1}{\sqrt{N_0 T / 2\tau^2}\sqrt{2\pi}} e^{-[n_0(t)]^2/2N_0T/2\tau^2}$$

$$\therefore \quad f_X[n_0(t)] = \frac{\tau}{\sqrt{\pi N_0 T}} e^{-[n_0(t)]^2/\frac{N_0 T}{\tau^2}} \quad ...(5.8.21)$$

This PDF can be plotted as shown in Fig. 5.8.4.



(E-446) **Fig. 5.8.4 : PDF of output noise n_0 (t) of integrate and dump receiver**

Step 5 : To calculate the error probability :

Error probability is given by the area under the shaded region shown in Fig. 5.8.4. Thus the probability of error is given by :

$$P_e = P\left[n_0(t) > \frac{AT}{\tau}\right] = P\left[n_0(t) < \frac{-AT}{\tau}\right] \quad ...(5.8.22)$$

$$\therefore \quad P_e = \int_{AT/\tau}^{\infty} f_X[n_0(t)]\; d[n_0(t)]$$

$$\therefore \quad P_e = \int_{AT/\tau}^{\infty} \frac{\tau}{\sqrt{\pi N_0 T}}\, e^{-[n_0(t)]^2/\frac{(N_0 T)}{\tau^2}}\; d[n_0(t)]$$

...(5.8.23)

Substitute $\frac{[n_0(t)]^2}{N_0 T/\tau^2} = x^2 \quad \therefore \quad \frac{[n_0(t)]}{\sqrt{N_0 T/\tau^2}} = x$

$$\therefore \quad d[n_0(t)] = \frac{\sqrt{N_0 T}}{\tau}dx$$

The limits of integration change as follows,

When $n_0(t) \to \infty, \quad x \to \infty$

and when $n_0(t) \to AT/\tau$

$$x = \frac{AT/\tau}{\sqrt{N_0 T/\tau}} = \sqrt{\frac{A^2 T^2}{N_0 T}} = \sqrt{\frac{A^2 T}{N_0}}$$

Substituting all these into Equation (5.8.23) we get,

$$P_e = \int_{\sqrt{\frac{A^2T}{N_0}}}^{\infty} \frac{\tau}{\sqrt{\pi N_0 T}} \cdot e^{-x^2} \cdot \frac{\sqrt{N_0 T}}{\tau} dx$$

$$\therefore \quad P_e = \frac{1}{\sqrt{\pi}} \int_{\sqrt{\frac{A^2T}{N_0}}}^{\infty} e^{-x^2} dx \quad ...(5.8.24)$$

Rearrange this equation to get,

$$P_e = \frac{1}{2}\left\{\frac{2}{\sqrt{\pi}} \int_{\sqrt{\frac{A^2T}{N_0}}}^{\infty} e^{-x^2} dx\right\} \quad ...(5.8.25)$$

The integration inside the bracket is called as "complementary error function" and it is evaluated using numerical methods. The complementary error function is defined as :

$$\operatorname{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_{u}^{\infty} e^{-x^2} dx \quad ...(5.8.26)$$

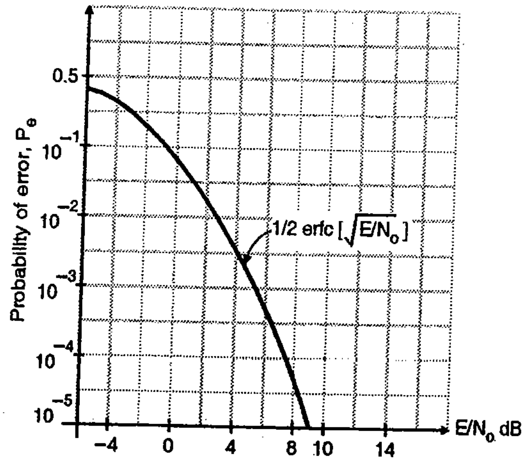
Therefore Equation (5.8.25) can be written as,

$$P_e = \frac{1}{2}\operatorname{erfc}\left[\sqrt{\frac{A^2T}{N_0}}\right] \quad ...(5.8.27)$$

But $A^2 T = E$ i.e. signal energy of a bit of duration T.

$$\therefore \quad P_e = \frac{1}{2}\operatorname{erfc}\left[\sqrt{E/N_0}\right] \quad ...(5.8.28)$$

- The complementary error function is a monotonically decreasing function.
- That means with increase in the value of $\sqrt{E/N_0}$, the value of complementary error function decreases and hence the probability of error will also decrease.



(E-447) **Fig. 5.8.5 : Variation of error probability with E / N_0**

- The graph of error probability P_e against the ratio (E/N_0) is as shown in Fig. 5.8.5 which shows that the error probability decreases rapidly as E/N_0 increases. This is a desirable result.
- The maximum value of $\sqrt{E/N_0}$ is 1 therefore the maximum value of P_e is 1 / 2.

Ex. 5.8.1 : **Show that for an input signal which is a sequence of rectangular positive and negative pulses, the integrator is a matched filter.**

Soln. :

Signal to noise ratio of an integrator :

- The output voltage $x_0(t)$ for a rectangular pulse of amplitude ± A and duration T is given by,

$x_0(T) = \pm\frac{AT}{\tau}$ ...[referring to Equation (5.8.3)] ...(1)

- Therefore the normalized output signal power of an integrator is given by,

$$[x_0(T)]^2 = \frac{A^2T^2}{\tau^2} \quad ...(2)$$

- The normalized output noise power of an integrator is given by,

$$(SNR_o) = \frac{[x_o(T)]^2}{[n_o(T)]^2} = \frac{A^2T^2/\tau^2}{N_o T/2\tau^2} = \frac{2A^2T}{N_o} \quad ...(4)$$

- The energy of a rectangular signal with amplitude ± A and duration T is given by,

$$E = \int_0^T x^2(t)\,dt = \int_0^T A^2\,dt = A^2 T \quad ...(5)$$

- Substituting Equation (5) into Equation (4) we get,

$$SNR_o = \frac{2E}{N_o} \quad ...(6)$$

This equation gives the signal to noise ratio for an integrator having rectangular pulses at its input.

- Now refer to Equation (5.10.17) which states that the maximum signal to noise ratio for a matched filter is given by,

For matched filter, $$\rho^2_{max} = \frac{2E}{N_o} \quad ...(7)$$

Equations (6) and (7) shows that the signal to noise ratio of an integrator is identical to that of a matched filter. Hence integrator acts as a matched filter for the rectangular pulses.

Ex. 5.8.2 : A signal which can take on voltages + V, 0 and – V with equal likelihood is transmitted. When received, it is embedded in white noise. The receiver integrates the signal and noise for a time T sec. Write an expression for the threshold voltage ± V_t so that the probability of error is independent of which signal is transmitted.

Soln. :

- This example is based on the integrate and dump receiver. The integrator output at sampling instant T is given by,

$$v_o(T) = x_o(T) + n_o(T) \quad ...(1)$$

where $x_o(T)$ is the signal output at t = T.

- For the input to be equal to ,

$$+V, x_o(T) = \frac{+VT}{\tau}$$

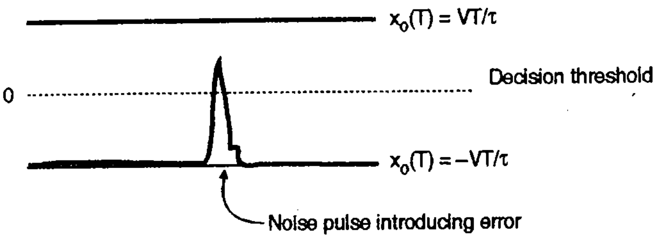
and for the input equal to , $-V, x_o(T) = \frac{-VT}{\tau}$

$$\therefore \quad v_o(T) = \frac{+VT}{\tau} + n_o(T) \text{ ...for input equal to +V} \quad ...(2)$$

$$\text{and } v_o(T) = -\frac{VT}{\tau} + n_o(T) \text{ ...for input equal to –V} \quad ...(3)$$

- From Equations (2) and (3) it is clear that the decision made by the receiver is based on whether the input signal is positive or negative.
- If the decision threshold is assumed to be at the centre of $\frac{VT}{\tau}$ and $-\frac{VT}{\tau}$ as shown in Fig. P. 5.8.2(a).
- Then the magnitude of output noise voltage $n_o(T)$ required to force a wrong decision is given by,

$$n_o(T) \geq \pm\frac{VT}{\tau} \quad \text{...for a wrong decision to be made}$$

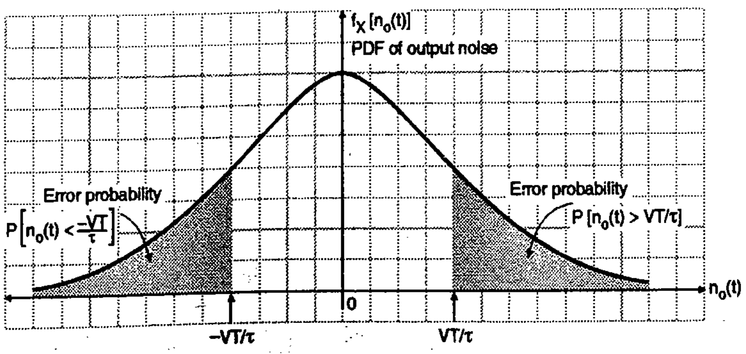


(E-448) Fig. P. 5.8.2(a)

- The output noise is assumed to be Gaussian. The PDF of $n_o(t)$ is given by,

$$f_X[n_o(t)] = \frac{1}{\sigma\sqrt{2\pi}} e^{-[n_o(t)]^2/2\sigma^2} \quad ...(4)$$

This is plotted as shown in Fig. P. 5.8.2(b).



(E-449) Fig. P. 5.8.2(b) : PDF of output noise of n_o (t) of integrate and dump filter

- As proved in section 5.8.3 the expression for error probability is given by,
- $P_e = \frac{1}{2} \text{erfc}\ [\sqrt{E/N_o}]$ ...[referring to Equation (5.8.28)]
- This is the expression for error probability when the decision threshold is exactly at the center of $\frac{VT}{\tau}$ and $-\frac{VT}{\tau}$ as shown in Fig. P. 5.8.2(a), i.e. at 0 Volts.
- This expression for P_e does not contain any input voltage term.

$\therefore$ To make the probability of error independent of the signal transmitted, the threshold voltage $V_t = 0$ Volts as shown in Fig. P. 5.8.2(a).

Ex. 5.8.3 : A received signal is either +2V or –2V held for a time T. The signal is corrupted by white Gaussian noise of power spectral density 10^{-4} volt^2 / Hz. If the signal is processed by an integrator and dump receiver, what is the minimum time T during which a signal must be sustained if the probability of error is not to exceed 10^{-4} ?

Dec. 10, 8 Marks

Soln. :

- The probability of error of integrate and dump receiver is given as :

$$P_e = \frac{1}{2} \text{erfc} \left[\frac{A^2 T}{N_o} \right]^{1/2} \quad \text{...[referring to Equation (5.8.27)]} \quad ...(1)$$

- It has been given that $\frac{N_o}{2} = 10^{-4}$

  $\therefore N_o = 2 \times 10^{-4}$ and $A = 2$.
- The error probability should not exceed 10^{-4}.

$$\therefore \quad P_e \leq 10^{-4}$$

$$\therefore \quad 10^{-4} \leq \frac{1}{2} \text{erfc} \left[\frac{A^2 T}{N_o} \right]^{1/2}$$

$$\therefore \quad 10^{-4} \leq \frac{1}{2} \text{erfc} \left[\frac{2^2 \times T}{2 \times 10^{-4}} \right]^{1/2}$$

- Using the sign of equality we get,

$$\text{erfc} \left[\frac{4\, T_{min}}{2 \times 10^{-4}} \right]^{1/2} = 2 \times 10^{-4}.$$

- Using the table of erfc we get,

$$\left[\frac{4\, T_{min}}{2 \times 10^{-4}} \right]^{1/2} \approx 2.6 \qquad \therefore \frac{4\, T_{min}}{2 \times 10^{-4}} = 6.76$$

$\therefore \quad T_{min} = 0.338 \times 10^{-3}$ sec. or 0.338 msec **...Ans.**

- Thus the minimum time for which the signal should be extended is 0.338 msec.

Ex. 5.8.4 : A bipolar binary signal is either (+ A) Volts or (– A) Volts pulse during the interval (0, T). It is applied to integrate and dump detector. Show that probability of error is given by, $P(e) = \frac{1}{2} \text{erfc} \left[\frac{A^2 T}{N_o} \right]^{1/2}$

May 2000, 8 Marks

Soln. : Refer to section 5.8.3 for this derivation.

Ex. 5.8.5 : A received signal is either +2V or –2V held for a time T. The signal is corrupted by white Gaussian noise of power spectral density 10^{-4} volt^2/Hz. If the signal is processed by an integrator and dump receiver, what is the minimum time T during which a signal must be sustained if the probability of error is not to exceed 10^{-4} ?

Dec. 06, 8 Marks, May 12, 8 Marks

Soln. : The probability of error of integrate and dump receiver is given as :

$$P_e = \frac{1}{2} \text{erfc} \left[\frac{A^2 T}{N_0} \right]^{1/2} \quad ...(1)$$

It has been given that $\frac{N_0}{2} = 10^{-4}$

$\therefore N_0 = 2 \times 10^{-4}$ and $A = 2$.

The error probability should not exceed 10^{-4}.

$$\therefore \quad P_e \leq 10^{-4} \quad \therefore 10^{-4} \leq \frac{1}{2} \text{erfc} \left[\frac{A^2 T}{N_0} \right]^{1/2}$$

$$\therefore \quad 10^{-4} \leq \frac{1}{2} \text{erfc} \left[\frac{2^2 \times T}{2 \times 10^{-4}} \right]^{1/2}$$

Using the sign of equality we get,

$$\text{erfc} \left[\frac{4\, T_{min}}{2 \times 10^{-4}} \right]^{1/2} = 2 \times 10^{-4}$$

Using the table of erfc we get,

$$\left[\frac{4\, T_{min}}{2 \times 10^{-4}} \right]^{1/2} \approx 2.6$$

$$\therefore \quad \frac{4\, T_{min}}{2 \times 10^{-4}} = 6.76$$

$\therefore \quad T_{min} = 0.338 \times 10^{-3}$ sec. or 0.338 msec **...Ans.**

Thus the minimum time for which the signal should be extended is 0.338 msec.

5.9 The Optimum Receiver :

SPPU : Dec. 06. Dec. 08

University Questions

Q. 1 Explain the operation of optimum receiver. **(Dec. 06, 8 Marks)**

Q. 2 Derive the expression for output of an optimum filter using correlator. **(Dec. 08, 8 Marks)**

- In the integrator and dump receiver of Fig. 5.8.1, the received signal was passed through an integrator so that at the sampling instant the signal voltage is received without error even in presence of noise.
- We can ask a question to ourselves whether the integrator is an "optimum filter" which will ensure a minimum probability of error ?
- For this we must know what exactly an optimum filter is.
- An optimum filter is a generalized filter which is used to receive binary coded signals with minimum probability of error. The block diagram of an optimum filter is shown in Fig. 5.9.1.
- The noise at the input of the optimum filter is assumed to have a Gaussian distribution, with a zero mean.
- We assume that the received signal is a binary waveform. One binary digit (bit) is represented by a signal waveform x_1 (t) which has a bit duration of T seconds and the other bit is represented by the waveform x_2 (t) which also has the same bit duration i.e. T. Let the received signal be represented by polar NRZ format.

$\therefore$ For binary 1 : $x_1 (t) = + A$ ... for a bit period T sec.

and for binary 0 : $x_2 (t) = - A$ ... for a bit period T sec.

- For other modulation systems, different waveforms are transmitted. For example for PSK signaling $x_1 (t) = A \cos \omega_c t$ and $x_2 (t) = - A \cos \omega_c t$.
- As shown in Fig. 5.9.1 the input which is x_1 (t) or x_2 (t) is corrupted by the additive noise n (t). This noise is Gaussian and has a spectral density S_n (f). Note that we are not assuming the noise to be the white noise. The input signal to the filter is a noisy signal which is represented by [x (t) + n (t)].
- This noisy signal is filtered and then sampled at the end of each bit interval. The output sample is denoted by v_0 (t) and is expressed as :

$$v_0 (T) = x_{01} (T) + n_0 (T)$$

$$\text{or} = x_{02} (T) + n_0 (T)$$

5.9.1 Probability of Error for an Optimum Filter : SPPU : Dec. 12

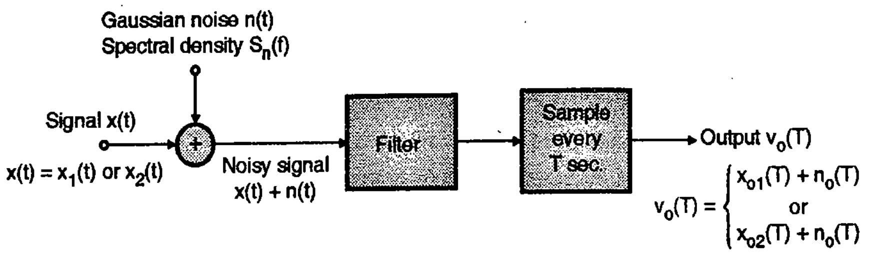
University Questions

Q. 1 Derive the expression for error probability for optimum filter. **(Dec. 12, 8 Marks)**

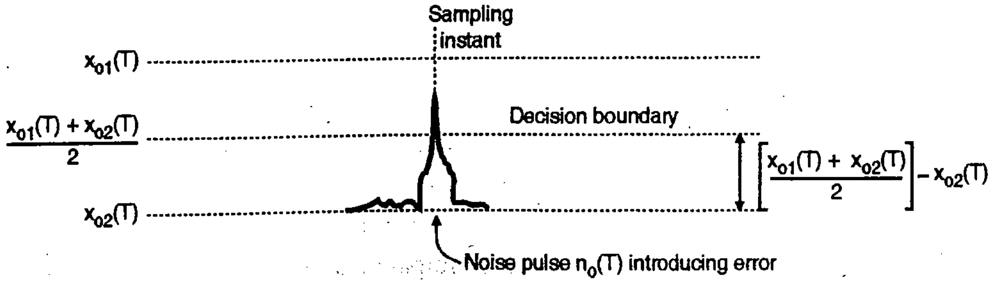
1. Decision boundary :

- In absence of noise the receiver output would be $v_0 (T) = x_{01} (T)$ or $x_{02} (T)$ and the error probability will be zero.
- When the noise is present to minimize the probability of error, the receiver should decide that x_1 (t) has been transmitted if $v_0 (T)$ is closer to $xo_1 (T)$ than to $x_{02} (T)$.
- Similarly it should decide that x_2 (t) has been transmitted if $v_0 (T)$ is closer to $x_{02} (T)$ than to $x_{01} (T)$.
- The decision boundary is therefore midway between $x_{01} (T)$ and $x_{02} (T)$ as shown in Fig. 5.9.2.

$$\therefore \quad \text{Decision boundary} = \frac{x_{01} (T) + x_{02} (T)}{2} \quad ...(5.9.1)$$



(E-450) Fig. 5.9.1 : Block diagram of an optimum receiver



(E-451) Fig. 5.9.2 : Decision boundary and noise pulse n_o (T) introducing an error

2. **Introduction of an error** : Suppose that $x_{o1}(T) > x_{o2}(T)$ as shown in Fig. 5.9.2. Assume that $x_2(t)$ was the transmitted signal. If at the time of sampling the noise $n_o(T)$ is positive and larger than the voltage difference $\left[\frac{x_{o1}(T) + x_{o2}(T)}{2}\right]$ $- x_{o2}(T)$ as shown in Fig. 5.9.2 then the error is introduced and an incorrect decision will be made by the receiver.

∴ For error introduction,

$$n_o(T) \geq \left[\frac{x_{o1}(T) + x_{o2}(T)}{2}\right] - x_{o2}(T)$$

$$\therefore \quad n_o(T) \geq \left[\frac{x_{o1}(T) - x_{o2}(T)}{2}\right] \quad ...(5.9.2)$$

3. Similarly if $x_1(t)$ was transmitted and at the time of sampling the noise $n_o(T)$ is negative and larger than the voltage difference $x_{o1}(T) - \left[\frac{x_{o1}(T) + x_{o2}(T)}{2}\right]$, then error will be introduced.

∴ For error introduction, $n_o(T) \leq \frac{x_{o2}(T) - x_{o1}(T)}{2}$ ...(5.9.3)

4. The noise is assumed to have a Gaussian distribution and we have already derived the expression for the PDF of $n_o(t)$ [see Equation (5.8.18)] in the prceeding section as :

$$f_X[n_o(t)] = \frac{1}{\sigma\sqrt{2\pi}} e^{-[n_o(t)]^2/2\sigma^2} \quad ...(5.9.4)$$

This is plotted as shown in Fig. 5.9.3.

5. The probability of error is equal to the probability that the noise $n_o(T)$ will have a magnitude larger than $\left[\frac{x_{o1}(T) - x_{o2}(T)}{2}\right]$ or less than $\left[\frac{x_{o2}(T) - x_{o1}(T)}{2}\right]$.

$$\therefore \quad P_e = P\left[n_o(T) \geq \left(\frac{x_{o1}(T) - x_{o2}(T)}{2}\right)\right]$$

$$= P\left[n_o(T) \leq \left(\frac{x_{o2}(T) - x_{o1}(T)}{2}\right)\right] \quad ...(5.9.5)$$

The probability of error is represented by the shaded area in Fig. 5.9.3.

6. Therefore probability of error is,

$$P_e = \int_{\frac{x_{o1}(T) - x_{o2}(T)}{2}}^{\infty} f_X[n_o(T)]\, d[n_o(t)]$$

$$= \int_{\frac{x_{o1}(T) - x_{o2}(T)}{2}}^{\infty} e^{-[n_o(t)]^2/2\sigma^2}\, d[n_o(t)] \quad ...(5.9.6)$$

Let $\frac{[n_o(t)]^2}{2\sigma^2} = z^2$

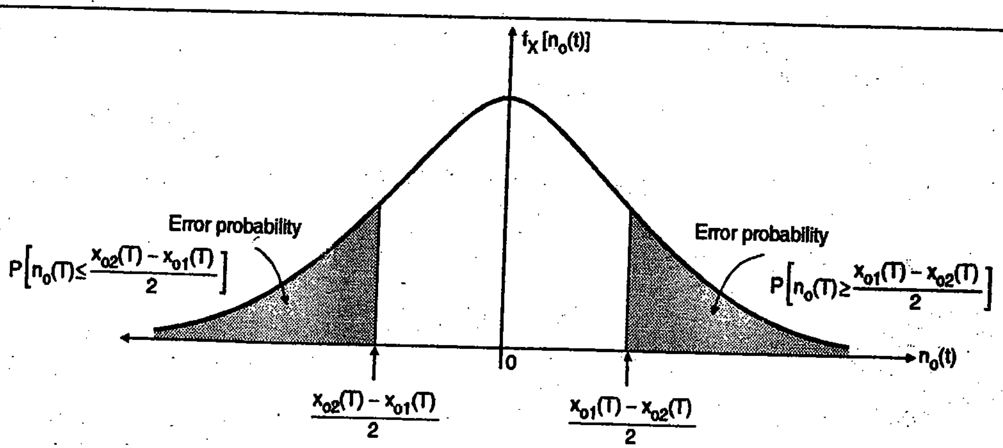
$$\therefore \quad \frac{[n_o(t)]}{\sqrt{2}\sigma} = z$$

$$\therefore \quad d[n_o(t)] = \sqrt{2}\sigma\, dz$$

For $n_o(t) \to \infty$ $\quad z \to \infty$

and for $n_o(t) = \frac{x_{o1}(T) - x_{o2}(T)}{2}$,

$$z = \frac{x_{o1}(T) - x_{o2}(T)}{2\sqrt{2}\sigma}$$



(E-452) **Fig. 5.9.3 : PDF of output noise $n_o(t)$**

Substituting all these values into Equation (5.9.6) we get,

$$P_e = \int_{\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2} \cdot \sqrt{2}\,\sigma\, dz \quad ...(5.9.7)$$

$$\therefore \quad P_e = \frac{1}{\sqrt{\pi}} \int_{\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}}^{\infty} e^{-z^2}\, dz$$

Rearranging this expression we get,

$$P_e = \frac{1}{2}\left[\frac{2}{\sqrt{\pi}} \int_{\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}}^{\infty} e^{-z^2}\, dz\right] \quad ...(5.9.8)$$

The expression inside the square bracket is the complementary error function.

$$\therefore \quad P_e = \frac{1}{2}\,\text{efrc}\left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}\right] \quad ...(5.9.9)$$

This is the expression for the error probability of an optimum filter.

Conclusions from Equation (5.9.9) :

1. The complementary error function "erfc" is a monotonic decreasing function. Therefore the error probability will reduce with increase in the difference $[x_{01}(T) - x_{02}(T)]$.
2. The error probability will reduce with decrease in the rms noise voltage σ. (The mean square value of noise is equal to standard deviation σ^2 since the mean value of Gaussian noise is zero).
3. Thus the optimum filter must maximize the ratio $\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]$ in order to minimize the error probability P_e.

5.9.2 Optimum Filter Transfer Function H (f) :

The transfer function H (f) of an optimum filter must be such that it will maximize the ratio.

$$\rho = \frac{x_{01}(T) - x_{02}(T)}{\sigma} \quad ...(5.9.10)$$

However for mathematical convenience, we shall actually maximize ρ^2 rather than ρ.

Assumptions :

The assumptions made to obtain the expression for the transfer function H (f) are as follows :

1. Let $x_{01}(T) - x_{02}(T) = x_0(T)$

$$\therefore \quad \rho = \frac{x_0(T)}{\sigma} \quad ...(5.9.11)$$

2. Let $x_o^2(T)$ be the normalized output signal power and $\sigma^2 = \overline{n_o^2(T)}$ = Mean square value of $n_0(T)$ = Normalized noise power.
3. Therefore in order to minimize the probability of error we will have to maximize the ratio.

$$\rho^2 = \frac{x_o^2(T)}{n_o^2(T)} = \frac{x_o^2(T)}{\sigma^2} \quad ...(5.9.12)$$

4. Let the input signal to the optimum filter be $x(t) = x_1(t) - x_2(t)$ and let the corresponding signal output be $x_0(t) = x_{01}(t) - x_{02}(t)$.

Derivation :

1. The relation between the output and input of the optimum filter is given by,

$$x_0(t) = x(t) * h(t) \quad ...(5.9.13)$$

Where h (t) = impulse response of the optimum filter.

2. Taking the Fourier transform of both the sides of Equation (5.9.13) we get,

$$X_0(f) = X(f) \cdot H(f) \quad ...(5.9.14)$$

Where H (f) = Transfer function of the optimum filter

$X_0(f)$ = F.T. $[x_0(t)]$ and $X(f)$ = F.T. $[x(t)]$

3. **To obtain $x_0(T)$:** To obtain the ratio ρ^2 we need to obtain the expression for $x_0(T)$.

$$\therefore \quad x_0(T) = \text{IFT}[X_0(f)] = \int_{-\infty}^{\infty} X_0(f) \cdot e^{j2\pi fT}\, df$$

...(5.9.15)

Substituting the value of $X_0(f)$ we get,

$$x_0(T) = \int_{-\infty}^{\infty} H(f)\, X(f) \cdot e^{j2\pi fT}\, df \quad ...(5.9.16)$$

4. **To obtain σ :** The noise at the filter input is n (t). Let the power spectral density of input noise be $S_{ni}(f)$. The output noise of the optimum filter is $n_o(t)$. Let the power spectral density of output noise be $S_{no}(f)$. The relation between $S_{no}(f)$ and $S_{ni}(f)$ is,

$$S_{no}(f) = |H(f)|^2 \cdot S_{ni}(f) \qquad ...(5.9.17)$$

Therefore the normalized noise power is given by,

$$\text{Normalized noise power} = \sigma^2 = \int_{-\infty}^{\infty} S_{no}(f)\, df \qquad ...(5.9.18)$$

This is because normalized power is equal to the area under the PSD curve. Substituting the expression for $S_{no}(f)$ from Equation (5.9.17) into Equation (5.9.18) we get,

$$\sigma^2 = \int_{-\infty}^{\infty} |H(f)|^2\, S_{ni}(f)\, df \qquad ...(5.9.19)$$

5. Substitute the values of $x_o(T)$ and σ^2 from Equations (5.9.16) and (5.9.19) into Equation (5.9.12) to get,

$$\rho^2 = \frac{x_o^2(T)}{\sigma^2}$$

$$\therefore \quad \rho^2 = \frac{\left| \int_{-\infty}^{\infty} H(f)\, X(f) \cdot e^{j2\pi fT}\, df \right|^2}{\int_{-\infty}^{\infty} |H(f)|^2 S_{ni}(f)\, df} \qquad ...(5.9.20)$$

To get the maximum value of ρ^2 we need to use the Schwarz inequality.

6. The Schwarz inequality states that given arbitrary complex functions Y (f) and Z (f) of a common variable f, then

$$\left| \int_{-\infty}^{\infty} Y(f)\, Z(f)\, df \right|^2 \le \int_{-\infty}^{\infty} |Y(f)|^2\, df \int_{-\infty}^{\infty} |Z(f)|^2\, df \qquad ...(5.9.21)$$

The equal sign applies when

$$Y(f) = KZ^*(f) \qquad ...(5.9.22)$$

where K is an arbitrary constant and $Z^*(f)$ is the complex conjugate of Z (f).

7. Let us now apply the Schwarz inequality to Equation (5.9.20) by substituting

$$Y(f) = \sqrt{S_{ni}(f)}\, H(f)$$

$$Z(f) = \frac{1}{\sqrt{S_{ni}(f)}} X(f) \cdot e^{j2\pi fT}$$

$$\therefore \quad \rho^2 = \frac{\left| \int_{-\infty}^{\infty} Y(f) \cdot Z(f)\, d(f) \right|^2}{\int_{-\infty}^{\infty} |Y(f)|^2\, df} \qquad ...(5.9.23)$$

Applying Schwarz inequality to the numerator we get,

$$\rho^2 \le \frac{\int_{-\infty}^{\infty} |Y(f)|^2\, df \int_{-\infty}^{\infty} |Z(f)|^2\, df}{\int_{-\infty}^{\infty} |Y(f)|^2\, df}$$

$$\therefore \quad \rho^2 \le \int_{-\infty}^{\infty} |Z(f)|^2\, df \qquad ...(5.9.24)$$

Now substitute the value of Z (f) to get,

$$\therefore \rho^2 \le \frac{1}{S_{ni}(f)} \int_{-\infty}^{\infty} |X(f) \cdot e^{j2\pi fT}|^2\, df \qquad ...(5.9.25)$$

But $|e^{j2\pi fT}| = [\cos^2(2\pi fT) + \sin^2(2\pi fT)]^{1/2} = 1$

$$\therefore \quad \rho^2 \le \frac{1}{S_{ni}(f)} \int_{-\infty}^{\infty} |X(f)|^2\, df \qquad ...(5.9.26)$$

8. We want the value of ρ^2 to be maximum. To maximize the value of ρ^2 we have to consider the sign of equality in Equation (5.9.26).

$$\text{i.e.} \quad \rho^2_{max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)}\, df$$

But this is possible only when the condition for equality in Schwarz inequality stated in Equation (5.9.22) is satisfied.

$$\text{i.e.} \quad \left| \int_{-\infty}^{\infty} Y(f) \cdot Z(f)\, df \right|^2$$

$$= \int_{-\infty}^{\infty} |Y(f)|^2\, df \int_{-\infty}^{\infty} |Z(f)|^2\, df$$

If and any if $Y(f) = KZ^*(f)$

Substituting the values of Y (f) and Z (f) we get,

$$\sqrt{S_{ni}(f)} \cdot H(f) = K \frac{1}{\sqrt{S_{ni}(f)}} X^*(f)\, e^{-j2\pi fT}$$

$$\therefore \quad H(f) = \frac{K X^*(f)}{S_{ni}(f)} \cdot e^{-j2\pi fT} \qquad ...(5.9.27)$$

This is the expression for the transfer function of an optimum filter which ensures minimum probability of error. Correspondingly, the maximum ratio is given by,

$$\rho^2_{max} = \left[\frac{x_o^2(T)}{\sigma^2}\right]_{max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)} df \quad ...(5.9.28)$$

Ex. 5.9.1 : If $s_1(t)$ and $s_2(t)$ are two signals being transmitted in a PSK system such that :

$s_1(t) = 10 \cos \omega_o(t) \quad s_2(t) = -10 \cos \omega_o(t)$

Calculate the SNR for the optimum filter provided the noise is white.

Soln. : When the noise is white, the optimum filter becomes matched filter. When a matched filter is used for detection of BPSK signal, the signal to noise ratio is given by,

$$\rho^2_{max} = \frac{8E}{N_o} \quad ...(1)$$

In this equation energy $E = P_s T_b$

$$\therefore \text{ Signal to noise ratio } \rho^2_{max} = \frac{8 P_s T_b}{N_o} \quad ...(2)$$

Let us calculate P_s and T_b.

To calculate P_s :

Given that $s_1(t) = -s_2(t) = 10 \cos \omega_o t$

$$\therefore \quad A = 10$$

$$\therefore \quad P_s = \frac{A^2}{2} = \frac{100}{2} = 50 \quad ...(3)$$

To calculate T_b :

Because the bit rate has not been given, we cannot calculate the value of T_b. So let it remain as it is. Substituting the values into Equation (2) we get,

$$(SNR)_o = \rho^2_{max} = \frac{8 \times 50 \times T_b}{N_o}$$

$$(SNR)_o = \frac{400\, T_b}{N_o} \quad \text{...Ans.}$$

5.10 Matched Filter :

SPPU : May 07

University Questions

Q. 1 Write a short note on matched filter. **(May 07, 8 Marks)**

Definition:

An optimum filter which yields a maximum ratio $[x_o^2(T)/\sigma^2]$ is called as the matched filter when the input noise is white. Then power spectral density (psd) of the input white noise is given by

$$S_{ni}(f) = N_o/2 \quad ...(5.10.1)$$

5.10.1 Impulse Response of a Matched Filter :

SPPU : Dec. 06, May 07, May 08, May 09, May 16

University Questions

Q. 1 State the various properties of matched filter. Explain the impulse response in detail. **(Dec. 06, 8 Marks)**

Q. 2 Write a short note on matched filter. **(May 07, 8 Marks)**

Q. 3 Show that for a matched filter the maximum signal component occurs at t = T and has magnitude E, that is the signal energy of x(t). **(May 08, 8 Marks)**

Q. 4 Show that the impulse response of a matched filter is a time reversed and delayed version of the input signal. **(May 08, May 09, 8 Marks)**

Q. 5 State the various properties of matched filter. Explain the impulse response in detail. **(May 16, 8 Marks)**

The impulse response h (t) of a matched filter can be obtained from its transfer function H (f), by taking an inverse Fourier transform (IFT). Let us use the expression for the transfer function of optimum filter as starting point.

1. Transfer function H (f) of an optimum filter is given by,

$$H(f) = \frac{K X^*(f)}{S_{ni}(f)} \cdot e^{-j2\pi fT} \quad \text{...(Referring to Equation (5.9.27).}$$

For the matched filter the input noise n (t) is assumed to be white noise. Therefore substitute $S_{ni}(f) = N_o/2$ to get,

$$H(f) = \frac{2 K X^*(f)}{N_o} e^{-j2\pi fT} \quad ...(5.10.2)$$

2. The conjugate property of Fourier transform states that,

$$X^*(f) = X(-f) \quad ...(5.10.3)$$

Substituting this in Equation (5.10.2) we get,

$$H(f) = \frac{2K}{N_o} \cdot X(-f) \cdot e^{-j2\pi fT} \quad ...(5.10.4)$$

3. The impulse response h (t) = IFT [H (f)]

$$\therefore \quad h(t) = \text{IFT}\left[\frac{2K}{N_o} X(-f) \cdot e^{-j2\pi fT}\right] \quad ...(5.10.5)$$

4. The inverse Fourier transform of X (– f) is x (– t) and the term $e^{-j2\pi fT}$ represents a time shift of T sec.

$$\therefore \quad FT[x(-t)] = X(-f)$$

$$\text{and} \quad FT[x(T-t)] = X(-f)\, e^{-j2\pi fT}$$

$$\text{Therefore} \quad h(t) = \frac{2K}{N_o}[x(T-t)] \quad ...(5.10.6)$$

5. But the input signal x (t) is given by,

$$x(t) = x_1(t) - x_2(t)$$

$$\therefore \quad h(t) = \frac{2K}{N_o}[x_1(T-t) - x_2(T-t)] \qquad ...(5.10.7)$$

This is the required expression for the impulse response of matched filter.

This expression shows that the impulse response of a matched filter is a time reversed and delayed version of the input signal. This is as shown in Fig. 5.10.1.

5.10.2 Probability of Error the Matched Filter :

SPPU : May 12, May 13, Dec. 14, Dec. 16

University Questions

Q. 1 Derive the expressions for signal to noise ratio and error probability of a matched filter in the presence of white gaussian noise. **(May 12, 8 Marks)**

Q. 2 Derive the error probability of Matched filter. **(May 13, 8 Marks)**

Q. 3 Derive the expression of probability error (P_e) for matched filter. **(Dec. 14, 8 Marks)**

Q. 4 Derive the expression for signal to noise ratio and error probability of a matched filter in the presence of white gaussian noise. **(Dec. 16, 8 Marks)**

1. The probability of error which results when we use the matched filter, can be obtained by evaluating the maximum signal to noise ratio $\left[\frac{x_o^2(T)}{\sigma^2}\right]_{max}$ given in Equation (5.9.28) for the optimum filter,

$$\therefore \quad \left[\frac{x_o^2(T)}{\sigma^2}\right]_{max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)} df \qquad ...(5.10.8)$$

2. For a matched filter, psd of input noise signal is $S_{ni}(f) = N_o / 2$

$$\therefore \quad \left[\frac{x_o^2(T)}{\sigma^2}\right]_{max} = \int_{-\infty}^{\infty} \frac{2|X(f)|^2}{N_o} df \qquad ...(5.10.9)$$

$$\text{But} \quad x_o(T) = x_{01}(T) - x_{02}(T) \qquad ...(5.10.10)$$

$$\therefore \left[\frac{x_{01}(T) - x_{02}(T)}{\sigma^2}\right]^2_{max} = \frac{2}{N_o} \int_{-\infty}^{\infty} |X(f)|^2 df. \qquad ..(5.10.11)$$

3. According to the Parseval's theorem we have,

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} x^2(t)\, dt = \int_{0}^{T} x^2(t)\, dt \quad ...(5.10.11(a))$$

In the last integral of Equation (5.10.11) the limits are 0 to T, this is because x (t) persists for only a time T.

$$\text{But } x(t) = x_1(t) - x_2(t)$$

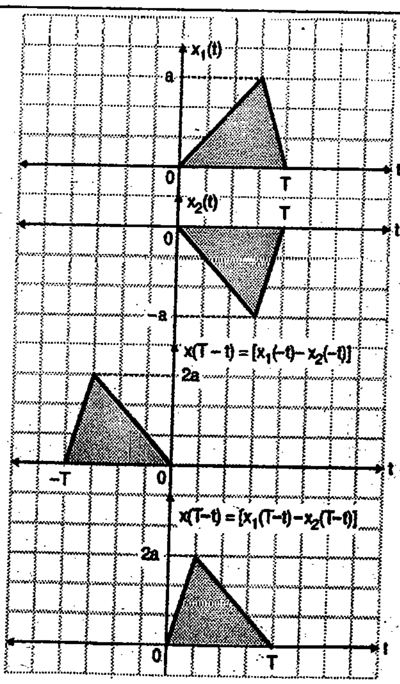
∴ Equation (5.10.11) can be written as,

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{0}^{T} [x_1(t) - x_2(t)]^2 dt$$

$$= \int_{0}^{T} \left[x_1^2(t) + x_2^2(t) - 2\,x_1(t)\,x_2(t)\right] dt$$

$$= \int_{0}^{T} x_1^2(t)\, dt + \int_{0}^{T} x_2^2(t)\, dt - \int_{0}^{T} 2\,x_1(t)\,x_2(t)\, dt$$

$$\therefore \quad \int_{-\infty}^{\infty} |X(f)|^2 df = E_1 + E_2 - 2\,E_{12} \quad ...(5.10.12)$$



(a) Signal $x_1(t)$

(b) Signal $x_2(t)$

(c) $x(T-t) = [x_1(-t) - x_2(-t)]$

(d) $x(T-t) = [x_1(T-t) - x_2(T-t)] = h(t)$, the impulse response of a matched filter

(E-453) Fig. 5.10.1

4. If we select $x_2(t) = -x_1(t)$ then we find that,

Where E_1 = Energy of $x_1(t)$

E_2 = Energy due to correlation between $x_1(t)$ and $x_2(t)$.

$$E_1 = E_2 = -E_{12} = E$$

Substituting this into Equation (5.10.12) we get,

$$\int_{-\infty}^{\infty} |X(f)|^2\, df = E + E + 2E = 4E \qquad \text{...(5.10.13)}$$

5. Now substitute this value into Equation (5.10.11) to get,

$$\therefore \left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{\max} = \frac{2}{N_o} \times 4E = \frac{8E}{N_o} \qquad \text{...(5.10.14)}$$

$$\therefore \left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]_{\max} = \sqrt{\frac{8E}{N_o}} = 2\sqrt{2}\sqrt{\frac{E}{N_o}} \qquad \text{...(5.10.15)}$$

6. The expression for the error probability of an optimum filter is given by,

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}\right] \qquad \text{...from Equation (5.9.9)}$$

Now substitute the value of $\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]_{\max}$ from Equation (5.10.15) to get the minimum error probability of a matched filter as :

$$P_{e(\min)} = \frac{1}{2}\,\text{erfc}\left[\frac{2\sqrt{2}\cdot\sqrt{E/N_o}}{2\sqrt{2}}\right]$$

$$\therefore P_{e(\min)} = \frac{1}{2}\,\text{erfc}\left[\sqrt{E/N_o}\right] \qquad \text{...(5.10.16)}$$

This is the required expression for the error probability of a matched filter.

Conclusions from Equation (5.10.16) :

1. The error probability depends only on the signal energy E and not on the shape of the signal.
2. The error probability of matched filter is same as that of the integrate and dump receiver. Therefore integrate and dump receiver is a matched filter.

5.10.3 Signal to Noise Ratio of a Matched Filter :

The signal to noise ratio at the output of the matched filter is given by,

$$\rho^2_{\max} = \left[\frac{x_o(T)}{\sigma}\right]^2_{\max} = \frac{2E}{N_o} \qquad \text{...(5.10.17)}$$

The proof of this has been given in property 3 of matched filter.

5.10.4 Properties of Matched Filter :

SPPU - Dec. 06, May 12, May 16, Dec. 16

University Questions

Q. 1 State the various properties of matched filter. Explain the impulse response in detail. **(Dec. 06, 8 Marks)**

Q. 2 Derive the expressions for signal to noise ratio and error probability of a matched filter in the presence of white gaussian noise. **(May 12, 8 Marks)**

Q. 3 State the various properties of matched filter. Explain the impulse response in detail. **(May 16, 8 Marks)**

Q. 4 Derive the expression for signal to noise ratio and error probability of a matched filter in the presence of white gaussian noise. **(Dec. 16, 8 Marks)**

- From the impulse response derived for the matched filter we may state that a filter which is matched to an input signal x (t) is characterized in the time domain by the impulse response.

$$h(t) = x(T - t) \qquad \text{...(5.10.18)}$$

- Here note that we have neglected the term $\frac{2K}{N_o}$ in Equation (5.10.6) for convenience. Thus the impulse response of a matched filter is a time reversed and delayed version of the input signal x (t). Taking the Fourier transform of both the sides of Equation (5.10.18) we obtain the transfer function of the matched filter as,

$$H(f) = \text{F.T.}[h(t)] = \text{F.T.}[x(T-t)]$$

$$\therefore \underbrace{H(f)}_{\text{Transfer function of matched filter}} = \underbrace{X^*(f)}_{\text{Complex conjugate of } X(f)}\;\underbrace{e^{-j2\pi fT}}_{\text{Delay factor}} \qquad \text{...(5.10.19)}$$

(E-1458)

- The transfer function of the matched filter is thus the complex conjugate of the spectrum of the input signal X (f), except for the delay factor indicated by the term $e^{-j2\pi fT}$.
- Based on these relations we are going to derive some important properties of the matched filter.

Property 1 :

The spectrum of the output signal of a matched filter is, except for a time delay factor, proportional to the energy spectral density of the input signal. That means

$$X_o(f) = \Psi(f)\cdot e^{-j2\pi fT}$$

Proof : Let the output signal be x_o (t) and its Fourier transform be X_o (f). Let the input signal be x (t) and its Fourier transform be X (f).

$$\therefore X_o(f) = H(f)\cdot X(f) \qquad \text{...(5.10.20)}$$

$$\text{But } H(f) = X^*(f)\cdot e^{-j2\pi fT}$$

$$\therefore X_o(f) = X^*(f)\cdot e^{-j2\pi fT}\cdot X(f)$$

$$\text{But } H(f) = X^*(f)\cdot e^{-j2\pi fT}$$

$$\therefore \quad X_o(f) = X^*(f)\cdot e^{-j2\pi fT}\cdot X(f)$$

$$= X(f)\cdot X^*(f)\, e^{-j2\pi fT}$$

$$\therefore \quad X_o(f) = |X(f)|^2 e^{-j2\pi fT} \qquad \text{...(5.10.21)}$$

But $|X(f)|^2 = \Psi(f) =$ Energy spectral density of input signal x (t)

$$\therefore \quad X_o(f) = \Psi(f)\cdot e^{-j2\pi fT} \qquad \text{...(5.10.22)}$$

Property 2 :

The output signal x_o (t) of a matched filter is proportional to the shifted version of the autocorrelation function of the input signal x (t) to which the filter is matched. That means

$$x_o(t) = R(t-T)$$

Proof :

- Consider Equation (5.10.22) which states that,

$$X_o(f) = \Psi(f)\, e^{-j2\pi fT}$$

- We can obtain the output signal x_o (t) from this expression by taking the inverse Fourier transform.

$$\therefore \quad x_o(t) = \text{IFT}[X_o(f)] = \int_{-\infty}^{\infty} X_o(f)\, e^{j2\pi fT}\, df$$

- Substituting the value of X_o (f) we get,

$$x_o(t) = \int_{-\infty}^{\infty} \Psi(f)\cdot e^{j2\pi f(t-T)}\, df \qquad \text{...(5.10.23)}$$

- Earlier we have proved that auto-correlation function R (τ) and energy spectral density Ψ (f) form a Fourier transform pair.

$$\text{i.e.} \quad R(\tau) \overset{F}{\leftrightarrow} \Psi(f)$$

- That means $R(\tau) = \int_{-\infty}^{\infty} \Psi(f)\cdot e^{j2\pi f\tau}\, df$

...(5.10.24)

- Comparing this equation with Equation (5.10.23) we conclude that,

$$x_o(t) = R(\tau) \text{ with } \tau = t-T$$

$$\therefore \quad x_o(t) = R(t-T) \qquad \text{...Proved.}$$

Property 3 :

The output signal to noise ratio of a matched filter depends only on the ratio of the signal energy to the power spectral density of the white noise at the filter input. That means :

$$\text{SNR}_o = \frac{E}{N_o/2}$$

Proof :

- The signal to noise ratio of an optimum filter is given by Equation (5.10.24) which states that,

$$\rho^2_{\max} = \left[\frac{x_o(T)}{\sigma}\right]^2_{\max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)}\, df \qquad \text{...(5.10.25)}$$

Where S_{ni} (f) = psd of input noise to the filter.

- For a matched filter, the input noise is white noise with a psd $S_{ni}(f) = N_o/2$

∴ Signal to noise ratio at the matched filter output $= \rho^2_{\max}$

$$= \frac{2}{N_o}\int_{-\infty}^{\infty} |X(f)|^2\, df$$

- But $\int_{-\infty}^{\infty} |X(f)|^2\, df = \text{Energy } E$

.....As per the Rayleigh's theorem.

$$\therefore \quad \rho^2_{\max} = \frac{2E}{N_o} \qquad \text{...(5.10.26)}$$

∴ Output signal to noise ratio $= \frac{2E}{N_o}$ **...Proved.**

- Equation (5.10.26) is perhaps the most important result required for the evaluation of performance of signal processing systems using matched filters.
- Equation (5.10.26) tells us that the matched filter does not depend on the shape of input signal x (t) but the judgement made by the matched filter depends only on the energy E of the input signal.

Ex. 5.10.1 : If the input to a matched filter is a rectangular pulse of amplitude A and duration T representing a logic "1" and no pulse representing a logic 0, draw the impulse response of the matched filter.

May 07, 4 Marks

Soln. : The rectangular pulse at the input of the matched filter be denoted as :

$$x_1(t) = \begin{cases} A & 0 \le t \le T. \\ 0 & \text{otherwise} \end{cases} \qquad \text{...(1)}$$

- And the absence of signal to represent a logic 0 be represented as :

$$x_2(t) = 0 \qquad \text{...(2)}$$

- But $x(t) = x_1(t) - x_2(t)$

$$\therefore \quad x(t) = x_1(t) - 0 = x_1(t) \qquad \text{...(3)}$$

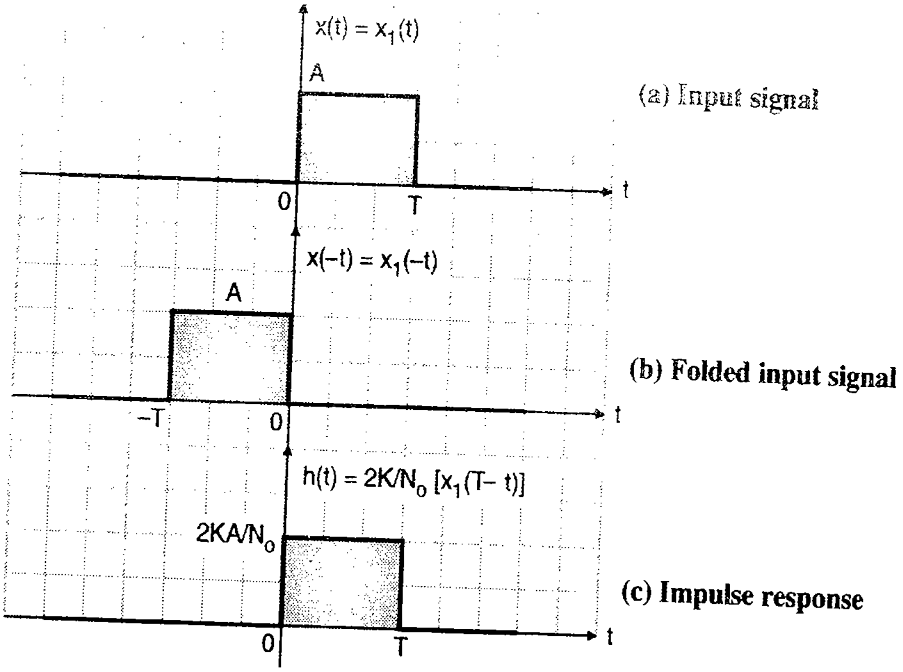
- Referring to Equation (5.10.6) the impulse response of a matched filter is given by,

$$h(t) = \frac{2K}{N_o}[x(T-t)] \qquad \text{...(4)}$$

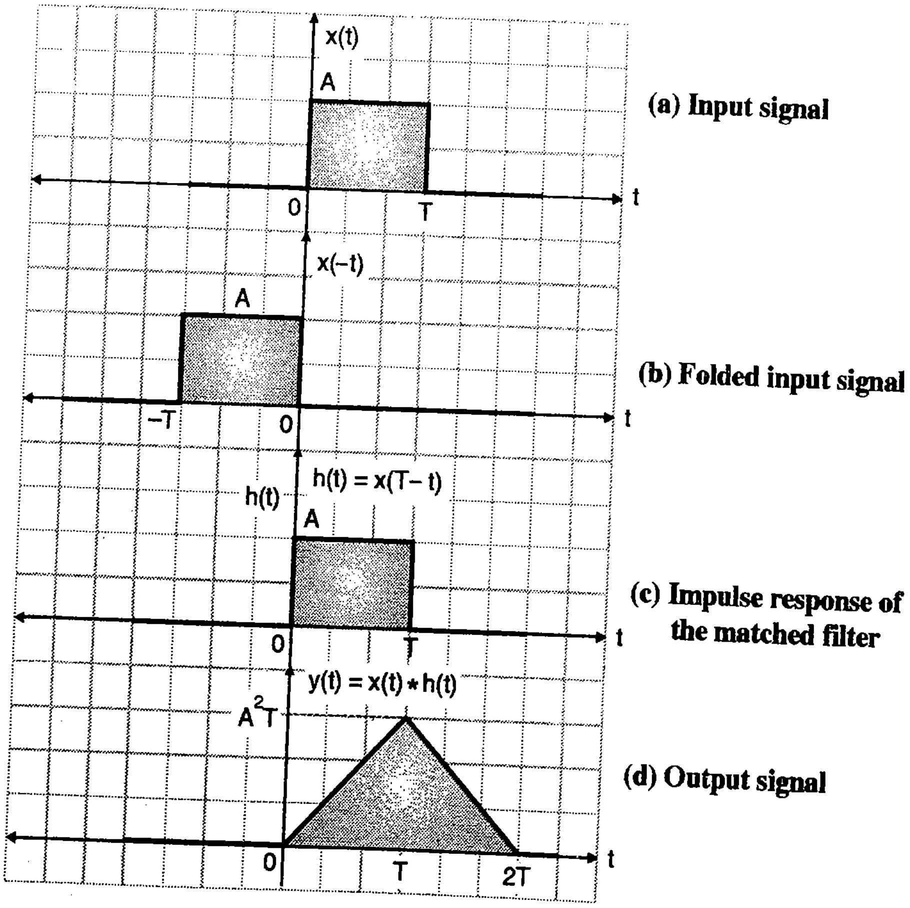
But $x(t) = x_1(t)$

$$\therefore \quad h(t) = \frac{2K}{N_o}[x_1(T-t)] \qquad \text{...(5)}$$

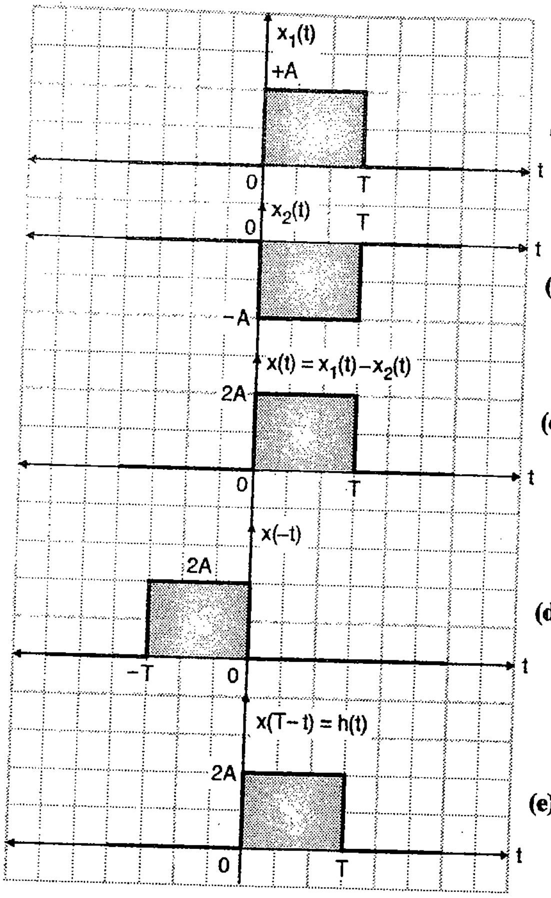
The impulse response is graphically plotted as shown in Fig. P. 5.10.1.



(E-454) Fig. P. 5.10.1



(E-456) Fig. P. 5.10.2(b).



(a) Input signal $x_1(t)$

(b) Input signal $x_2(t)$

(c) Difference signal $x(t) = x_1(t) - x_2(t)$

(d) Folded difference signal $x(-t)$

(e) Impulse response of matched filter

(E-457) **Fig. P. 5.10.3**

Ex. 5.10.2 : For the same data as in Ex. 5.10.1, calculate the maximum signal to noise ratio and draw the waveform for the output signal of the matched filter. **May 07, 4 Marks**

Soln. :

Maximum signal to noise ratio :

Referring to Equation (5.10.17) the maximum signal to noise ratio is given by,

$$\rho^2_{\max} = \left[\frac{x_0(T)}{\sigma}\right]^2_{\max} = \frac{2E}{N_0} \quad ...(1)$$

- But we do not know the value of energy E. So let us calculate E first.

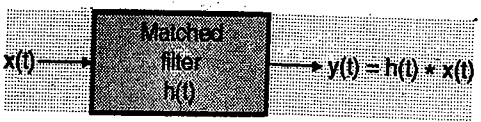
$$E = \int_0^T x^2(t)\,dt = \int_0^T A^2\,dt = A^2[t]_0^T$$

$$\therefore \quad E = A^2 T \quad ...(2)$$

Substituting this value into Equation (1) we get,

$$\therefore \text{ Maximum signal to noise ratio} = \rho^2_{\max} = \frac{2A^2T}{N_0} \quad \text{...Ans.}$$

Output of the matched filter :



(E-455) **Fig. P. 5.10.2(a)**

Looking at Fig. P. 5.10.2(a), the output y (t) of a matched filter is given by :

$$y(t) = h(t) * x(t) \quad ...(3)$$

where * represents convolution of h (t) and x (t).

- The process of convolution has been discussed earlier. Assuming $\frac{2K}{N_0} = 1$. We get, h (t) = x (t). Therefore the result of convolution is a triangular wave as shown in Fig. P. 5.10.2(b).
- From the output voltage waveform it is clear that the output of the matched filter is maximum at t = T and it is equal to the energy $E = A^2 T$.

Ex. 5.10.3 : A polar NRZ signal is applied at the input of a matched filter. The binary 1 is represented by a rectangular pulse of amplitude A and duration T and binary 0 is represented by a rectangular pulse of amplitude – A and duration T. Obtain the impulse response of the matched filter and sketch it. **May 94, 8 Marks**

Soln. :

- From the information given in the example.

$$x_1(t) = +A \quad ...\text{for } 0 \le t \le T$$

$$\text{and} \quad x_2(t) = -A \quad ...\text{for } 0 \le t \le T.$$

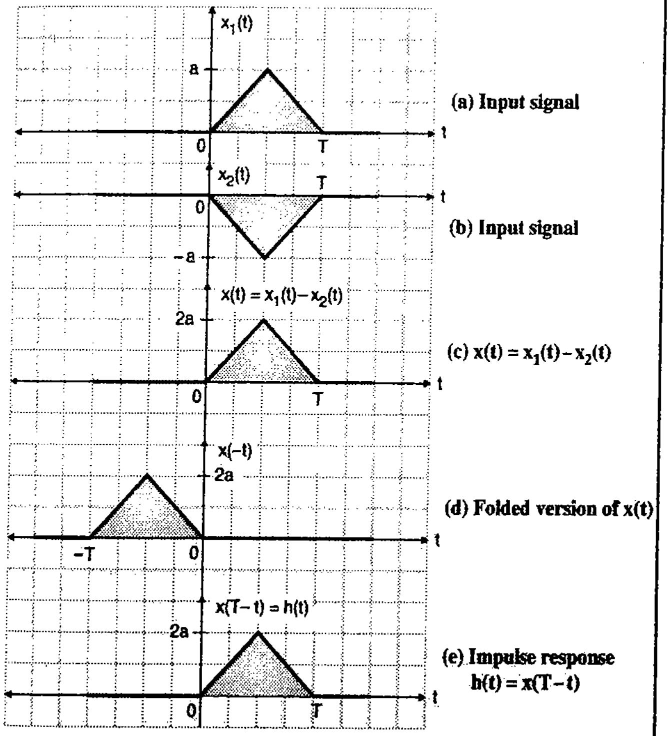
- The impulse response of a matched filter is given by,

$$h(t) = \frac{2K}{N_0}[x_1(T-t) - x_2(T-t)]$$

- Assuming that $\frac{2K}{N_0} = 1$ the waveforms are plotted in Fig. P. 5.10.3.

Ex. 5.10.4 : For two symbol transmission system, assume that $s_1(t) = s_1(t)$ or $s_2(t)$ and $s_1(t) = -s_2(t)$. If $s_1(t)$ happens to be a triangular symmetrical wave (with positive and negative ramps characterized by equal slope), calculate SNR at the receiver output, given that $\eta / 2 = 100$ mW / Hz. The receiver is developed to be an optimum filter. Sketch the filter impulse response and comment on probability of error. Derive the expressions used in the numerical. **May 99, 12 Marks**

Soln. :



(a) Input signal

(b) Input signal

(c) $x(t) = x_1(t) - x_2(t)$

(d) Folded version of x(t)

(e) Impulse response $h(t) = x(T-t)$

(E-458) **Fig. P. 5.10.4**

The psd of noise is given to be equal to $\eta / 2$ i.e. $N_0 / 2$ which shows that the noise is a white Gaussian noise. As discussed in section 5.10.3 the signal to noise ratio of a matched filter is,

$$\rho^2_{max} = \frac{[x_0(T)]^2}{[n_0^2(T)]} = \frac{2E}{N_0} \quad \text{[refer to Equation (5.10.17)]}$$

Where $E = P_s T_b$.

This expression is valid for any shape of the waveform.

Filter impulse response : The impulse response of a matched filter is given by,

$$h(t) = \frac{2K}{N_0}[x_1(T-t) - x_2(T-t)]$$

...[referring to Equation (5.10.7)]

Assuming $\frac{2K}{N_0} = 1$ we draw the waveforms of $x_1(t)$, $x_2(t)$, $x(t)$, $x(-t)$, $x(T-t) = h(t)$ as shown in Fig. P. 5.10.4.

Ex. 5.10.5 : A signal is either $s_1(t) = A\cos(2\pi f_0 t)$ or $s_2(t) = 0$ for an interval $T = \frac{n}{f_0}$ with 'n' an integer. The signal is corrupted by white noise with PSD $= \frac{N_0}{2}$. Find the transfer function of the matched filter for this signal. Write an expression for the probability of error P_e.

Soln. :

- The given signal is an BASK signal.
- It has been given that,

$$x_1(t) = A\cos(2\pi f_0 t) = \sqrt{2P_s}\cos(2\pi f_0 t)$$

$$\text{and } x_2(t) = 0$$

$$A = \sqrt{2P_s} \text{ or } P_s = \frac{A^2}{2}$$

- We have already derived the expressions for the transfer function and error probability. They are as follows.

$$H(f) = \frac{2k}{N_0} X^*(f)\, e^{-j2\pi fT}$$

$$\text{and } P_e = \frac{1}{2}\operatorname{erfc}\sqrt{\frac{E}{4N_0}}$$

$$\text{where } X(f) = \text{F.T. of } x(t) \text{ and } E = P_s T = \frac{A^2}{2}T$$

Expression for transfer function :

$$H(f) = \frac{2k}{N_0} X^*(f)\, e^{-j2\pi fT}$$

$$x(t) = x_1(t) - x_2(t) = s_1(t) - s_2(t) = A\cos(2\pi f_0 t)$$

$$\therefore X(f) = FT[x(t)] = FT[A\cos 2\pi f_0 t]$$

$$= \frac{A}{2}\{\delta(f-f_0) + \delta(f+f_0)\}$$

As the imaginary part in the above expression is zero,

$$X^*(f) = X(f) = \left\{\frac{A}{2}\delta(f-f_0) + \delta(f+f_0)\right\}$$

Substituting we get, $H(f) = \frac{2k}{N_0} \times \frac{A}{2}$

$$\{\delta(f-f_0) + \delta(f+f_0)\, e^{-j2\pi fT}\}$$

$$= \frac{kA}{N_0}\{\delta(f-f_0) + \delta(f+f_0)\}e^{-j2\pi fT}$$

This is the desired result.

Expression for P_e :

$$P_e = \frac{1}{2}\operatorname{erfc}\sqrt{\frac{E}{4N_0}}$$

Ex. 5.10.6 : A matched filter has time response given by (triangular shape)

$h(t) = [\,1000 \times t\,]$ Volts for $0 \le t \le 0.01$ msec

$= 0$ otherwise

1. What should be the shape of input signal for this matched filter ? **(4 Marks)**
2. What is the sampling instance for decision purpose ? **(4 Marks)**
3. What is the maximum possible data rate at the input ? **(5 Marks)**
4. If $N_0 = 0.625 \times 10^{-10}$ W/Hz, estimate the probability of error.

(Dec. 05, 5 Marks)

Soln. :

Given : $h(t) = 1000\,t$ .... $0 \le t \le 10\,\mu S$.

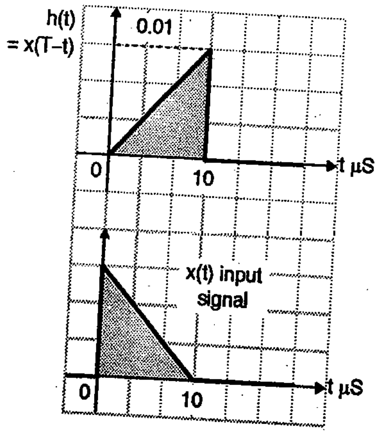
$= 0$ ...otherwise.

1. Shape of the input signal :

- The impulse response of a matched filter is a time reversed and delayed version of the input signal x(t).

$$h(t) = \frac{2K}{N_0}\, x\,(T - t)$$

- Let $2K / N_0 = 1$. The impulse response and input are as shown in Fig. P. 5.10.6(a).



(E-888) **Fig. P. 5.10.6(a) : Input signal**

2. Sampling instant :

Obtain the output of the matched filter as

$$y(t) = x(t) * h(t)$$

- The output is a convolution of x(t) and h(t) as shown in Fig. P. 5.10.6(b). The sampling instant corresponds to the instant at which maximum output is obtained as shown in Fig. P. 5.10.6(b).

So the sampling instant is $t = 10\,\mu S$.

3. Maximum data rate :

$$\text{The maximum data rate} = \frac{1}{20\,\mu S} = \frac{10^6}{20}$$

$$= 5 \times 10^4 \text{ bits / sec}$$

$$= 50 \text{ kbps.}$$

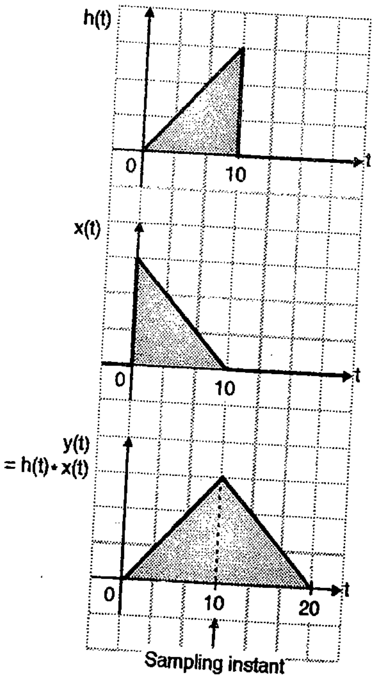
4. Error probability :

$$P_e = \frac{1}{2}[\sqrt{E / N_0}\,] = \frac{1}{2} \text{erfc} \sqrt{2} \left[\frac{\sqrt{E / N_0}}{\sqrt{2}}\right]$$

$$= \frac{1}{2} Q\,[\sqrt{2} \times \sqrt{E} / N_0\,]$$

But $$E = P_s T_b = \frac{A^2}{2} \times T_b$$

$$= \frac{(0.01)^2}{2} \times \frac{1}{50 \times 10^3} = 1 \times 10^{-9}$$



(E-889) **Fig. P. 5.10.6(b)**

$$\therefore \quad P_e = \frac{1}{2} Q\left[\sqrt{2} \times \sqrt{\frac{1 \times 10^{-9}}{0.625 \times 10^{-10}}}\right] = \frac{1}{2} Q\,[\sqrt{2} \times 4\,]$$

$$\therefore \quad P_e = \frac{1}{2} Q\,[\,5.65\,] = \frac{1}{2} \times 1 \times 10^{-7} = 5 \times 10^{-8}$$

Ex. 5.10.7 : Find the impulse response of a matched filter whose input is given by,

$$g(t) = A \sin\left(\frac{2\pi t}{T}\right) \quad 0 \le t \le T$$

$= 0$ Otherwise **May 06, 4 Marks**

Soln. :

Given : Input signal $g(t) = A \sin\left(\frac{2\pi}{T} t\right)$ ...$0 \le t \le T$

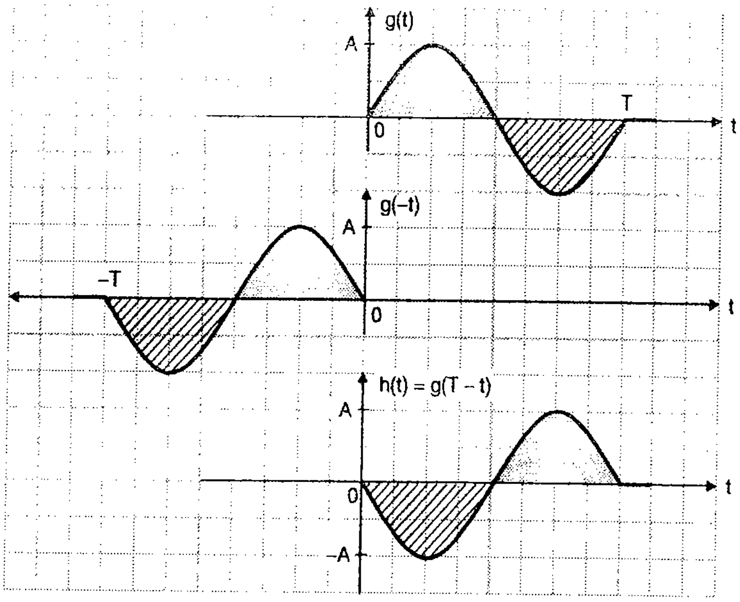
$= 0$ ...otherwise

To find : h(t) of matched filter :

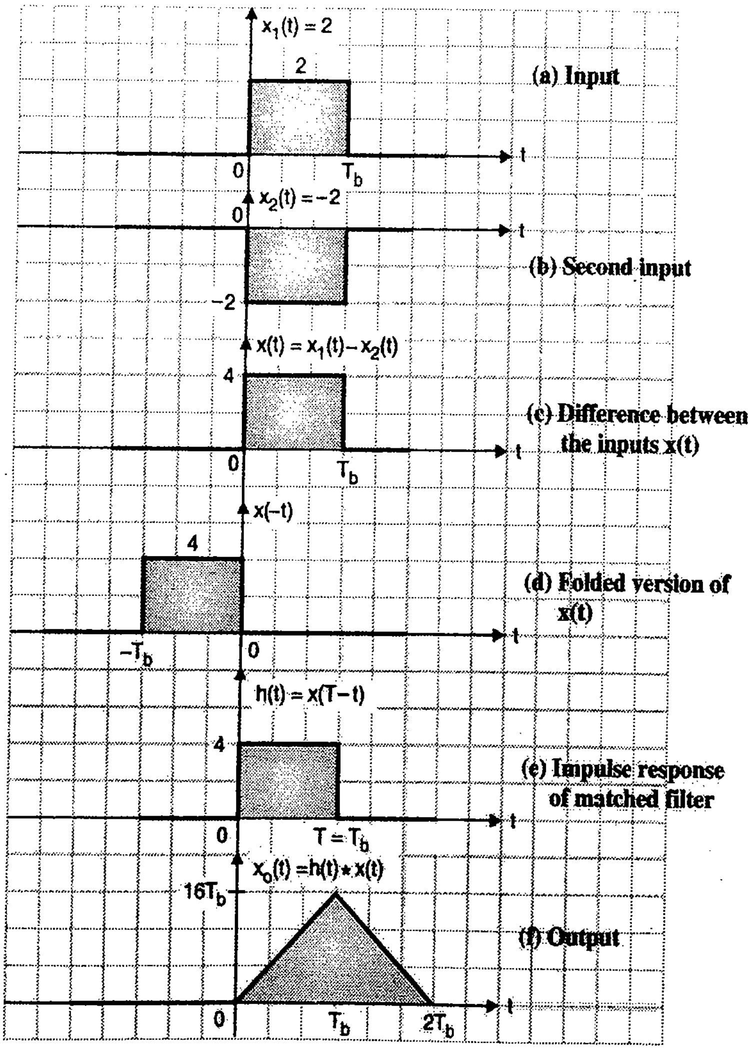
The impulse response of a matched filter is the time reversed and delayed version of the input signal. It is as shown in Fig. P. 5.10.7.

The impulse response is given by the delay version of folded input signal

$$\therefore \quad h(t) = g\,(T - t)$$



(E-890) Fig. P. 5.10.7



(E-463) Fig. P. 5.10.8

Ex. 5.10.8 : A binary data transmission scheme transmits $s_1(t) = +2V$ or $s_2(t) = -2V$, both held for a time T_b. The signal is corrupted by noise of white Gaussian nature with power spectral density 10^{-6} Volt^2 / Hz. If the probability of error is to be no larger than 10^{-4}. Find the minimum allowable interval T_b. Draw the signal waveform at the output of the matched filter and derive the expression for the transfer function of the matched filter.

May 07, 8 Marks

Soln. :

Given that $s_1(t) = -s_2(t) = 2V$.

Noise is white Gaussian with psd $\frac{N_0}{2} = 10^{-6}$,

$\therefore N_0 = 2 \times 10^{-6}$

$$P_{e(max)} = 10^{-4}$$

The probability of error of a matched filter is given by Equation (5.10.16) as

$$P_e = \frac{1}{2} \text{erfc} [E / N_0]^{1/2} \quad ...(1)$$

$$\text{But} \quad E = P_s T_b \therefore \; P_e = \frac{1}{2} \text{erfc} \left[\frac{P_s T_b}{N_0} \right]^{1/2} \quad ...(2)$$

$$\text{But} \quad P_s = \frac{A^2}{2} = \frac{2^2}{2} = 2.$$

Substituting all the values in Equation (2) we get,

$$10^{-4} \leq \frac{1}{2} \text{erfc} \left[\frac{2\, T_b}{2 \times 10^{-6}} \right]^{1/2}$$

$\therefore \quad \text{erfc} [T_b \times 10^6]^{1/2} \geq 2 \times 10^{-4}$

Using the equality sign we get,

$\text{erfc} [T_{b(min)} \times 10^6]^{1/2} = 2 \times 10^{-4}$

Using the table for erfc we get,

$\therefore \quad [T_{b(min)} \times 10^6]^{1/2} \approx 2.5 \therefore \; T_{b(min)} \times 10^6 = 6.25$

$\therefore T_{b(min)} = 6.25 \times 10^{-6} = 6.25 \,\mu S$ **...Ans.**

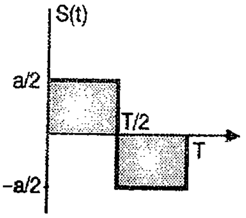
Thus the minimum allowable interval is 6.25 μS.

The waveforms for the input, impulse response and output of a matched filter are as shown in Fig. P. 5.10.8. We have assumed that $\frac{2K}{N_0} = 1$ while drawing the waveform for the impulse response of the matched filter.

The output waveform is obtained by taking the convolution of the input signal with the impulse response.

$$\text{i.e.} \; x_o(t) = x(t) * h(t)$$

Ex. 5.10.9 : Consider the signal S(t) shown in Fig. P. 5.10.9.



(E-1368) **Fig. P. 5.10.9**

Determine the impulse response of a filter matched to this signal and sketch it as a function of time, plot the matched filter output as a function of time.

May 08, May 09, Dec. 13, Dec. 15, 8 Marks

Soln. :

The impulse response of a matched filter is given by,

$$h(t) = \frac{2k}{N_0} x(T-t)$$

- Assuming $\frac{2k}{N_0} = 1$, we get,

$$h(t) = x(T-t)$$

The given signal s (t) can be mathematically expressed as,

$$s(t) = -\frac{a}{2} \quad ...\text{for } 0 \leq t \leq \frac{T}{2}$$

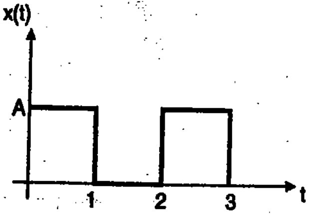
$$= -\frac{a}{2} \quad ...\text{for } \frac{T}{2} \leq t \leq T$$

The output y (t) of the matched filter and it is obtained by convolution of input s (t) and the impulse response h (t).

$$\therefore \quad y(t) = s(t) * h(t)$$

The input, impulse response and output are shown in Fig. P. 5.10.9(a), (c) and (d) respectively.

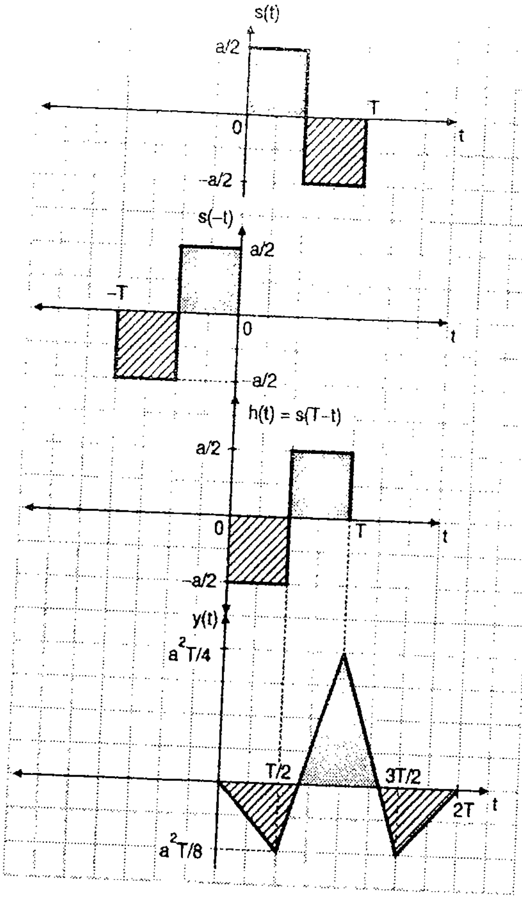
Ex. 5.10.10 : Find the impulse response and output of matched filter for the given signal shown in Fig. P. 5.10.10(a). **May 11, 7 Marks**



(E-1346) **Fig. P. 5.10.10(a)**

Soln. :

The required waveforms are as shown in Fig. P. 5.10.10.



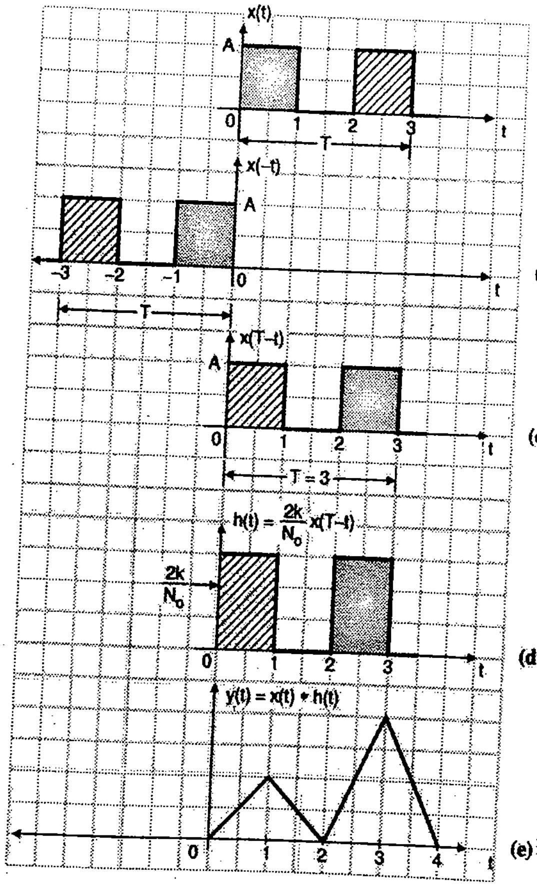
(a) Input signal

(b) Time folded version of input signal

(c) Impulse response of matched filter

(d) Output y(t)

(E-1374) **Fig. P. 5.10.9**



(a) Input signal

(b) Time folded version of input signal

(c) Folded signal delayed by T = 3

(d) Inpulse response of matched filter

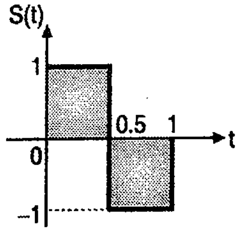
(e) Output of matched filter

(E-1365) **Fig. P. 5.10.10**

Ex. 5.10.11 : Consider the signal s (t) shown in Fig. P. 5.10.11 :

1. Determine the impulse response of a filter matched to this signal and sketch it as a function of time
2. Plot the matched filter output as a function of time
3. What is the peak value of the output

Dec. 12. 8 Marks



(E-1347) **Fig. P. 5.10.11**

Soln. :

- The impulse response of a matched filter is given by

$$h(t) = \frac{2k}{N_0} x(T-t)$$

- Assuming $\frac{2k}{N_0} = 1$ we get

$$h(t) = x(T-t)$$

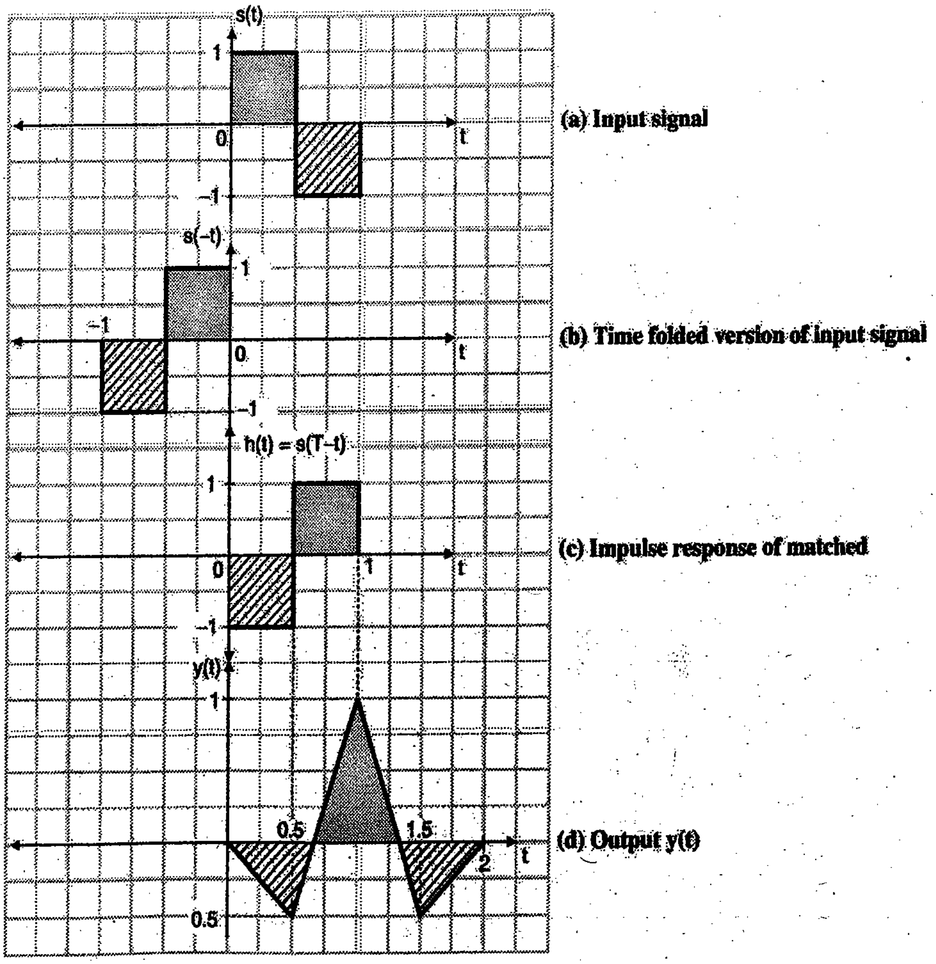
- The given signal s(t) can be mathematically expressed as

$$s(t) = -1 \ldots \text{for } 0 \le t \le 0.5$$
$$= -1 \ldots \text{for } 1 \le t \le 1$$

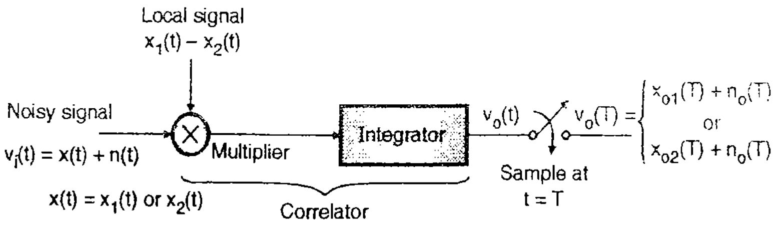
- The output y(t) of the matched filter and it is obtained by convolution of input s(t) and the impulse response h(t).

$$\therefore \; y(t) = s(t) * h(t)$$

- The input, impulse response and output are shown in Fig. P. 5.10.11(a), (c) and (d) respectively.



(E-1366) **Fig. P. 5.10.11**



(E-459) Fig. 5.11.1 : Block diagram of a correlator

5.11 Coherent Reception (Correlation) :

- In this section we are going to discuss an alternative type of receiving system which has an identical performance to that of the matched filter receiver.
- The block diagram of a correlator or a coherent receiver is as shown in Fig. 5.11.1.
- As shown in Fig. 5.11.1, the input signal is a binary data waveform $x_1(t)$ or $x_2(t)$ corrupted by noise $n(t)$. The bit duration of each input signal is T.
- The received signal $v_i(t)$ is a noisy signal i.e. $x(t) + n(t)$. This signal is multiplied by the locally generated signal $x_1(t) - x_2(t)$.
- The output of the multiplier $f(t)$ is then passed through an integrator whose output is sampled at instant $t = T$. The sampling switch is closed at the end of each bit interval. At the beginning of each bit interval, the integrator capacitor is discharged as discussed for the integrator and dump receiver.
- This type of receiver is called as a "correlator" because here we are correlating the received signal and noise, with the locally generated signal $[x_1(t) - x_2(t)]$.
- The output signal and noise of the correlator shown in Fig. 5.11.1, are :

$$\text{Output signal, } v_o(T) = \frac{1}{RC}\int_0^T x(t)\cdot[x_1(t) - x_2(t)]\,dt \qquad ...(5.11.1)$$

Where $x(t) = x_1(t)$ or $x_2(t)$.

$$\text{Output noise, } n_o(T) = \frac{1}{RC}\int_0^T n(t)\cdot[x_1(t) - x_2(t)]\,dt \qquad ...(5.11.2)$$

Where RC is the time constant of the integrator.

- Substitute $[x_1(t) - x_2(t)] = z(t)$ into

Equation (5.11.1) to get,

$$\text{output signal } v_o(T) = \frac{1}{RC}\int_0^T x(t)\,z(t)\,dt \qquad ...(5.11.3)$$

5.11.1 Correlator Realization of a Matched Filter :

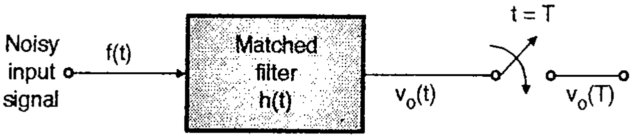
SPPU : May 06, Dec. 07, May 08, Dec. 08

University Questions

Q. 1 Show that performances of a correlator and matched filter are identical with the help of suitable output expressions.

(May 06, Dec. 07, May 08, Dec. 08, 8 Marks)

- Consider the block diagram of the matched filter shown in Fig. 5.11.2.



(E-460) Fig. 5.11.2 : Matched filter

- The output voltage of the matched filter is given by,

$$v_o(t) = f(t) * h(t) \qquad ...(5.11.4)$$

- But the impulse response of the matched filter is given by,

$$h(t) = \frac{2K}{N_o}[x(T - t)] \qquad ...(5.11.5)$$

- The convolution of $f(t)$ and $h(t)$ given in Equation (5.11.4) can be written as,

$$v_o(t) = \int_{-\infty}^{\infty} f(\tau)\,h(t-\tau)\,d\tau \qquad ...(5.11.6)$$

$$\text{Here } h(t-\tau) = \frac{2K}{N_o}[x(T - t + \tau)]$$

- Substituting this value into Equation (5.11.6) we get

$$v_o(t) = \int_{-\infty}^{\infty} f(\tau)\cdot\frac{2K}{N_o}[x(T - t + \tau)\cdot d\tau] \qquad ...(5.11.7)$$

- But the integration is to be performed over only one bit duration

$$\therefore \quad v_o(t) = \int_0^T f(\tau) \cdot \frac{2K}{N_o} [x(T - t + \tau) d\tau] \quad ...(5.11.8)$$

- At the end of each bit duration the output of matched filter is given by,

$$v_o(T) = \int_0^T f(\tau) \cdot \frac{2K}{N_o} [x(T - T + \tau) d\tau] \quad ...(5.11.9)$$

$$= \int_0^T f(\tau) \cdot \frac{2K}{N_o} x(\tau) d\tau$$

$$\therefore \quad v_o(T) = \frac{2K}{N_o} \int_0^T f(\tau) \cdot x(\tau) d\tau \quad ...(5.11.10)$$

- Substituting $t = \tau$ in the above expression we get,

$$v_o(T) = \frac{2K}{N_o} \int_0^T f(t) \cdot x(t) d\tau \quad ...(5.11.11)$$

Compare this expression with Equation (5.11.3). You will find that except for the multiplying factor $\frac{2K}{N_o}$ these equations are identical.

Conclusion :

Thus we have proved that the output of the correlator and that of a matched filter are identical.

- The expressions for $x_o(T)$ and $n_o(T)$ obtained at the output of the correlator are exactly identical to those obtained for the matched filter. Hence the performance of the correlator is exactly identical to that of a matched filter receiver.
- Therefore we conclude that the matched filter and correlator are not simply two distinct independent techniques which produce the same results. In fact they are two techniques of synthesizing the optimum filter.

5.11.2 Solved Examples :

Ex. 5.11.1 : In a binary transmission, one of the message is represented by a rectangular pulse x (t). The other message is transmitted by the absence of the pulse. Calculate the signal to noise ratio at t = T. Assuming white noise with psd $\frac{N_o}{2}$. Also sketch the impulse response of the matched filter and output of the matched filter.

Soln. :

Let the signal x (t) be represented as follows :

$x(t) = x_1(t)$ or $x_2(t)$

$x_1(t) = A \quad 0 \le t \le T$ and $x_2(t) = 0 \quad 0 \le t \le T$

For the solution to this problem, **refer Ex. 5.10.1.**

Ex. 5.11.2 : Show that the impulse response of a matched filter could be obtained as $h_{opt}(t) = Kg(T - t)$. Here g (t) is the applied signal, K and T are arbitrary constants.

Soln. :

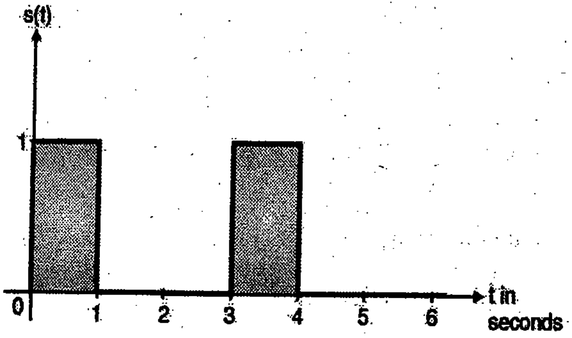
Refer section 5.10.1, in which we have proved that,

$h(t) = \frac{2K}{N_o} [x(T - t)]$ ...Equation (5.10.6).

here x (t) is the input signal. Since $\frac{2K}{N_o}$ is constant we can write that,

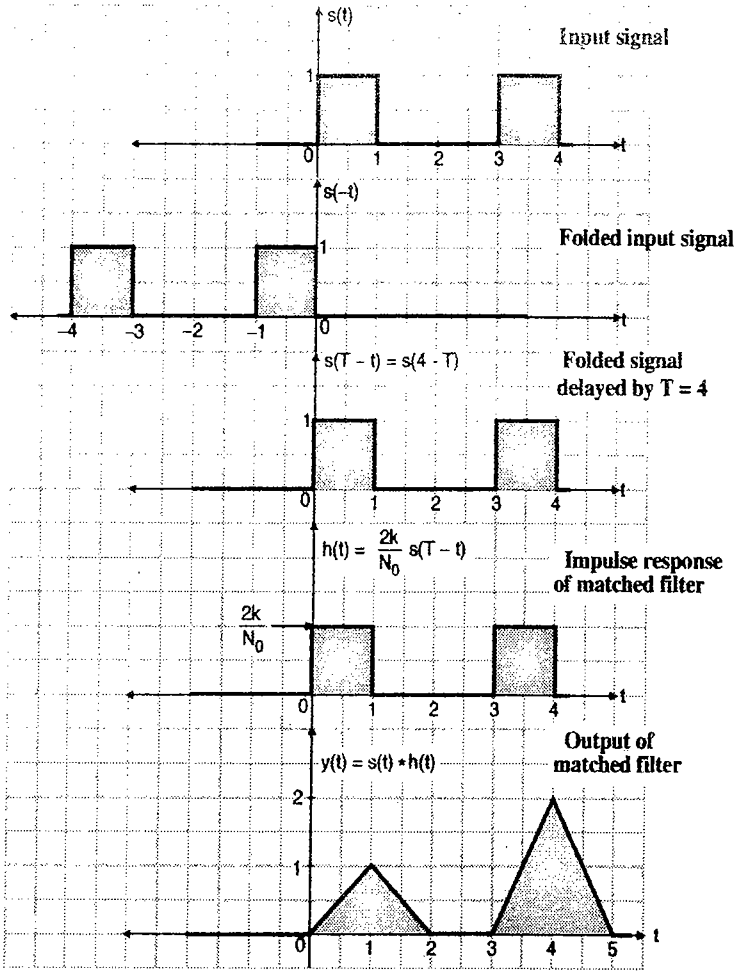
$$h(t) = kx(T - t)$$

Ex. 5.11.3 : Specify a matched filter receiver for the signal shown in Fig. P. 5.11.3 and sketch the filter output as a function of time.



(E-461) Fig. P. 5.11.3

Soln. :



(E-462) **Fig. P. 5.11.3(a) : Waveforms**

The period of given signal is T = 4. The impulse response of a matched filter is given by,

$$h(t) = \frac{2K}{N_o} x(T - t)$$

But here the input signal is x (t). Hence

$$h(t) = \frac{2K}{N_o} s(T - t)$$

Output of matched filter is, y (t) = h (t) * s (t)

It is shown in Fig. P. 5.11.3(a).

Ex. 5.11.4 : A bipolar signal S_i (t) is a +1 V or – 1 V pulse during the interval (0, T). Additive white Gaussian noise of $\eta/2 = 10^{-5}$ W/Hz is added to the signal. Determine the maximum bit rate that can be sent with $P_e \le 10^{-4}$. Take Q [3.71] = 10^{-4}.

May 06. 6 Marks. May 16. 8 Marks

Soln. :

Given : This is a binary PCM system.

PSD of white Gaussian noise $\frac{\eta}{2} = \frac{N_o}{2} = 10^{-5}$ W/Hz

Pulse amplitude A = ± 1 V, Error probability $P_e \le 10^{-4}$

$$Q[3.71] = 10^{-4}$$

Step 1 : Write expression for P_e in terms of Q function :

$$P_e = \frac{1}{2} \text{erfc} \sqrt{\frac{A^2 T}{4 N_o}}$$

And the relation between erfc and Q functions is as follows :

$$Q(x) = \frac{1}{2} \text{erfc} (x / \sqrt{2})$$

So rearrange the expression for P_e as follows :

$$P_e = \frac{1}{2} \text{erfc} \left[\frac{\sqrt{A^2 T / 2 N_o}}{\sqrt{2}} \right]$$

$$P_e = Q \left[\sqrt{\frac{A^2 T}{2 N_o}} \right] \qquad ...(1)$$

Step 2 : Calculate T :

$$P_e = 10^{-4} \qquad ...\text{Given}$$

$$\therefore \quad 10^{-4} = Q[3.71]$$

$$\therefore \quad \sqrt{\frac{A^2 T}{2 N_o}} = 3.71$$

$$\therefore \frac{A^2 T}{2 N_o} = (3.71)^2 = 13.7641$$

$$\therefore \quad T = \frac{13.7641 \times 2 N_o}{A^2}$$

$$\text{But} \quad N_o = 2 \times 10^{-5} \text{ and } A^2 = 1$$

$$\therefore \quad T = \frac{13.7641 \times 2 \times 2 \times 10^{-5}}{1} = 5.2 \times 10^{-4}$$

Step 3 : Calculate the bit rate :

$$\text{Bit rate} = \frac{1}{\text{Bit duration}} = \frac{1}{T} = \frac{1}{5.5 \times 10^{-4}}$$

$$= 1.8163 \text{ kbits/sec}$$ ...**Ans.**

Ex. 5.11.5 : A binary data transmission scheme transmits $s_1(t) = +2V$ or $s_2(t) = -2V$, both held for a time T_b. The signal is corrupted by noise of white Gaussian nature with power spectral density 10^{-6} Volt^2 / Hz. If the probability of error is to be no larger than 10^{-4}. Find the minimum allowable interval T_b. Draw the signal waveform at the output of the matched filter and derive the expression for the transfer function of the matched filter.

Soln. :

Given : $s_1(t) = -s_2(t) = 2V.$

Noise is white Gaussian with psd $\frac{N_o}{2} = 10^{-6}$,

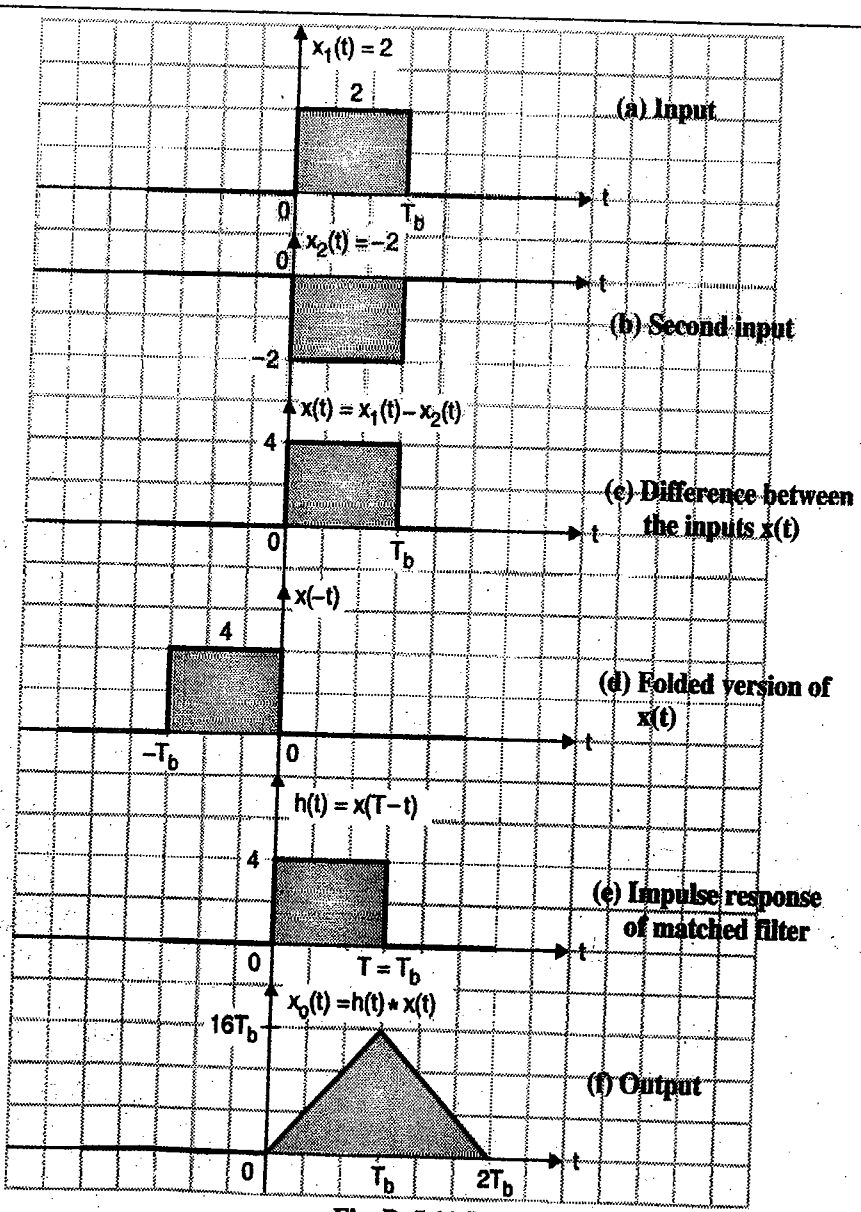
$$\therefore N_o = 2 \times 10^{-6}$$

$$P_{e(\max)} = 10^{-4}$$

The probability of error of a matched filter is given by Equation (5.10.16) as

$$P_e = \frac{1}{2} \operatorname{erfc} [E / N_o]^{1/2} \quad ...(1)$$

$$\text{But} \quad E = P_s T_b \therefore \quad P_e = \frac{1}{2} \operatorname{erfc} \left[\frac{P_s T_b}{N_o}\right]^{1/2} \quad ...(2)$$



(E-463) **Fig. P. 5.11.5**

But $P_s = \frac{A^2}{2} = \frac{2^2}{2} = 2.$

Substituting all the values in Equation (2) we get,

$$10^{-4} \le \frac{1}{2} \text{erfc} \left[\frac{2\, T_b}{2 \times 10^{-6}} \right]^{1/2}$$

$$\therefore \quad \text{erfc}\, [T_b \times 10^6\,]^{1/2} \ge 2 \times 10^{-4}$$

Using the equality sign we get,

$$\text{erfc}\, [T_{b\,(\min)} \times 10^6\,]^{1/2} = 2 \times 10^{-4}$$

Using the table for erfc we get,

$$\therefore \quad [T_{b\,(\min)} \times 10^6\,]^{1/2} \approx 2.5$$

$$\therefore \quad T_{b\,(\min)} \times 10^6 = 6.25$$

$$\therefore T_{b\,(\min)} = 6.25 \times 10^{-6} = 6.25\ \mu S \qquad \text{...Ans.}$$

Thus the minimum allowable interval is 6.25 μS.

The waveforms for the input, impulse response and output of a matched filter are as shown in Fig. P. 5.11.5. We have assumed that $\frac{2K}{N_o} = 1$ while drawing the waveform for the impulse response of the matched filter.

The output waveform is obtained by taking the convolution of the input signal with the impulse response.

i.e. $x_o(t) = x(t) * h(t)$

5.12 Maximum-Likelihood Receiver :

SPPU : Dec. 16

University Questions

Q. 1 Explain the principle of Maximum likelihood receiver with the help of various methods of detection of signal. **(Dec. 16, 8 Marks)**

- For maximum likelihood receiver, we assume that all the messages (transmitted signals) have equal probability.

$$\therefore p(m_1) = p(m_2) = ... = p(m_M) = 1/M$$

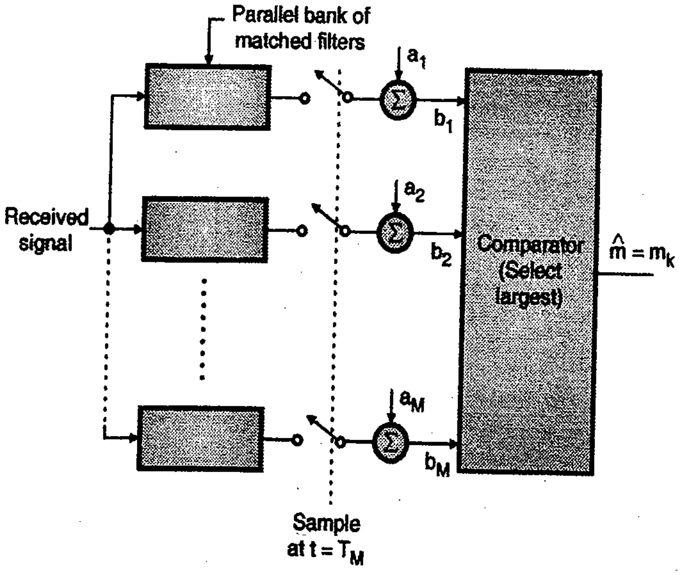
- The maximum likelihood receiver operates on the Baye's rule of probability.
- The decision at receiver will be based on the following statement. Set $\hat{m} = m_k$ if

$$p(q/m_k) > p(q/m_i) \qquad \text{for all } i \ne k$$

where $p(q/m_k)$ = Probability of observing q when m_k is transmitted and

$p(q/m_i)$ = Probability of observing q when m_i is transmitted.

- Thus this receiver will choose that signal which when transmitted will maximize the likelihood (probability) of observing the received signal "q". Therefore this receiver is called as maximum likelihood receiver.
- The practical implementation of the maximum likelihood receiver is as shown in Fig. 5.12.1.
- The incoming signal is applied to a parallel bank of matched filters.
- The output of each filter is sampled at instant $t = T_M$.



(E-880) **Fig. 5.12.1 : Realization of maximum likelihood receiver**

- Then a constant (a_1, a_2 .... a_M) is added to the filter output sample and the resultants are applied to a comparator.
- The decision is made in favour of the signal for which the comparator output is largest.

5.13 Correlator Receiver :

- For an AWGN channel and when the transmitted signals s_1 (t), s_2 (t) .....S_M (t) are equally likely, the optimum receiver consists of two subsystems which are shown in Figs. 5.13.1(a) and (b).

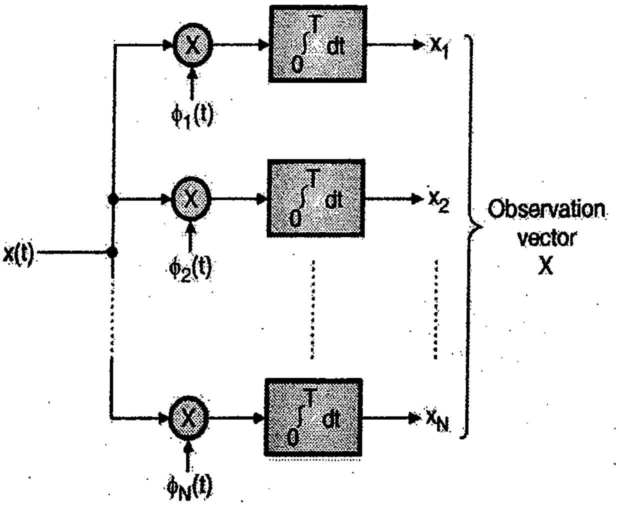
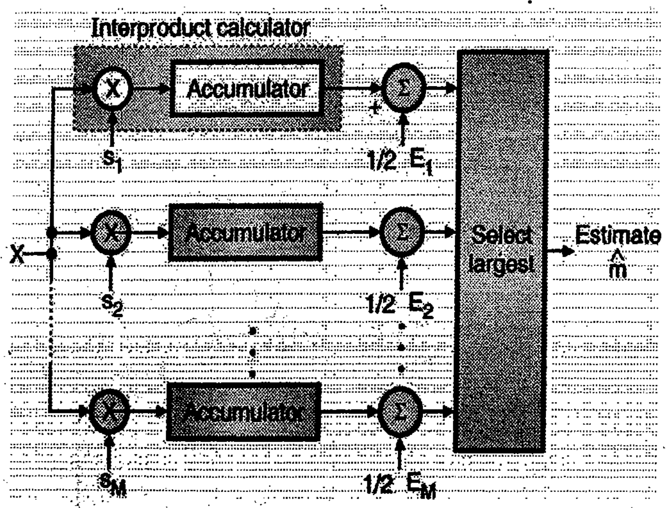


Fig. 5.13.1(a) : Detector or demodulator

- Fig. 5.13.1(a) shows the detector or demodulator part of the optimum receiver. It is made of N number of correlators.
- The required N orthogonal basis functions ϕ_1, ϕ_2 ....ϕ_N (t) are generated locally. This correlator bank operates on the input signal x(t) to produce the observation vector X.
- Fig. 5.13.1(b) shows the other part of this receiver called signal transmission decoder. It is implemented in the form of maximum likelihood detector discussed in the previous section.
- The input to signal transmission decoder is the observation vector X and it produces an estimate $\hat{m}$ of the transmitted symbol m_i with i = 1, 2 ....M in such a way that would minimize the average probability of symbol error.
- The optimum receiver of Fig. 5.13.1 is often called as a **correlation receiver.**



(b) Signal transmission decoder

(E-1393) **Fig. 5.13.1 : Two subsystems of an optimum receiver**

Review Questions

Q. 1 Define white Gaussian noise.

Q. 2 Derive the expression for auto correlation function of a white Gaussian noise.

Q. 3 What is geometric representation of signals.

Q. 4 Define and explain signal space.

Q. 5 Explain the principle of maximum likelihood receiver.

Q. 6 With the help of block schematic explain the principle of optimum filter.

Q. 7 What is matched filter ?

Q. 8 Explain correlation.

Q. 9 Prove that the matched filter and correlator yield the same results

Q. 10 What is matched filter receiver.

Q. 11 Explain the maximum likelihood receiver.

Q. 12 State the requirements of a detection technique.

Q. 13 Explain the correlator receiver.

Q. 14 Explain the use of integrate and dump filter for the reception of a digital signal.

Q. 15 Derive the expression for probability of error for integrate and dump receiver.

Q. 16 Derive the expression for SNR of integrate and dump filter.

Q. 17 Derive the expression for error probability of an optimum receiver.

Q. 18 Derive the expression for the optimum filter transfer function.

Q. 19 Derive the expression for impulse response of matched filter.

Q. 20 Derive the expression for probability of error of a matched filter.

Q. 21 State and explain the properties of a matched filter.

5.14 University Questions and Answers :

Q. 1 Derive the expression of SNR for optimum filter.

(Dec. 2015, 8 Marks)

Ans. :

$\text{SNR}_0 = \sigma^2_{\max}$. Refer Section 5.9.2.

❑❑❑

CHAPTER 6

Passband Digital Transmission

Unit V

Syllabus :

Passband transmission model, Signal space diagram, Generation and detection, Error probability derivation and power spectra of coherent BPSK, BFSK and QPSK, Geometric representation, Generation and detection of M-ary PSK, M-ary QAM and their error probability, Non-coherent BFSK, DPSK.

6.1 Introduction :

- We have discussed the baseband pulse transmission earlier. In baseband pulse transmission, the input data is represented in the form of a discrete PAM signal (line codes). These signals are transmitted over the communication channel.
- However in the digital passband transmission which is discussed in this chapter, the digital input data is used to modulate a sinusoidal carrier. These signals are transmitted over a bandpass channel.
- The examples of bandpass channels are microwave radio link or a satellite channel.
- There are three basic signaling schemes used in passband data transmission :
  1. Amplitude shift keying (ASK)
  2. Phase shift keying (PSK)
  3. Frequency shift keying (FSK)
- These are similar to amplitude modulation (AM), phase modulation (PM) and frequency modulation (FM) respectively.

6.1.1 Hierarchy of Digital Modulation Techniques :

SPPU : Dec. 11

University Questions

Q. 1 What is coherent detection ? Draw the block diagram of BPSK receiver and explain its operation with proper mathematical expressions. (Dec. 11, 8 Marks)

The digital modulation techniques are classified into two categories as :

1. Coherent techniques 2. Noncoherent techniques.

1. **Coherent techniques :**
   - In the coherent digital modulation techniques, we have to use a phase synchronized locally generated carrier at the receiver to recover the information signal.
   - The frequency and phase of this carrier produced at the receiver should be perfectly synchronized with that at the transmitter.
   - Coherent techniques are complex but guarantee better performance.
2. **Noncoherent techniques :**
   - In the noncoherent techniques, no phase synchronized local carrier is needed at the receiver.
   - These techniques are less complex.
   - But the performance is not as good as that of coherent techniques.

6.1.2 Binary and M-ary Schemes :

- There are two types of digital modulation schemes :

  1. Binary schemes 2. M-ary schemes.
- In the binary schemes there only two possible messages (0011). We send any onc of thc two possible signals during each signaling interval of duration T_b. Examples are ASK, FSK and PSK.
- Whereas in M-ary schemes there are M-different possible messages. We can send any one of the M possible signals during each signaling interval of duration T_b.
- Examples of M-ary schemes are M-ary PSK, M-ary FSK, QPSK, MSK, QASK or QAM etc.
- M-ary schemes need less bandwidth as compared to the binary schemes.
- But the error performance of M-ary schemes is poor as compared to the binary schemes.

6.1.3 Probability of Error (P_e) :

- The most important goal of passband data transmission systems is to design the receiver having minimum value of average probability of error in presence of additive white Gaussian noise (AWGN).
- The value of error probability P_e of a system indicates its performance in presence of AWGN. The value of P_e should be as small as possible.

6.1.4 Power Spectra :

- The features of every method can be completely understood if and only if we study the power spectra of the modulated signal.
- It is a graph of power spectral density plotted on Y axis versus frequency (on X axis).
- It gives us information about the bandwidth requirement and cochannel interference.

6.1.5 Bandwidth Efficiency :

- The channel bandwidth and transmitted power are the two primary communication resources.
- Every communication systems should be spectrally efficient.
- The **bandwidth efficiency** is defined as the ratio of the data rate (bits/sec) to the effectively utilized channel bandwidth. It is denoted by ρ.

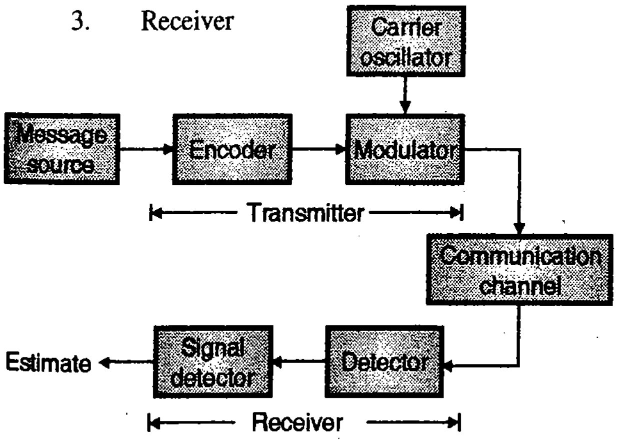
$$\therefore \quad \rho = \frac{R_b}{B} \text{ bits / Hz}$$

- The bandwidth efficiency is dependent on the following factors :

1. Multichannel encoding 2. Spectral shaping.

6.1.6 Passband Transmission Model :

- Fig. 6.1.1 shows the model of passband data transmission system.
- The model of Fig. 6.1.1 consists of three parts :
  1. Transmitter
  2. Communication channel
  3. Receiver



(E-356) **Fig. 6.1.1 : Passband transmission model**

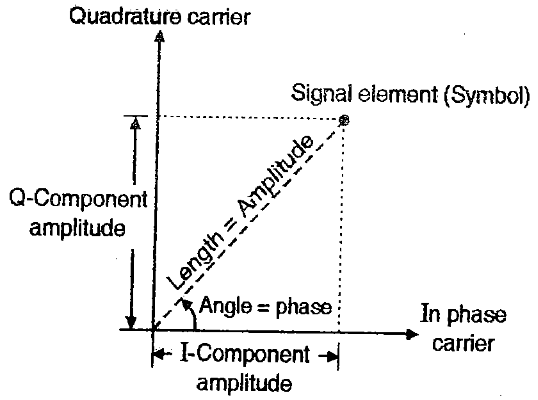
- The message source produces one symbol per T_b seconds. Let these symbols be denoted by $m_1, m_2 \ldots. m_M$. The probabilities of these symbols are $P(m_1), P(m_2), \ldots\ldots P(m_M)$.
- If all the M symbols from the message source are equally likely, then

$$P(m_1) = P(m_2) = \ldots.. P(m_M) = 1/M$$

- The message source output is applied to the **encoder** which produces a corresponding vector S_i made up of N real elements. Thus the encoder converts the message signal into an N element codeword.
- The modulator produces a distinct signal $S_i(t)$ of duration T_b seconds to represent the symbol m_i produced by the message source.
- The output of modulator $S_i(t)$ is transmitted over the communication channel.
- The modulation can be ASM, FSK, PSK etc.
- The communication channel is a bandpass channel and has the following two characteristics :
  1. It is a linear channel, and it has a sufficient bandwidth.
  2. The channel noise ω (t) is the sample function of a White Gaussian noise which has a zero mean and a spectral density equal to $N_0 / 2$.
- The receiver consists of a detector and a signal detector, as shown in Fig. 6.1.1.
- The receiver is designed to perform the following functions :
  1. To reverse all the operations carried out by the transmitter.
  2. To minimize the effect of channel noise on the estimate $(\hat{m})$ of the transmitted symbol m, produced at the output of the receiver.
- We recover the estimate $(\hat{m})$ of the transmitted signal m_i, at the output of the receiver.

6.1.7 Constellation Diagram or Signal Space Diagram :

- A constellation diagram is a diagram which can help us to define the amplitude and phase of each symbol or signal element in a given system (ASK, FSK, PSK, ... etc.). This is particularly helpful when we are using two carriers (one in phase and the other in quadrature) or when we are studying the multilevel ASK, PSK or QAM.
- In this diagram each signal element (i.e. symbol) is represented by a dot as shown in Fig. 6.1.2. This diagram has two axes i.e. it is a two dimensional graphical representation of a symbol.
- On the X-axis we plot the in phase carrier while on the Y-axis we plot the quadrature (90° phase shifted) carrier. The concept of constellation diagram has been illustrated in Fig. 6.1.2.



(L-786) **Fig. 6.1.2 : Concept of constellation diagram**

- Each point on the constellation diagram gives us the following information :
  1. Projection of the symbol point on X-axis gives amplitude of I (in phase) component.
  2. Projection of the symbol point on Y-axis gives amplitude of Q (Quadrature) component.
  3. The line that joins the origin and the signal element is called as signal vector.
  4. The length of the signal vector represents its peak amplitude.
  5. The angle made the signal vector with X-axis represents the phase angle of the signal element.

6.2 Coherent Binary Modulation Techniques :

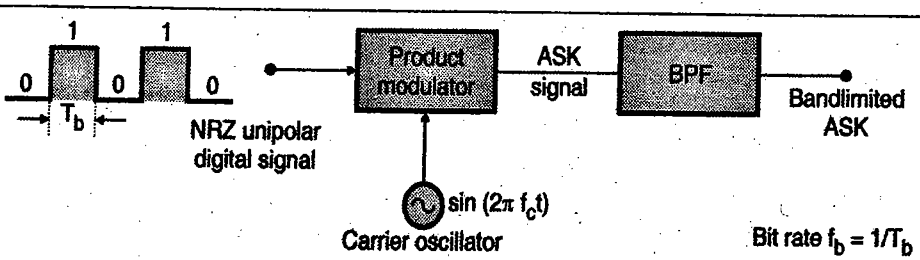
- As we have mentioned earlier, the binary modulation has three basic forms namely :
  1. Amplitude shift keying (ASK)
  2. Phase shift keying (PSK)
  3. Frequency shift keying (FSK).
- In this chapter we are going to analyze these systems for coherent detection assuming an additive white Gaussian noise (AWGN) model.
- For analysis, we use the signal constellations. A **signal constellation** is defined as the set of possible message points.

6.3 Binary Amplitude Shift Keying (BASK) :

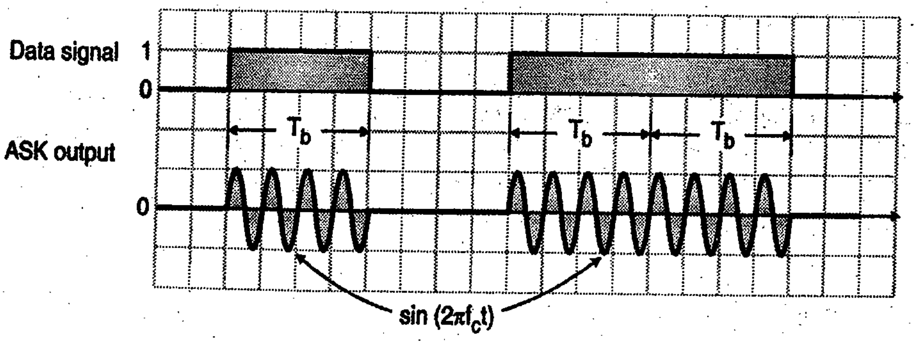
- Amplitude shift keying (ASK) is the simplest type of digital CW modulation. Here the carrier is a sine wave of frequency f_c. We can represent the carrier signal mathematically as follows :

$$e_c = \sin(2\pi f_c t) \qquad \text{...(6.3.1)}$$

- The digital signal coming from a digital source is a unipolar NRZ signal which acts as the modulating signal.
- The ASK modulator is nothing but a multiplier followed by a bandpass filter as shown in Fig. 6.3.1(a).
- This multiplier will multiply the NRZ digital and the sinusoidal carrier signal to produce the binary ASK signal at its output.
- Due to the multiplication, the ASK output will be present only when a binary "1" is to be transmitted.
- The ASK output corresponding to a binary "0" is zero as shown in Fig. 6.3.1(b).



(a) ASK generator



(b) ASK waveforms

(E-361) **Fig. 6.3.1**

- From the waveforms of Fig. 6.3.1(b) we can conclude that, the carrier is transmitted when a binary "1" is to be sent and no carrier is transmitted when a binary "0" is to be sent.
- The ASK signal can be mathematically expressed as follows :

$$V_{ASK}(t) = d(t)\sin(2\pi f_c t) \qquad ...(6.3.2)$$

where d (t) = Data bit which can take values "1" or "0".

$$\left.\begin{aligned} \therefore\ V_{ASK}(t) &= \sin(2\pi f_c t) \quad \text{when } d(t) = 1 \\ \text{and } V_{ASK}(t) &= 0 \quad \text{when } d(t) = 0 \end{aligned}\right\} \qquad ...(6.3.3)$$

- The BASK signal is obtained by multiplying signals d (t) and sin $2\pi f_c t$.
- Therefore its spectrum can be obtained by taking the Fourier transform of the product d (t) × sin ($2\pi f_c$ t).
- Using the modulation theorem in Fourier transform we get,

$$F[d(t)\times\sin(2\pi f_c t)] = \frac{1}{2}X(f+f_c)+\frac{1}{2}X(f-f_c)$$

Where X (f) is the Fourier transform of d (t). Remember that X (f) is a sinc function as d (t) is a rectangular function.

6.3.1 Transmission Bandwidth of the ASK Signal :

- The bandwidth of ASK signal is dependent on the bit rate f_b. Where bit rate $f_b = 1 / T_b$ and T_b is one bit period as shown in Fig. 6.3.1(a).
- The bandwidth required for an ASK signal is,

$$BW = 2 f_b \text{ Hz} \qquad ...(6.3.4)$$

- The frequency spectrum of an ASK signal is shown in Fig. 6.3.2 which shows that the spectrum consists of the carrier frequency f_c with upper and lower sidebands.
- The transmission bandwidth BW of the ASK signal can be restricted by using a filter connected after ASK generator. The restricted value of bandwidth is given as,

$$BW = (1+r)f_b \qquad ...(6.3.5)$$

where "r" is a factor related to the filter characteristics and its value lies between "0" and "1". For r = 1 the maximum bandwidth is obtained equal to 2 f_b.

6.3.2 Merits and Demerits of ASK :

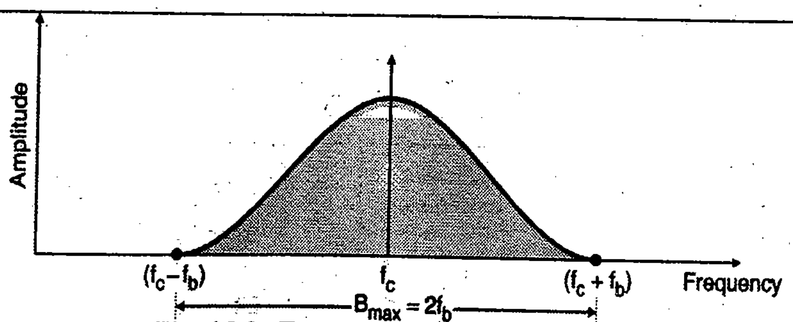
1. The advantage of using ASK is its simplicity. It is easy to generate and detect.
2. However its disadvantage is that it is very sensitive to noise, therefore it finds limited application in data transmission. It is used at very low bit rates, upto 100 bits/sec.

6.3.3 Probability of Error :

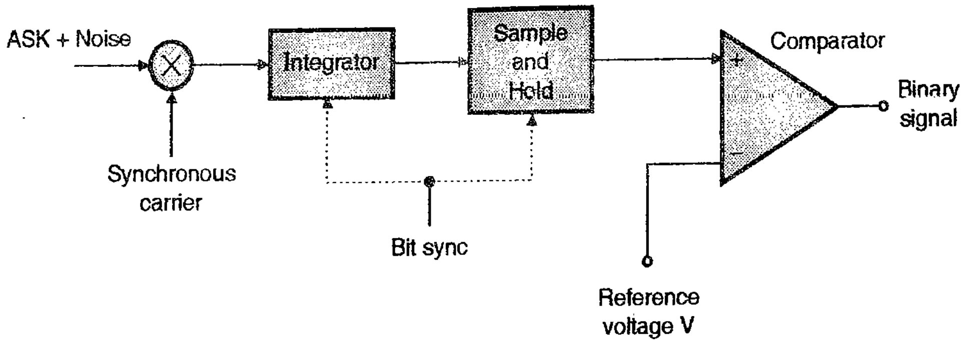
- Probability of error (P_e) is an important parameter which can be used for judging the performance of a digital communication system.
- P_e should be as small as possible for better performance of a system.

6.3.4 Coherent Detection of ASK :

- The coherent receiver for an ASK signal is shown in Fig. 6.3.3 in which a locally produced synchronized carrier is applied to a multiplier.
- The ASK signal alongwith noise is also applied to the multiplier.
- The multiplier output is then applied to an integrator which integrated over one bit duration T_b.
- The integrator output is sampled at a particular instant corresponding to the maximum possible value of output and the sampled value is held by the sample and hold circuit.
- The output of sample and hold circuit is compared with a reference voltage V by a comparator.
- If the S/H output is less than V, then comparator output is low which indicates that the received ASK signal is 0.
- If the S/H output is greater than V, then comparator output is high which indicates that the received ASK signal corresponds to 1.



(E-362) Fig. 6.3.2 : Frequency spectrum of an ASK signal



(E-363) **Fig. 6.3.3 : Coherent ASK receiver**

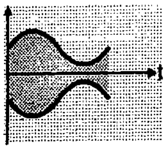
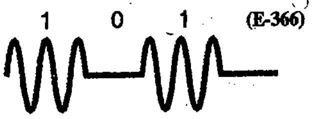
- Thus at the receiver output we recover the original binary signal.

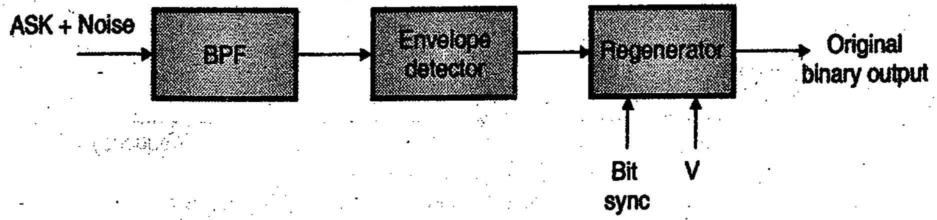
6.3.5 Noncoherent Detection of ASK :

- The noncoherent ASK detector is as shown in Fig. 6.3.4. It consists of a bandpass filter followed by an envelope detector and a regenerator.
- The bandpass filter is a matched filter.
- The envelope detector is basically a rectifier. If the ASK signal is present (corresponding to a logic 1) then the output of envelope detector is high and a binary 1 is obtained at the output.
- If ASK is absent (corresponding to 0) then only the noise is present at the receiver input and envelope detector output is not of a large amplitude.
- Because it is less than the reference voltage V, a zero is produced at the output.

6.3.6 Comparison of AM and ASK :

Table 6.3.1 : Comparison of AM and ASK

Sr. No.	Parameter	AM	ASK
1.	Variable characteristics of the carrier.	Amplitude	Amplitude
2.	Nature of modulating signal.	Modulating signal is analog	Modulating signal is digital
3.	Modulated signal shape.	 (E-365)	 (E-366)
4.	Variation in the carrier amplitude.	Continuous variation in accordance with the amplitude of modulating signal.	Carrier ON or OFF depending on whether a 1 or 0 is to be transmitted.
5.	Number of sidebands produced.	Two	Two
6.	Bandwidth	$2 f_m$	$(1 + r) R$
7.	Noise immunity	Poor	Poor
8.	Application	Radio broadcasting	Data transmission at low bit rate
9.	Detection Method	Envelope	Envelope



(E-364) **Fig. 6.3.4 : Noncoherent ASK receiver**

6.3.7 Error Probability of ASK :

- We know that the ASK (amplitude shift keying) signal is represented as :

Binary 1 : $x_1(t) = \sqrt{2P_s} \cos \omega_c t$

Binary 0 : $x_2(t) = 0$

Where P_s = Average normalized signal power

$$P_s = \frac{A^2}{2}$$

- Now let us obtain the expression for probability of error using the matched filter. We have already derived the expression **for error probability with optimum filter.**

$$P_e = \frac{1}{2} \operatorname{erfc}\left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}\right] \quad ...(6.3.6)$$

- We have also derived the expression for maximum output signal to noise ratio of an optimum filter as,

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{\max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)}\, df \quad ...(6.3.7)$$

- The value of psd for a white noise input is $S_{ni}(f) = N_0 / 2$. Substituting this into Equation (6.3.7) we get the expression for maximum signal to noise ratio of a matched filter as :

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{\max} = \frac{2}{N_0} \int_{-\infty}^{\infty} |X(f)|^2\, df \quad ...(6.3.8)$$

- According to the Rayleigh's energy theorem,

$$\int_{-\infty}^{\infty} |X(f)|^2 df = E = \int_{-\infty}^{\infty} x^2(t)\, dt \quad ...(6.3.9)$$

- Substituting this value into Equation (6.3.8) we get,

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{\max} = \frac{2}{N_0} \int_{-\infty}^{\infty} x^2(t)\, dt \quad ...(6.3.10)$$

- But as the signal x (t) is present only over a bit interval T, the limits of integration will change to 0 to T.

$$\therefore \left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{\max} = \frac{2}{N_0} \int_0^T x^2(t)\, dt \quad ...(6.3.11)$$

where $x(t) = x_1(t) - x_2(t)$

- But in case of ASK, $x_2(t) = 0$. Therefore $x(t) = x_1(t) = \sqrt{2P_s} \cos \omega_c t$

$$\therefore \left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{\max} = \frac{2}{N_0} \int_0^T 2P_s \cos^2 \omega_c t\, dt \quad ...(6.3.12)$$

$$\therefore \left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{\max} = \frac{4P_s}{N_0} \int_0^T \cos^2 \omega_c t\, dt \quad ...(6.3.13)$$

- But $\cos^2 \omega_c t = \frac{1 + \cos 2\omega_c t}{2}$

$$\therefore \left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{\max} = \frac{4P_s}{N_0} \int_0^T \frac{1 + \cos 2\omega_c t}{2} \cdot dt$$

$$= \frac{2P_s}{N_0} \left\{ \int_0^T 1\, dt + \int_0^T \cos 2\omega_c t\, dt \right\}$$

$$= \frac{2P_s}{N_0} \left\{ [t]_0^T + \frac{1}{2\omega_c} [\sin 2\omega_c t]_0^T \right\}$$

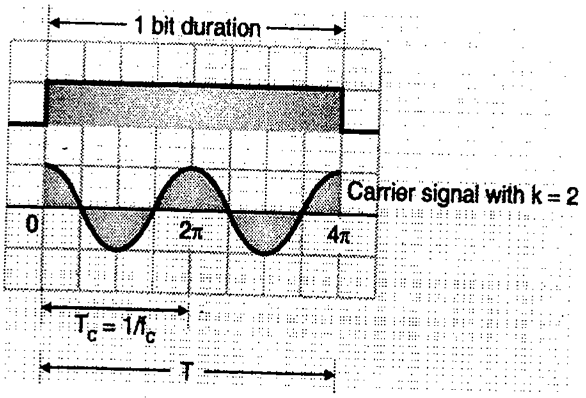
$$\therefore \left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{\max} = \frac{2P_s}{N_0} \left\{ T + \frac{\sin 2\omega_c T}{2\omega_c} \right\} \quad ...(6.3.14)$$

- Consider the second term on the right hand side of Equation (6.3.14) i.e.,

$$\sin 2\omega_c T = \sin (2 \times 2\pi f_c T)$$

$$= \sin 4\pi f_c T \quad ...(6.3.15)$$

- We assume that the frequency of the carrier signal (f_c) is selected such that there are k number of complete cycles of the carrier during one bit duration T i.e. k = 2, as shown in Fig. 6.3.5.



(E-464) **Fig. 6.3.5 : Relation between bit period T and carrier frequency**

- From Fig. 6.3.5 it is clear that,

One bit period $T = 2T_c = \frac{2}{f_c}$ ... as $T_c = \frac{1}{f_c}$

$$\therefore f_c T = 2 \quad ...(6.3.16)$$

- Here k = 2 therefore Equation (6.3.16) can be written in general as,

$$f_c T = k \quad ...(6.3.17)$$

Substituting this value into Equation (6.3.15) we get,

$$\sin 2\,\omega_c T = \sin 4\,\pi\,k \quad \text{... where } k = 1, 2, \qquad \text{...(6.3.18)}$$

$$\therefore \sin 2\,\omega_c\,T = \sin 4\,\pi, \sin 8\,\pi, \sin 12\,\pi \text{ ...etc.}$$

for $k = 1, 2, 3, ...$

$$\therefore \sin 2\,\omega_c\,T = 0 \text{ for all values of k.}$$

- Therefore Equation (6.3.14) gets modified as,

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2\,P_s\,T}{N_0} \qquad \text{...(6.3.19)}$$

Taking square root of both the sides we get,

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]_{max} = \sqrt{\frac{2\,P_s\,T}{N_0}} \qquad \text{...(6.3.20)}$$

- Now substitute Equation (6.3.20) into Equation (6.3.6) to get the error probability for ASK

$$\therefore \quad P_e = \frac{1}{2}\,\text{erfc}\left[\frac{\sqrt{\frac{2\,P_s\,T}{N_0}}}{2\sqrt{2}}\right]$$

$$\therefore \quad P_e = \frac{1}{2}\,\text{erfc}\left[\sqrt{\frac{P_s\,T}{4\,N_0}}\right] \qquad \text{...(6.3.21)}$$

- This is the expression for the bit error probability denoted by P_B,

$$\therefore \quad P_B = \frac{1}{2}\,\text{erfc}\left[\sqrt{\frac{P_s\,T}{4\,N_0}}\right]$$

But $P_s\,T = E$ = Energy of the signal. (Energy per bit)

$$\therefore \quad P_B = \frac{1}{2}\,\text{erfc}\,[\sqrt{E/4\,N_0}\,] \qquad \text{...(6.3.22)}$$

- This is the expression for bit error probability of ASK using the matched filter.
- The complementary error function "erfc(x)" is a monotonically decreasing function of x. That means as x increases, the value of erfc x decreases. Hence the bit error probability P_B decreases with increase in the ratio $E / 4\,N_0$. Thus the error probability depends only on the signal energy and not its shape or any other parameter.

6.4 Binary Phase Shift Keying (BPSK) :

SPPU : Dec. 05

University Questions

Q. 1 Describe the scheme to recover the baseband signal in BPSK with the help of block diagram and signals marked at input/output of these blocks. Also write the functional names inside the blocks. **(Dec. 05, 6 Marks)**

- Binary phase shift keying (BPSK) is the most efficient of the three modulation methods, i.e. ASK, FSK and PSK. Therefore it is used for high bit rate applications.
- In BPSK, phase of the sinusoidal carrier is changed according to the data bit to be transmitted.
- Fig. 6.4.1 shows the simplest form of PSK called Binary PSK (BPSK). The carrier phase is changed between 0° and 180° by the bipolar digital signal.
- A bipolar NRZ signal is used to represent the digital data coming from the digital source.

6.4.1 Mathematical Representation of BPSK Signal :

- Let the peak amplitude of a sinusoidal carrier be "A". Then the normalized power contained in this sinusoidal carrier is given by,

$$P_s = A^2/2 \qquad \text{...(6.4.1)}$$

$$\therefore \quad A = \sqrt{2\,P_s} \qquad \text{...(6.4.2)}$$

- Then the transmitted BPSK signal is given by,

$$V_{BPSK}(t) = \sqrt{2\,P_s}\,\cos\omega_c\,t$$

......when binary "1" is being transmitted

- But the signal will be shifted in phase by 180° to represent a binary 0.

$$\begin{aligned} \therefore \quad V_{BPSK}(t) &= \sqrt{2\,P_s}\,\cos(\omega_c\,t + \pi) \\ &= \sqrt{2\,P_s}\,[\cos\omega_c\,t\cos\pi - \sin\omega_c\,t\sin\pi\,] \\ &= -\sqrt{2\,P_s}\,\cos\omega_c\,t \end{aligned}$$

......when binary "0" is being transmitted.

Combining the above two conditions we can write that,

$$V_{BPSK}(t) = b(t) \times \sqrt{2\,P_s}\,\cos(\omega_c\,t) \qquad \text{...(6.4.3)}$$

where $b(t) = \pm 1$ corresponding to logic 1 or 0 levels respectively.

- The energy and power of a signal are related in the following way,

Energy = Power × Time

∴ Power = Energy / Time

$$\therefore \quad P_s = E_b / T_b \qquad \text{...(6.4.4)}$$

Where E_b is the signal energy and T_b is the bit duration.

- Substituting this value of P_s into Equation (6.4.4) we get the expression for BPSK signal as follows :

$$V_{BPSK}(t) = b(t) \times \sqrt{\frac{2\,E_b}{T_b}}\,\cos(\omega_c\,t) \text{...(6.4.5)}$$

where $b(t) = \pm 1$

6.4.2 Graphical Representation of BPSK Signal :

The graphical representation of a BPSK signal is shown in Fig. 6.4.1(a). Note the reversal of phase taking place corresponding to every changeover of the data from "0" to "1" or from "1" to "0".

6.4.3 BPSK Generation :

SPPU : May 07, Dec. 08, Dec. 10

University Questions

Q. 1 Explain generation and reception of BPSK signal with mathematical expression. **(May 07, 8 Marks)**

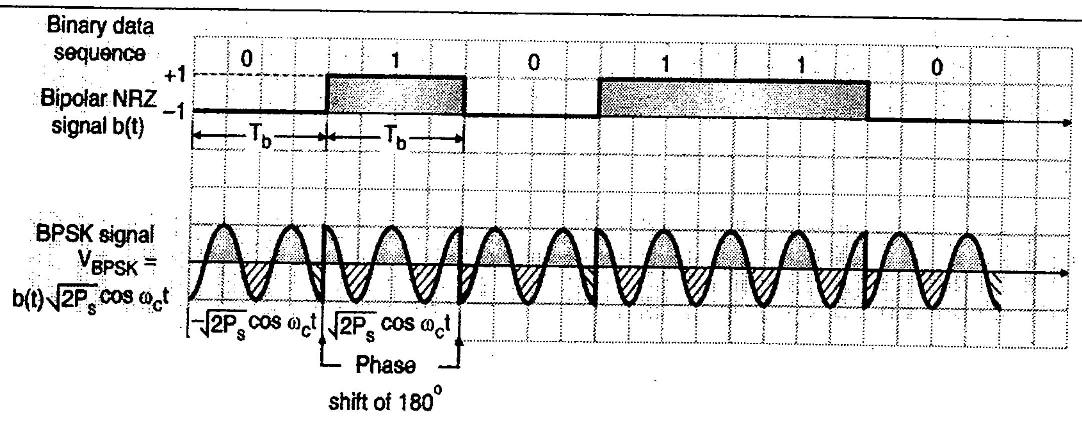
Q. 2 Draw the block diagram and with the help of mathematical expression explain in detail the BPSK transmitter and receiver. Diagram the geometric representation and draw its power spectral density, along with its expression thereby comment on its Euclidian distance and bandwidth. **(Dec. 08, 8 Marks)**

Q. 3 Explain coherent BPSK transmitter and receiver. Derive the expression for receiver output considering effect of noise. Draw the spectrum of BPSK signal and comment on bandwidth required. **(Dec. 10, 8 Marks)**

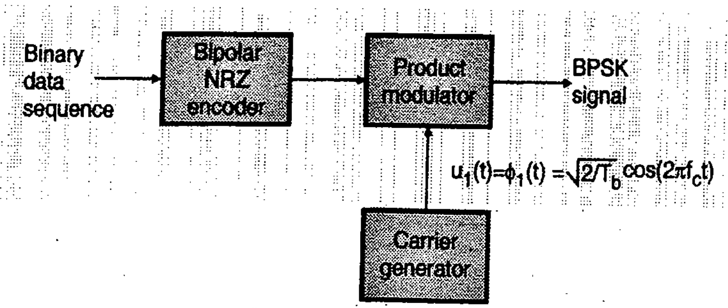
- The BPSK generation takes place as shown in Fig. 6.4.1(b).
- The binary data signal (0s and 1s) is converted into a NRZ bipolar signal by an NRZ encoder, which is then applied to a multiplier (product modulator).
- The other input to the multiplier is the basis function $\phi_1(t) = \sqrt{2/T_b} \cos \omega_c t = u_1(t)$, which is produced by the carrier generator.
- The data bits 0s and 1s are first converted into a bipolar NRZ signal b (t) as shown in Table 6.4.1.

Table 6.4.1

Digital signal	Bipolar NRZ signal b (t)	BPSK output
Binary 0	$b(t) = -1$	$-\sqrt{2P_s} \cos(\omega_c t)$
Binary 1	$b(t) = +1$	$+\sqrt{2P_s} (\cos \omega_c t)$



(a) Binary phase shift keying (BPSK) signal



(b) BPSK generation

(E-367) **Fig. 6.4.1**

Where $P_s = E_b / T_b$ as mentioned earlier and $\omega_c = 2 \pi f_c$.

- Sometimes BPSK is represented in terms of a sine wave as follows :

Digital signal	Bipolar NRZ signal b (t)	BPSK output
0	$b(t) = -1$	$-\sqrt{2 P_s} \sin \omega_c t$
1	$b(t) = +1$	$+\sqrt{2 P_s} \sin \omega_c t$

6.4.4 Balanced Ring Modulator :

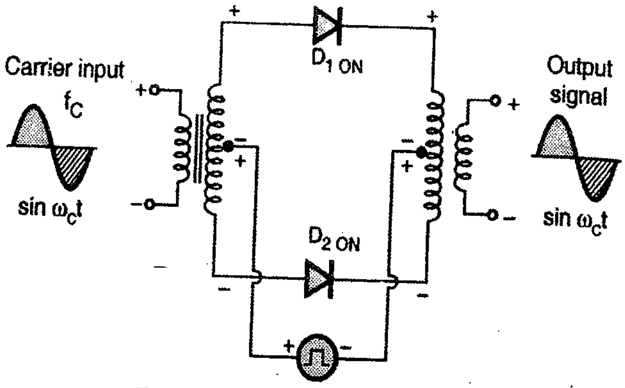
- In the BPSK generator we have a product modulator which is actually a balanced ring modulator with two inputs. One input is the carrier frequency while other input is the digital data.
- Fig. 6.4.2 shows the balanced ring modulator.

Operation of the circuit :

- The operation is explained with the assumptions that the diodes act as perfect switches and that they are switched on and off by the digital data signal.
- The operation can be divided into different modes.

Operation with binary input = Logic 1 :

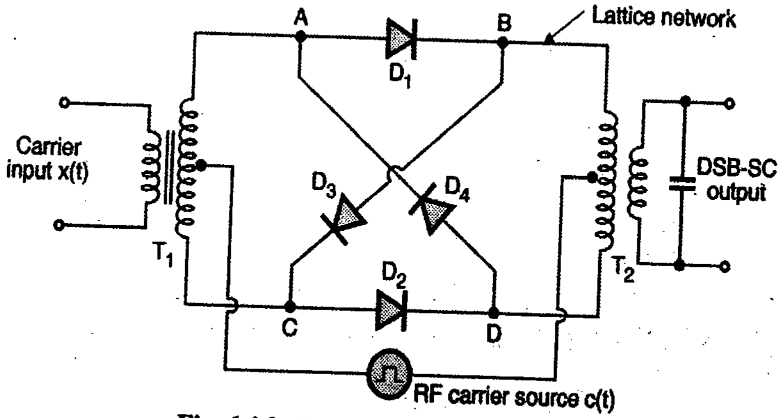
- If the binary input is equal to logic 1 then the equivalent circuit is as shown in Fig. 6.4.3(a). The diodes D_1 and D_2 are on (forward biased) while D_3 and D_4 are reverse biased and therefore in their off state.
- With the polarity shown, the carrier voltage is developed across the transformer T_2 in phase with the carrier voltage across T_1.
- Hence the output signal is in phase with the carrier input signal as shown in Fig. 6.4.3(a).



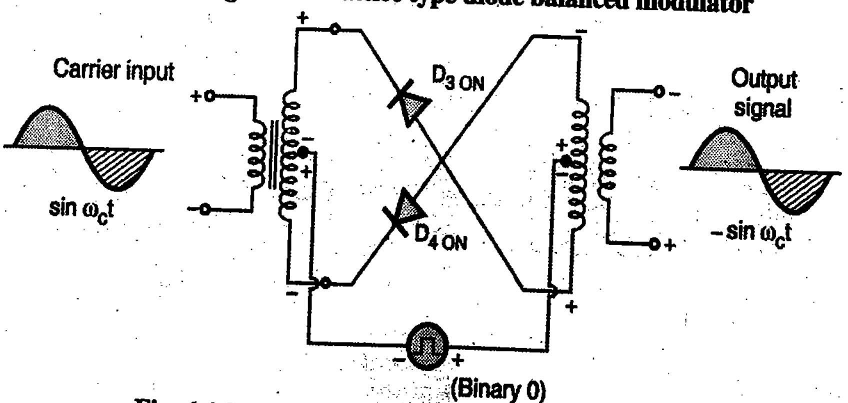
(E-369) Fig. 6.4.3(a) : Equivalent circuit with binary input = logic 1

Operation with binary input = logic 0 :

- If the binary input is equal to 0 (actually / V) then D_1, D_2 are reverse biased and remains off whereas D_3 and D_4 are forward biased and hence conduct.
- Fig. 6.4.3(b) shows the equivalent circuit.



(E-368) Fig. 6.4.2 : Lattice type diode balanced modulator

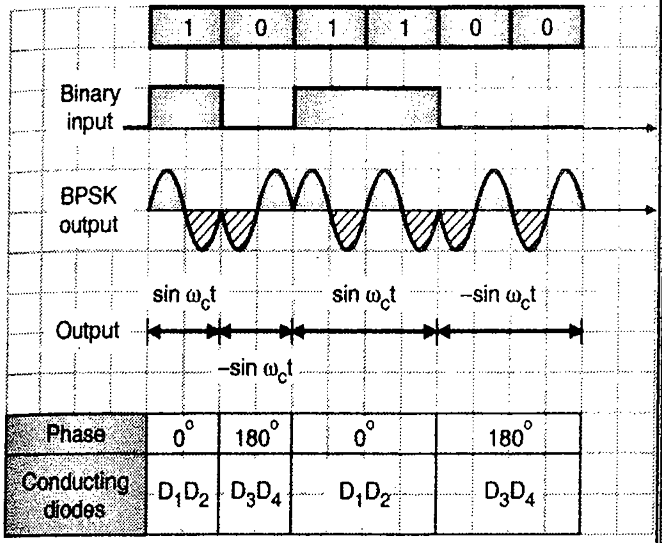


(E-370) Fig. 6.4.3(b) : Equivalent circuit with binary input = logic 0

- The output voltage across T_2 is 180° out of phase with respect to the carrier input. Hence the output voltage is – sin (ω_c t).

Waveforms :

- Fig. 6.4.4 shows the input output waveforms for the diode ring modulator circuit.



(E-371) **Fig. 6.4.4 : Waveforms of diode ring modulator**

Truth table :

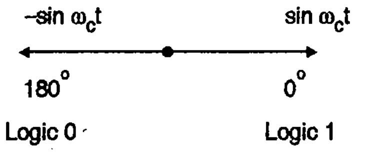
Table 6.4.2 gives the truth table of the diode ring modulator.

Table 6.4.2 : Truth table

Binary Input	Output Phase
Logic 0	0°
Logic 1	180°

Phasor diagram :

Fig. 6.4.5(a) shows the phasor diagram of BPSK signal.



(E-372) **Fig. 6.4.5(a) : Phasor diagram of BPSK**

6.4.5 BPSK Receiver :

SPPU : Dec. 05, May 07, Dec. 08, Dec. 10, Dec. 11

University Questions

Q. 1 Describe the scheme to recover the baseband signal in BPSK with the help of block diagram and signals marked at input, output of these blocks. Also write the functional names inside the blocks. **(Dec. 05, 6 Marks)**

Q. 2 Explain generation and reception of BPSK signal with mathematical expression. **(May 07, 8 Marks)**

Q. 3 Draw the block diagram and with the help of mathematical expression explain in detail the BPSK transmitter and receiver. Diagram the geometric representation and draw its power spectral density, along with its expression thereby comment on its Euclidian distance and bandwidth. **(Dec. 08, 8 Marks)**

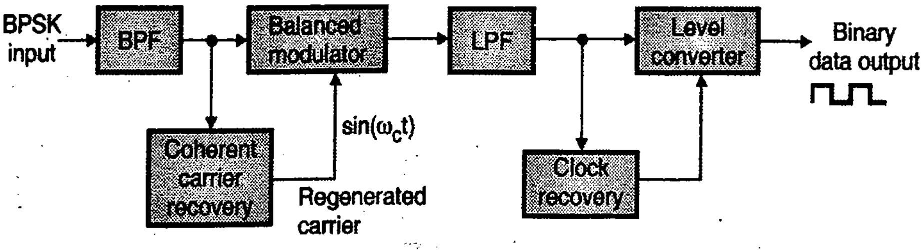
Q. 4 Explain coherent BPSK transmitter and receiver. Derive the expression for receiver output considering effect of noise. Draw the spectrum of BPSK signal and comment on bandwidth required. **(Dec. 10, 8 Marks)**

Q. 5 What is coherent detection ? Draw the block diagram of BPSK receiver and explain its operation with proper mathematical expressions. **(Dec. 11, 8 Marks)**

- The block diagram of a BPSK receiver is shown in Fig. 6.4.5(b).
- The input BPSK signal can be either + sin ω_c t or – sin ω_c t representing either logic 1 or 0 respectively.

Operation :

- The coherent carrier recovery circuit detects and regenerates a carrier signal sin ω_ct. This regenerated carrier has the same frequency and phase as the carrier used at the transmitter.
- So the regenerated carrier is known as coherent carrier, which is phase and frequency synchronized with the transmitter.



(E-373) **Fig. 6.4.5(b) : BPSK receiver**

- The filtered BPSK signal alongwith the regenerated carrier is applied to a balanced modulator which acts as a product detector.

$$\therefore \quad \text{B.M. output} = \text{BPSK} \times \text{Regenerated carrier}$$

$$= (\pm \sin \omega_c t \times \sin \omega_c t = \pm \sin^2 \omega_c t)$$

$$\text{But} \quad \sin^2 \theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta$$

(E-1459)

$$\therefore \text{B.M. output} = \underbrace{\pm\frac{1}{2}}_{\text{DC term}} \underbrace{\mp\frac{1}{2} \cos 2\omega_c t}_{\text{Second harmonic}}$$

- The BM output consist of a dc term and a term having frequency twice the carrier frequency (Second harmonic term).
- The BM output is passed through LPF which allows only the second harmonic term to pass through and blocks the dc component.

$$\therefore \quad \text{LPF output} = \mp\frac{1}{2} \cos 2\omega_c t$$

- The LPF output is applied to the level detector and clock recovery circuit. At the output of level detector we get the following output.

$$-\frac{1}{2} \cos \omega_c t \rightarrow \frac{1}{2} \text{ V (logic 1)}$$

$$\frac{1}{2} \cos \omega_c t \rightarrow -\frac{1}{2} \text{ V (logic 0)}$$

Thus the binary signal is obtained at the output.

Detail operation of BPSK receiver :

The detailed block diagram of a BPSK receiver is shown in Fig. 6.4.6.

- The received BPSK signal at the receiver input is in the form :

$$V_{BPSK}(t) = b(t)\sqrt{2P_s} \cos(\omega_c t + \theta) \quad ...(6.4.6)$$

where θ is the phase shift corresponding to the time delay (θ / ω_c) which depends on the distance travelled by the signal to reach the receiver from the transmitter.

- The demodulation technique used in the BPSK receiver of Fig. 6.4.6 is called "synchronous detection".
- The synchronous demodulation technique uses a locally generated synchronous carrier signal $\cos(\omega_c t + \theta)$. This carrier signal is "recovered" from the received BPSK signal as shown in Fig. 6.4.6. The carrier recovery takes place as follows :
- The received signal is passed through a square law device. We get a squared signal at the output of the square law device.

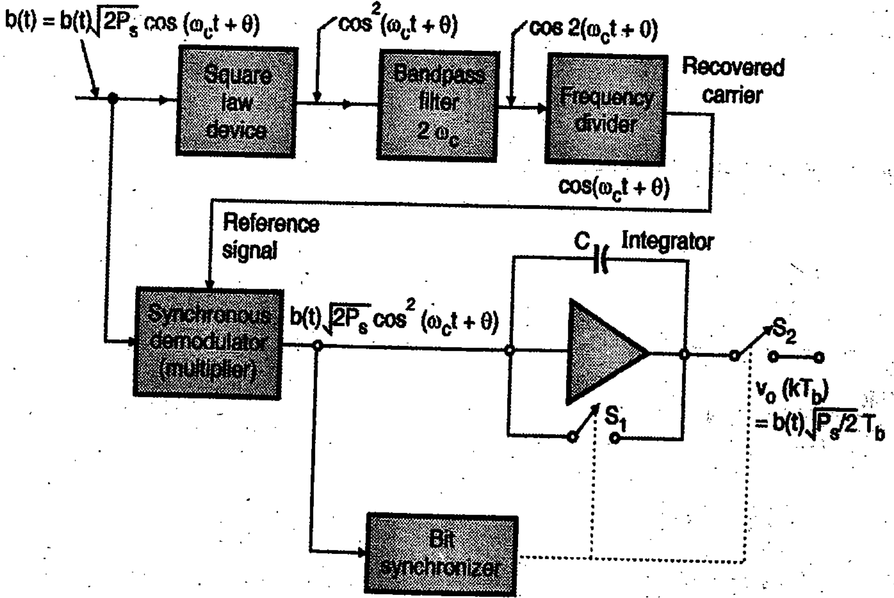
$$\text{Output of the square law device} = \cos^2(\omega_c t + \theta) \quad ...(6.4.7)$$

Note that we have not considered the amplitude because, in the synchronous demodulation the amplitude is not important.

- But, $\cos^2(\omega_c t + \theta) = \frac{1}{2}[1 + \cos 2(\omega_c t + \theta)]$

$$= \frac{1}{2} + \frac{1}{2} \cos 2(\omega_c t + \theta) \quad ...(6.4.8)$$

The RHS of this expression shows that the output of the square law device consists of a dc term i.e. 1/2 and the second harmonic of the carrier.



(E-856) Fig. 6.4.6 : BPSK receiver

- Only the second harmonic component is allowed to pass through by the bandpass filter. the center frequency of which is adjusted. to 2 ω_c. Thus the output of the bandpass filter is cos 2 (ω_c t + θ), neglecting the amplitude. The dc component is blocked by the filter.
- The bandpass filter output is then passed through a frequency divider to recover the carrier signal i.e. cos (ω_c t + θ). Thus carrier is recovered or regenerated from the received BPSK signal.
- The input BPSK signal and the recovered carrier signal are multiplied in the synchronous modulator which is nothing but a multiplier.

$$\therefore \text{ Output of multiplier} = b(t)\sqrt{2P_s}\cos(\omega_c t+\theta) \times \cos(\omega_c t+\theta)$$

$$= b(t)\sqrt{2P_s}\cos^2(\omega_c t+\theta)$$

$$= b(t)\sqrt{2P_s}\left[\frac{1}{2}+\frac{1}{2}\cos 2(\omega_c t+\theta)\right] \quad ...(6.4.9)$$

- This signal is applied to an integrator. The bit synchronizer circuit will close the switch S_1, momentarily to discharge the integrator capacitor at the beginning of every bit interval. This switch S_1 is then left open for the entire bit duration, so that the integrator can produce an output proportional to its input voltage.
- The output of the integrator circuit is the output of the BPSK receiver. This integrator output is made available to the user by closing switch S_2 at the end of each bit interval. This switch is called as the sampling switch and it is operated by the bit synchronizer circuit.
- To obtain the output of the integrator, let us assume that the bit interval T_b is equal to the periods of integral number of carrier cycles "n".

$$\therefore \quad n \times 2\pi = \omega_c T_b$$

- Hence the output voltage v_o (k T_b) at the end of the bit interval which extends from instant (k – 1) T_b to k T_b is given by,

Integrator output,

$$v_o(kT_b) = b(t)\sqrt{2P_s}\int_{(k-1)T_b}^{kT_b}\frac{1}{2}dt$$

$$+ b(t)\sqrt{2P_s}\int_{(k-1)T_b}^{kT_b}\frac{1}{2}\cos 2(\omega_c t+\theta)\,dt \quad ...(6.4.10)$$

- Thus the integrator integrates only over one bit interval T_b.

$$\therefore \quad \text{Integrator output} = b(kT_b)\sqrt{\frac{P_s}{2}}\,T_b \quad ...(6.4.11)$$

- The integration of the second term in Equation (6.4.10) is zero. This is because the integration of a sinusoidal signal over a whole number of cycles has a zero value. Equation (6.4.11) shows that the integrator output successfully reproduces the bit stream b (t) which was originally transmitted.
- The integrator circuit alongwith the two switches S_1, S_2 and the bit synchronizer is called as the integrator and dump circuit.

6.4.6 Spectrum of BPSK :

SPPU : Dec. 05, Dec. 07, Dec. 10, May 12

University Questions

Q. 1 Express mathematically and plot the power spectral density of NRZ data b(t) and BPSK. Plot on graph paper one below other with common scale.

(Dec. 05, 10 Marks)

Q. 2 For rectangular data pulses calculate the second null-to-null bandwidth of BPSK, QPSK, MSK, 16 PSK and 16 QAM. Discuss advantages and disadvantages of using each of these methods.

(Dec. 07, 6 Marks)

Q. 3 Explain coherent BPSK transmitter and receiver. Derive the expression for receiver output considering effect of noise. Draw the spectrum of BPSK signal and comment on bandwidth required.

(Dec. 10, 8 Marks)

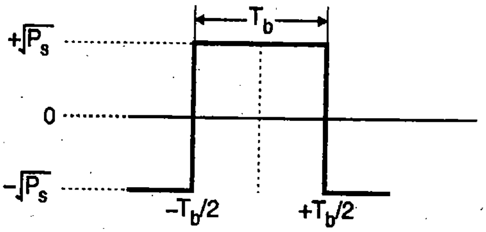
Q. 4 Derive and draw the spectrum of BPSK, QPSK and BFSK signal and compare their bandwidths.

(May 12, 8 Marks)

- In BPSK, the waveform b (t) is a non-return to zero (NRZ) waveform which has two peaks at $+\sqrt{P_s}$ and $-\sqrt{P_s}$. If we assume that each pulse is T_b seconds wide and is extended from $-\frac{T_b}{2}$ to $+\frac{T_b}{2}$ around the center as shown in Fig. 6.4.7(a), then the Fourier transform of such a pulse is given by,

$$X(f) = \sqrt{P_s}\;T_b\,\frac{\sin(\pi f T_b)}{(\pi f T_b)} \quad ...(6.4.12)$$

$$= \sqrt{P_s}\;T_b\,\text{sinc}(f T_b) \quad ...(6.4.13)$$



(E-857) **Fig. 6.4.7(a) : NRZ signal b (t)**

- But the NRZ signal b (t) will consist of a large number of such pulses. Therefore the power spectral density S (f) of such a signal is given by :

$$S(f) = \frac{|\overline{X(f)}|^2}{T_b} \qquad ...(6.4.14)$$

where $\overline{X(f)}$ represents the average value of X (f), due to a large number of pulses in b (t), and T_b is the bit duration.

- Substitute the value of X (f) from Equation (6.4.12) into Equation (6.4.14) to get,

$$\text{Power spectral density, } S(f) = P_s\, T_b \left[\frac{\sin(\pi f T_b)}{(\pi f T_b)}\right]^2 \qquad ...(6.4.15)$$

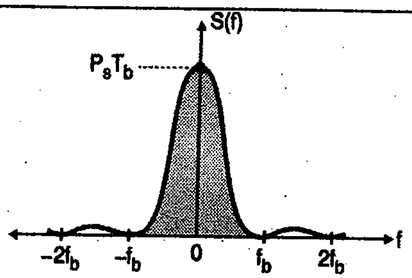
- This equation gives the power spectral density (PSD) of the signal b (t). This spectrum is shown in Fig. 6.4.7(b).
- In order to obtain the BPSK signal, the NRZ waveform b (t) is multiplied by the carrier signal $\sqrt{2}\cos\omega_c t$. Due to this modulation process, the spectrum of baseband signal b (t) gets translated about + f_c and – f_c and its amplitude reduces to half of the baseband spectrum of Equation (6.4.15). Therefore power spectral density of a BPSK signal is given by,

$$S_{BPSK}(f) = \frac{P_s T_b}{2} \left\{\left[\frac{\sin\pi(f_c - f)T_b}{\pi(f_c - f)T_b}\right]^2 + \left[\frac{\sin\pi(f_c + f)T_b}{\pi(f_c + f)T_b}\right]^2\right\} \qquad ...(6.4.16)$$

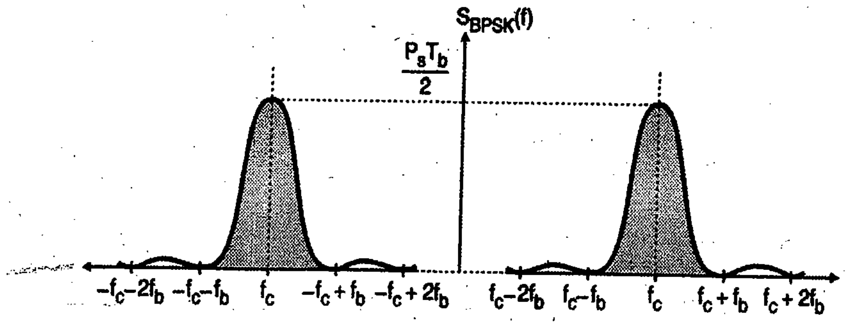
- The PSD of BPSK signal is plotted in Fig. 6.4.7(c).

6.4.7 Possibility of Overlapping (Interchannel Interference and ISI) :

- If we try to multiplex multiple BPSK signals using different carrier frequencies for different baseband signals, then due to the nature of frequency spectrum of BPSK signal shown in Fig. 6.4.7(b), there will be overlap in the spectra of the adjacent BPSK signals.
- Therefore a receiver tuned to one carrier will also receive a signal from the adjacent channel.
- This overlapping is unwanted and called as "interchannel interference". Such an interference takes place due to overlapping side lobes of the adjacent spectrums.
- In order to reduce this interference, the side-lobes produced in BPSK should be reduced below a certain specific level.
- This can be achieved by employing a filter to restrict the bandwidth allowed to the NRZ baseband signal b (t), before applying it to a modulator.
- This low pass filter will allow only the main lobe of the spectrum to pass through to the modulator.
- Due to such spectrum suppression the modulating signal is distorted and there is a partial overlap of a bit (symbol) and its adjacent bits in a single channel.
- This overlap is called as "intersymbol interference" (ISI). Intersymbol interference can be reduced by using "equalizers" at the receiver.
- They are filter like structures used to nullify the adverse effects of filters introduced at other places in a communication channel.



(a) Power spectral density of the NRZ data b(t)



(b) Spectrum of BPSK

(E-374) Fig. 6.4.7

6.4.8 Bandwidth of BPSK :

SPPU : Dec. 07, Dec. 10, May 12

University Questions

Q. 1 For rectangular data pulses calculate the second null-to-null bandwidth of BPSK, QPSK, MSK, 16 PSK and 16 QAM. Discuss advantages and disadvantages of using each of these methods. **(Dec. 07, 6 Marks)**

Q. 2 Explain coherent BPSK transmitter and receiver. Derive the expression for receiver output considering effect of noise. Draw the spectrum of BPSK signal and comment on bandwidth required. **(Dec. 10, 8 Marks)**

Q. 3 Derive and draw the spectrum of BPSK, QPSK and BFSK signal and compare their bandwidths. **(May 12, 8 Marks)**

- From the frequency spectrum of BPSK signal, shown in Fig. 6.4.7(c), we can come to a conclusion that the bandwidth of a BPSK signal is given by,

BW = Highest frequency - Lowest frequency in main lobe

$$= (f_c + f_b) - (f_c - f_b)$$

$$\therefore \quad BW = 2 f_b \qquad ...(6.4.17)$$

where $f_b = 1 / T_b$

- Thus the minimum bandwidth of BPSK signal is equal to twice the highest frequency contained in the baseband signal.

6.4.9 Geometrical Representation of BPSK Signals :

SPPU : May 11, Dec. 14, Dec. 15

University Questions

Q. 1 Diagram the geometric representation of :

(a) Orthogonal and non-orthogonal BFSK.

(b) M-ary FSK.

State the Euclidean distance of above mentioned systems by explaining the importance of Euclidean distance. **(May 11, 6 Marks)**

Q. 2 Compare BPSK, QPSK and M'ary PSK with the help of equations, signal space representation, symbol rate and bandwidth. **(Dec. 14, 9 Marks)**

Q. 3 Draw and explain signal space representation of following signal. BPSK **(Dec. 15, 4 Marks)**

- We know that a BPSK signal is mathematically expressed as,

$$V_{BPSK}(t) = b(t)\sqrt{2P_s}\cos\omega_c t \qquad ...(6.4.18)$$

- Let us represent this signal in terms of one orthonormal signal $u_1(t)$ which is defined as,

$$\phi_1(t) = u_1(t) = \sqrt{(2/T_b)}\ \cos\ \omega_c t \qquad ...(6.4.19)$$

- Note that $u_1(t)$ is same as the basis function $\phi_1(t)$ defined earlier.

$$\therefore \quad \cos\omega_c t = \frac{u_1(t)}{\sqrt{(2/T_b)}} = u_1(t) \times \sqrt{\frac{T_b}{2}}$$

- Substituting this value into Equation (6.4.18) we get,

$$V_{BPSK}(t) = b(t) \times \sqrt{2P_s} \times \sqrt{T_b/2} \times u_1(t)$$

$$= \left[b(t) \times \sqrt{P_s T_b}\right] \times u_1(t) \qquad ...(6.4.20)$$

- But $b(t) = \pm 1$

$$\therefore \quad V_{BPSK}(t) = \pm\sqrt{P_s T_b} \times u_1(t) \qquad ...(6.4.21)$$

That means

$$V_{BPSK}(t) = +\sqrt{P_s T_b}\, u_1(t) \text{ or } -\sqrt{P_s T_b}\ u_1(t)$$

- We can show these two values as two distinct points on a single axis of $u_1(t)$. The first point is located at $+\sqrt{P_s T_b}$ while the second point is located at $-\sqrt{P_s T_b}$ as shown in Fig. 6.4.8.
- When the BPSK signal is received at the receiver, the $-\sqrt{P_s T_b}$ represents (or identified as) a logic 0 signal and $+\sqrt{P_s T_b}$ is identified as a logic 1 signal. The distance "d" between these signals is given by :

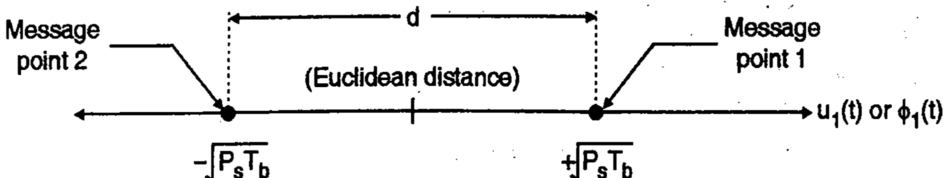
$$d = +\sqrt{P_s T_b} - (-\sqrt{P_s T_b}) = 2\sqrt{P_s T_b} \qquad ...(6.4.22)$$

But $P_s T_b = E_b$ i.e. the signal energy

$$\therefore \quad d = 2\sqrt{E_b} \qquad ...(6.4.23)$$

This distance is also called as **Euclidean distance.**

- The importance of this distance is that, the error probability of BPSK is dependent on the value of d. The error probability decreases with increase in the value of "d". The geometric representation is also called as signal space representation.



(E-375) **Fig. 6.4.8 : Geometrical representation of BPSK signal**

6.4.10 Advantages of BPSK :

1. BPSK has a bandwidth which is lower than that of a BFSK signal.
2. BPSK has the best performance of all the systems in presence of noise. It gives the minimum possibility of error.
3. BPSK has a very good noise immunity.

6.4.11 Disadvantage of BPSK :

The only disadvantage of BPSK is that generation and detection of BPSK is not easy. It is quite complicated, because the synchronous (coherent) demodulation is used to recover the original signal from BPSK signal.

Ex. 6.4.1 : If the data bit sequence consists of the following string of bits, what will be the nature of waveform transmitted by BPSK transmitter ? The data bit sequence is 1 0 1 1 1 0 1 0.

Soln. : We know that the BPSK signal is given by,

$$V_{BPSK}(t) = \sqrt{2 P_s}\, b(t) \cos \omega_c t$$

where b (t) = ± 1 depending on the digital input signal. Table P. 6.4.1 lists the values of b (t) and the transmitted signal V_{BPSK} for different bit intervals.

- The transmitted BPSK signal is as shown in Fig. P. 6.4.1.

6.4.12 Error Probability of BPSK (with Coherent Detection) :

SPPU : Dec. 06, Dec. 07, May 08, May 09, Dec. 10, May 11, May 13, Dec. 13, May 15, May 16

University Questions

Q. 1 Derive the expression for error probability if BPSK signal is detected using optimum receiver. **(Dec. 06, 8 Marks)**

Q. 2 Derive the error probability expression for BPSK and BFSK. **(Dec. 07, 10 Marks, May 08, May 09, May 11, 8 Marks)**

Q. 3 Derive the expression for error probability of BPSK receiver. **(Dec. 10, 8 Marks)**

Q. 4 Show that the probability of error of QPSK is same as that of BPSK for 1 bit duration. **(May 11, 8 Marks)**

Q. 5 Derive the expression for the probability of error of a BPSK system. **(May 13, Dec. 13, 8 Marks)**

Q. 6 Derive an expression for error probability of BPSK using matched filter. **(May 15, 8 Marks)**

Q. 7 Derive the expression for error probability of BPSK system. **(May 16, 8 Marks)**

- The steps that we are going to follow in order to obtain the expression for the error probability, are exactly same as those followed to obtain error probability of an ASK system.
- We know that the BPSK signal is represented as follows :

Binary 1 : $x_1(t) = \sqrt{2 P_s} \cos \omega_c t$

Binary 0 : $x_2(t) = -\sqrt{2 P_s} \cos \omega_c t$

Therefore $x_2(t) = -x_1(t)$

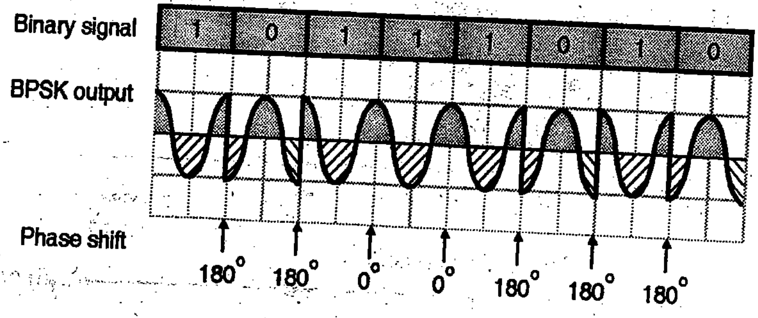
- We are going to **use the matched filter for detection of BPSK signal**. The expression for error probability of an optimum filter is, (See chapter 4)

$$P_e = \frac{1}{2} \operatorname{erfc}\left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}\right] \quad ...(6.4.24)$$

- The expression for the signal to noise ratio of a matched filter is given by,

Table P. 6.4.1

Binary signal	1	0	1	1	1	0	1	0
b (t)	+ 1	− 1	+ 1	+ 1	+ 1	− 1	+ 1	− 1
$V_{BPSK}(t)$	cos ω_c t	− cos ω_c t	cos ω_c t	− cos ω_c t	cos ω_c t	− cos ω_c t	cos ω_c t	− cos ω_c t



(E-376) **Fig. P. 6.4.1 : Waveforms**

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{\max} = \frac{2}{N_0}\int_{-\infty}^{\infty} |X(f)|^2\, df \quad ...(6.4.25)$$

- Using the Rayleigh's energy theorem,

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} x^2(t)\, dt = \int_{0}^{T} x^2(t)\, dt \quad ...(6.4.26)$$

- The limits of integration of the last term in Equation (6.4.26) are 0 to T because x (t) is present only over **one bit interval T**. Substituting Equation (6.4.26) into Equation (6.4.25) we get,

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{\max} = \frac{2}{N_0}\int_{0}^{T} x^2(t)\, dt \quad ...(6.4.27)$$

- But $x(t) = x_1(t) - x_2(t)$

  and for BPSK, $x_2(t) = -x_1(t)$

  $\therefore \quad x(t) = 2x_1(t) = 2\sqrt{2P_s}\cos\omega_c t$

- Substituting this value of x (t) into Equation (6.4.27) we get,

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{\max} = \frac{2}{N_0}\int_{0}^{T} 8P_s\cos^2\omega_c t\, dt$$

$$= \frac{16P_s}{N_0}\int_{0}^{T}\cos^2\omega_c t\, dt \quad ...(6.4.28)$$

But $\cos^2\omega_c t = \frac{1+\cos 2\omega_c t}{2}$

- Substitute this into Equation (6.4.28) to get,

$$\therefore \left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{\max} = \frac{16P_s}{N_0}\int_{0}^{T}\frac{1+\cos 2\omega_c t}{2}\cdot dt$$

$$= \frac{8P_s}{N_0}\left[\int_{0}^{T} 1\, dt + \int_{0}^{T}\cos 2\omega_c t\, dt\right]$$

$$= \frac{8P_s}{N_0}\left\{[t]_0^T + \frac{1}{2\omega_c}(\sin 2\omega_c t)_0^T\right\}$$

$$= \frac{8P_s}{N_0}\left\{T + \frac{\sin 2\omega_c T}{2\omega_c}\right\} \quad ...(6.4.29)$$

- The value of second term in the RHS of Equation (6.4.29) is zero, as proved earlier in section 6.3.7. (Refer to Fig. 6.3.5).

$$\therefore \left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{\max} = \frac{8P_sT}{N_0} \quad ...(6.4.30)$$

But P_sT = Energy E.

$$\therefore \left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{\max} = \frac{8E}{N_0}$$

- Taking the square root of both sides

$$\therefore \left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]_{\max} = \sqrt{\frac{8E}{N_0}} \quad ...(6.4.31)$$

- Substitute this expression into Equation (6.4.24) to get the error probability for BPSK as :

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{1}{2\sqrt{2}}\cdot\sqrt{\frac{8E}{N_0}}\right]$$

$$\therefore \quad P_e = \frac{1}{2}\,\text{erfc}\left[\sqrt{\frac{E}{N_0}}\right] \quad ...(6.4.32)$$

- This is the expression for error probability of BPSK with matched filter receiver.

  This is the expression for the bit error probability P_B.

It indicates that the probability of error depends on the energy contents of the signal "E". It does not depend on the shape of the signal. As the energy increases, the value of erfc function will decrease and the probability of error will also reduce.

- This result can be expressed in terms of the Q function as :

$$P_B = Q\sqrt{\frac{2E}{N_0}} \quad ...(6.4.33)$$

- This is because the relation between erfc and Q function is,

$Q(x) = \frac{1}{2}\,\text{erfc}\,(x/\sqrt{2})$ and $\text{erfc}\,(x) = 2Q(x\sqrt{2})$

Note : The "erfc" function is a monotonic decreasing function. Therefore the value of erfc $[\sqrt{E/N_0}]$ will be less than erfc $[\sqrt{E/4N_0}]$. Hence the error probability with BPSK technique is less than that with the ASK technique. Hence BPSK system will be superior in performance as compared to the ASK system.

Ex. 6.4.2 : Find the bit error probability for a BPSK system having a bit rate of 1 Mbits/s. The receiver receives the waveforms $s_1(t) = A\cos\omega_c t$ and $s_2(t) = -A\cos\omega_c t$. The received signals are co-herently detected using a matched filter. If A = 10 mV and single sided noise power spectral density is $N_0 = 10^{-11}$ W/Hz. Assume that the signal power and energy per bit are normalized.

Dec. 11, Dec. 13, Dec. 16. 8 Marks

Soln. :

1. We know that $A = \sqrt{\frac{2\,E_b}{T}} = 10 \times 10^{-3}\ V$

But T = One bit period $= \frac{1}{1 \times 10^6} = 1\ \mu S$

$$\therefore\ E_b = \frac{A^2\,T}{2} = \frac{(10 \times 10^{-3})^2 \times 1 \times 10^{-6}}{2}$$

$\therefore\ E_b = 5 \times 10^{-11}$ Joules

This is the energy per bit.

2. $\therefore \sqrt{\frac{E_b}{N_0}} = \sqrt{\frac{5 \times 10^{-11}}{1 \times 10^{-11}}} = 2.24$

$\therefore$ Bit error probability $P_B = \frac{1}{2}\,\text{erfc}\left[\sqrt{\frac{E_b}{N_0}}\right]$

$$= \frac{1}{2}\,\text{erfc}\ [2.24] = \frac{1}{2} \times 0.00041$$

$$P_B = 2.05 \times 10^{-4}$$

Ex. 6.4.3 : A bandpass data transmission scheme uses PSK with bit interval 0.2 msec. The carrier amplitude at the receiver input is 1 mV and PSD of AWGN is 10^{-11} Watt/Hz. Calculate the probability of error of the receiver.

May 15. 10 Marks

Soln. :

The amplitude of the carrier is,

$$A = 1 \times 10^{-3}\ V$$

$\therefore$ psd of white noise is, $\frac{N_0}{2} = 10^{-11}$ Watt/Hz

Bit duration, $T_b = 0.2 \times 10^{-3}$ sec.

The normalized power of the carrier in 1 Ω load is,

$$P_s = \frac{1}{2} A^2$$

$\therefore$ Bit energy is, $E_b = P_s\,T_b$ (i.e. Power × Bit duration)

$$= \frac{1}{2} A^2\,T_b$$

$$= \frac{1}{2} \times (1 \times 10^{-3})^2 \times 0.2 \times 10^{-3} = 1 \times 10^{-10}$$

Error probability of PSK is,

$$P_e = \frac{1}{2}\,\text{erfc}\sqrt{\frac{E_b}{N_0}} = \frac{1}{2}\,\text{erfc}\sqrt{\frac{1 \times 10^{-10}}{2 \times 10^{-11}}}$$

Here, E_b = Bit energy

$$= \frac{1}{2}\,\text{erfc}\sqrt{5} = \frac{1}{2}\ \text{erfc}\ (2.24) \qquad \text{...Ans.}$$

6.5 Differential PSK (DPSK) :

SPPU : May 06, Dec. 10, May 11, May 16

University Questions

Q. 1 What is differential PSK ? Draw the block diagrams of DPSK transmitter and receiver. **(May 06, 4 Marks)**

Q. 2 What is non-coherent version of BPSK ? Explain with suitable block diagram and waveforms. **(Dec. 10, 8 Marks)**

Q. 3 What is DPSK and DEPSK ? **(May 11, 4 Marks)**

Q. 4 Explain with the help of block diagram and waveforms DPSK modulation. **(May 16, 8 Marks)**

- The differential phase shift keying can be treated as the noncoherent version of PSK.
- It combines two basic operations, namely :
  1. The differential encoding and
  2. Phase shift keying
- Hence the name differential phase shift keying (DPSK).
- In the BPSK receiver of 6.5.1(b), the process of carrier recovery began by squaring the received BPSK signal $\sqrt{2\,P_s}\ b\,(t) \cos\,(\omega_c\,t + \theta)$.
- Even if the received signal is $-\,b\,(t) \times \sqrt{2\,P_s}\ \cos\,(\omega_c\,t + \theta)$, the squared signal will remain same. Therefore we shall not be able to determine at the receiver whether the received signal is the transmitted signal $b\,(t)$ or its inverted version $-\,b\,(t)$.
- This type phase reversal takes place when the signal passes through the telephone switching networks.
- Differential PSK i.e. DPSK will eliminate this ambiguity about whether the received data was inverted or not.
- Another advantage of DPSK is that it does not require the synchronous carrier at the demodulator for its detection.

6.5.1 DPSK Generator :

SPPU : May 06, Dec. 10. Dec. 12. Dec. 14, May 16

University Questions

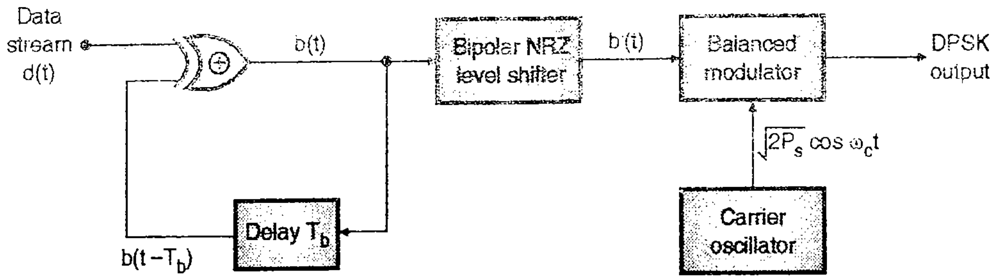
Q. 1 What is differential PSK ? Draw the block diagrams of DPSK transmitter and receiver. **(May 06, 4 Marks)**

Q. 2 What is non-coherent version of BPSK ? Explain with suitable block diagram and waveforms. **(Dec. 10, 8 Marks)**

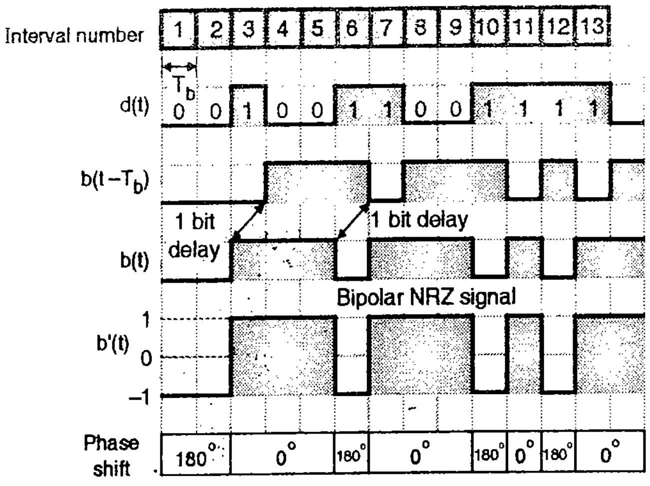
Q. 3 For an input stream of 110100010. Explain the encoding and decoding process for DPSK with the help of waveforms and expressions. **(Dec. 12, 8 Marks)**

Q. 4 Draw the block diagram of DPSK transmitter and explain its operation with proper waveforms. **(Dec. 14, 6 Marks)**

Q. 5 Explain with the help of block diagram and waveforms DPSK modulation. **(May 16, 8 Marks)**



(a) DPSK generator



(b) Waveforms of DPSK

(E-377) **Fig. 6.5.1**

- The differential phase shift keying (DPSK) is a modification of BPSK. Fig. 6.5.1(a) shows the block diagram of the DPSK generator and the relevant waveforms are as shown in Fig. 6.5.1(b).

Operation :

Operation of the DPSK generator is as follows :

- d (t) represents the data stream obtained from a digital source. This data is to be transmitted. It is applied to one input of an EX-OR logic gate.
- The EX-OR gate output "b (t)" is delayed by one bit period T_b and applied to the other input of the EX-OR gate. The delayed EX-OR gate output is represented by "$b(t - T_b)$".
- Depending on the values of "d (t)" and "$b(t - T_b)$", the EX-OR gate produces the output sequence "b (t)". The waveforms for DPSK generator are as shown in Fig. 6.5.1(b). These waveforms have been drawn by arbitrarily assuming that in the first interval b (0) = 0.
- Output of EX-OR gate is then applied to a bipolar NRZ level shifter, which converts "b (t)" to a bipolar level signal b′ (t) as shown in Fig. 6.5.1(a).

b (t)	b′ (t)
0	–1
1	+1

- This bipolar NRZ signal b′ (t) is then multiplied with the carrier signal to produce the DPSK signal. The DPSK output signal is mathematically expressed as :

$$V_{DPSK}(t) = b'(t) \times \sqrt{2P_s} \cos \omega_c t \qquad ...(6.5.1)$$

So when b (t) = 1, b′ (t) = 1 hence

$$V_{DPSK}(t) = \sqrt{2P_s} \cos \omega_c t \qquad ...(6.5.2)$$

- This expression indicates that no phase shift has been introduced.

But when b (t) = 0, b′ (t) = –1 hence

$$V_{DPSK}(t) = -\sqrt{2P_s} \cos \omega_c t \qquad ...(6.5.3)$$

This shows that 180° phase shift is introduced to represent b (t) = 0.

Important observations :

- From the waveforms of Fig. 6.5.1(b) we can observe that the signal b (t) changes its level at the beginning of each interval in which d (t) = 1. There is no change in the level of b (t) when d (t) = 0. Therefore in Fig. 6.5.1 (b), b (t) changes its level at the beginning of intervals 3, 6, 7, 10, 11, 12 and 13.
- Also note that when d (t) = 1, sometimes b (t) = 1 and sometimes b (t) = 0. Thus there is absolutely no definite relationship between the values of d (t) and b (t). The only thing that we can conclude is that b (t) will change (sometimes up and sometimes down) whenever d (t) = 1 and b (t) will not change when d (t) = 0.

- The combined expression for DPSK signal is given by,

$$V_{DPSK}(t) = \pm\sqrt{2P_s}\cos\omega_c t$$

$$= b'(t)\sqrt{2P_s}\cos\omega_c t \qquad ...(6.5.4)$$

- Instead of using b′(t), for convenience, let us use b (t) in place of b′(t).

$$\therefore \quad V_{DPSK}(t) = b(t)\sqrt{2P_s}\cos\omega_c t \qquad ...(6.5.5)$$

6.5.2 DPSK Reception :

SPPU : May 06, Dec. 10, Dec. 12, May 16

University Questions

Q. 1 What is differential PSK ? Draw the block diagrams of DPSK transmitter and receiver. **(May 06, 4 Marks)**

Q. 2 What is non-coherent version of BPSK ? Explain with suitable block diagram and waveforms. **(Dec. 10, 8 Marks)**

Q. 3 For an input stream of 110100010. Explain the encoding and decoding process for DPSK with the help of waveforms and expressions. **(Dec. 12, 8 Marks)**

Q. 4 Explain with the help of block diagram and waveforms DPSK modulation. **(May 16, 8 Marks)**

- The block diagram of a DPSK receiver is shown in Fig. 6.5.2.
- The received DPSK signal is denoted by x (t) and it is applied through a bandpass filter to a correlator.
- The correlator receives another input from the delay unit which introduces a delay of one bit duration (T_b).
- The output of correlator (m) is applied to a decision device which compares it with the threshold level of 0 volts.
- If m > 0, then the receiver makes a decision that a 1 was transmitted and if m < 0, then the decision is made in favour of a 0.

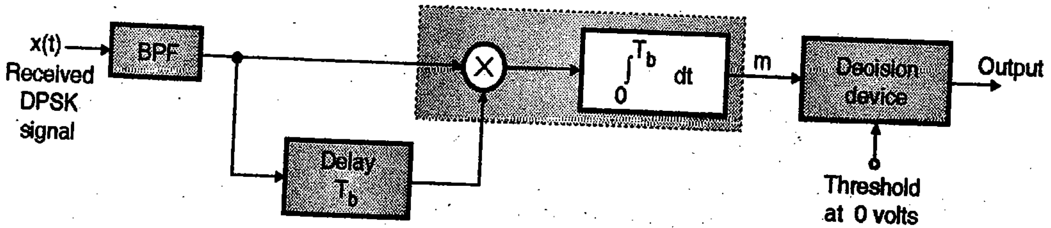
Detail operation of DPSK reception :

- The block diagram of a DPSK receiver is shown in Fig. 6.5.3.
- As shown in Fig. 6.5.3, the received DPSK signal is applied to a synchronous demodulator alongwith its delayed version. The delayed signal is obtained by using the delay circuit which introduces a delay of one bit duration T_b.
- The output of the multiplier is given by,

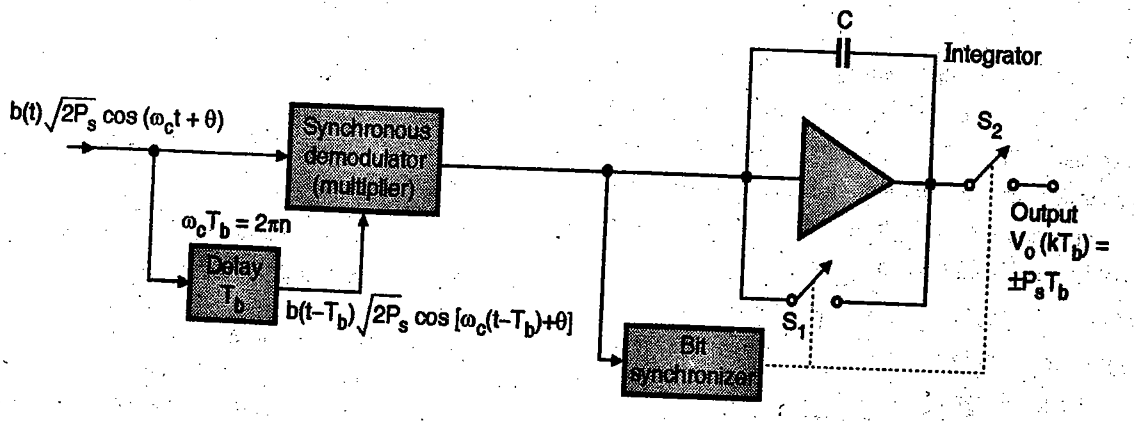
$$\text{Multiplier output} = b(t)\sqrt{2P_s}\cos(\omega_c t+\theta) \times b(t-T_b)\sqrt{2P_s}\cos[\omega_c(t-T_b)+\theta]$$

$$= b(t)\,b(t-T_b)\,P_s \times 2\cos(\omega_c t+\theta) \times \cos[\omega_c(t-T_b)+\theta] \qquad ...(6.5.6)$$

We know that 2 cos A cos B = cos (A + B) + cos (A − B)



(E-378) **Fig. 6.5.2 : Block diagram of DPSK receiver**



(E-1057(a)) **Fig. 6.5.3 : DPSK receiver**

$$\therefore \text{ Multiplier output } = b(t)\, b(t-T_b)\, P_s \times \left\{\cos(\omega_c T_b) + \cos\left[2\,\omega_c\left(t-\frac{T_b}{2}\right)+2\,\theta\right]\right\} \quad ...(6.5.7)$$

- The multiplier output is then applied to a combination of integrator and bit synchronizer discussed for BPSK receiver.
- We select $\omega_c T_b = 2\,n\pi$ where n is an integer. Substituting this value into Equation (6.5.7) we get,

$$\text{Multiplier output } = b(t)\, b(t-T_b)\, P_s \left\{\cos 2\,\pi n + \cos\left[2\,\omega_c\left(t-\frac{T_b}{2}\right)+2\,\theta\right]\right\} \quad ...(6.5.8)$$

But $\cos 2\pi n = 1$

$$\therefore \text{ Multiplier output} = b(t)\, b(t-T_b)\, P_s \left\{1 + \cos\left[2\,\omega_c\left(t-\frac{T_b}{2}\right)+2\,\theta\right]\right\} \quad ...(6.5.9)$$

- When the multiplier output is applied to the integrator and when the integration is carried out over one bit period, the second term of Equation (6.5.9) will produce a zero output voltage whereas the first term will produce a non-zero output.

$$\text{Integrator output } = V_o(k\,T_b)$$

$$= \int_{(k-1)\,T_b}^{k\,T_b} b(t) \times b(t-T_b)\, P_s \times dt$$

$$= b(k\,T_b)\, b[(k-1)\,T_b] \times P_s\,[k\,T_b - k\,T_b + T_b]$$

$$\therefore \quad V_o(k\,T_b) = b(k\,T_b)\, b[(k-1)\,T_b]\, P_s\, T_b \quad ...(6.5.10)$$

- In Equation (6.5.10), the product $b(k\,T_b)\, b[(k-1)\,T_b]$ will decide the sign of $P_s\,T_b$, because if $b(t) = b(t-T_b)$ then $d(t) = 0$.

$\therefore$ If $b(t)\, b(t-T_b) = 1$ ...then $d(t) = 0$

And if $b(t) \neq b(t-T_b)$ ...then $d(t) = 1$

$\therefore$ if $b(t) \times b(t-T_b) = -1$ ... then $d(t) = 1$

- Thus we obtain $V_o(k\,T_b) = \pm P_s\,T_b$ depending on the value of the product $b(t) \times b(t-T_b)$.

$\therefore \quad V_o(k\,T_b) = +P_s\,T_b$ ... for $d(t) = 0$

and $\quad V_o(k\,T_b) = -P_s\,T_b$ ... for $d(t) = 1$

6.5.3 Bandwidth of DPSK Signal :

- From the discussion made till now, it is clear that the phase shift of DPSK signal is dependent on the existing bit and one previous bit. Thus we can say that in DPSK two successive bits form a symbol.

$$\therefore \quad \text{Symbol duration } T_s = 2\,T_b \quad ...(6.5.11)$$

where T_b = Duration of one bit

- Therefore the bandwidth of the DPSK signal is given by,

$$BW = 2/T_s = 2/2\,T_b$$

$$\therefore \quad BW = f_b \quad ...(6.5.12)$$

Thus the bandwidth of DPSK is half the bandwidth of the BPSK.

6.5.4 Advantages of DPSK :

1. DPSK does not need a synchronous carrier at the demodulator for detecting a DPSK signal. This makes the receiver circuit simple. A synchronous carrier is required for the BPSK demodulation.
2. DPSK has a lower bandwidth requirement as compared to that of a BPSK.

6.5.5 Disadvantages of DPSK :

1. In DPSK a bit determination at the receiver is made on the basis of the signal received in two successive bit intervals. Hence noise in one bit interval may cause errors to the decision making of two bits.
2. The error rate or probability of error in DPSK is higher than that in BPSK.
3. The effect of noise is higher in DPSK than that in BPSK.

6.5.6 Error Probability :

- The symbol duration is DPSK is equal to two bit duration i.e. $T_s = 2\,T_b$.
- The energy associated with each symbol is $2\,E_b$.
- Hence the error probability is given by,

$$P_e = \frac{1}{2} e^{-(E_b/N_0)} \quad ...(6.5.13)$$

This is same as that of the noncoherent BFSK.

6.6 Differentially Encoded PSK (DEPSK) :

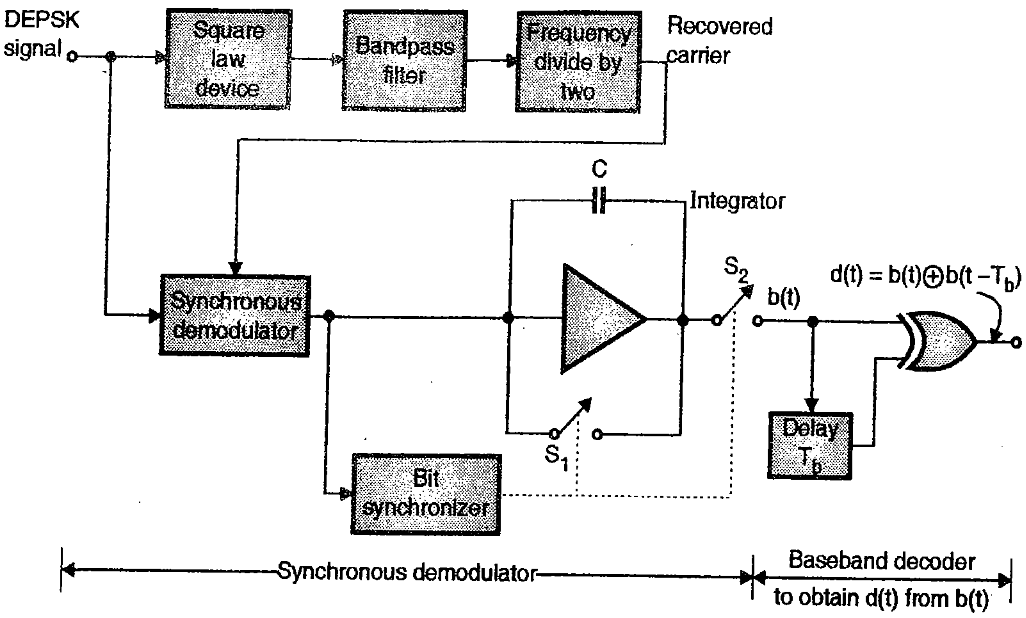
SPPU : May 11, May 13

University Questions

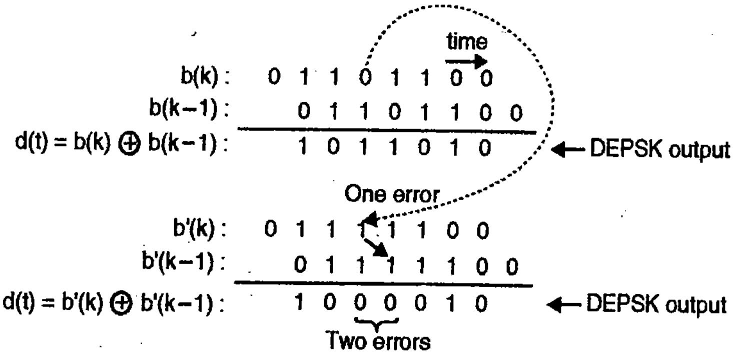
Q. 1 What is DPSK and DEPSK ? **(May 11, 4 Marks)**

Q. 2 Write a notes on : DEPSK. **(May 13, 2 Marks)**

- The transmitter of a DEPSK system is identical to the DPSK transmitter shown in Fig. 6.6.1, but the receiver is completely different.
- The block diagram of DEPSK receiver is shown in Fig. 6.6.1. It shows that the signal b (t) is recovered from the received signal, using the synchronous demodulation technique.
- This is same as the BPSK detection. Once the signal b (t) is recovered, it is applied to one input of an EX-OR gate.



(E-379) **Fig. 6.6.1 : Receiver block diagram of DEPSK system**



(E-380) **Fig. 6.6.2 : Errors in differentially encoded PSK occur in pairs**

- The signal b (t) is also applied to a time delay circuit and the delayed signal b (t − T_b) is applied to the other input of the EX-OR gate as shown in Fig. 6.6.1.

  If b (t) = b (t − T_b) then output of the EX-OR gate will be 0.

  $\therefore \; d(t) = 0 \quad \ldots \text{if } b(t) = b(t-T_b).$

  And if b (t) = $\overline{b(t-T_b)}$ then output of the EX-OR gate will be 1.

  $\therefore \; d(t) = 1 \quad \ldots \text{if } b(t) = \overline{b(t-T_b)}$

6.6.1 Advantages of DEPSK :

1. The DEPSK system uses the synchronous or coherent type of detection. Therefore the probability of error is reduced. It is less than the probability of error for the DPSK system because DPSK system uses the noncoherent type demodulation technique.
2. In DPSK demodulator, the delay generating device (T_b) has to operate at the carrier frequency but in DEPSK demodulator, the delay device operates at the baseband frequency f_b. This reduces the hardware cost of the delay device.

6.6.2 Disadvantages :

1. DEPSK needs a complex demodulator. The complexity of the receiver increases because the synchronous demodulation is being used.
2. The errors occur in pairs as explained below. Thus one error present in b (t) will give rise to two errors in d (t).

6.6.3 Errors in DEPSK System :

- We have seen that in DPSK there is a tendency for bit errors to occur in pairs but the single bit errors are also possible. However in DEPSK the errors will always occur in pairs. This is shown in Fig. 6.6.2.
- In Fig. 6.6.2 the signals b (k), b (k − 1) and d (k) = b (k) ⊕ b (k − 1) are error free signals, whereas signal b′(k) is the same signal b(k) with one error.
- Therefore its delayed version b′(k − 1) will also have one error. When b′(k) and b′ (k − 1) are added together (modulo-2 addition) in an EX-OR gate the resultant signal d′(k) has two errors as compared to the original error free signal d (k).

6.6.4 Comparison of BPSK and DPSK :

Sr. No.	Parameters	BPSK	DPSK
1.	Variable characteristics	Phase	Phase
2.	Bandwidth	f_b	f_b
3.	Error probability	Low	Higher than BPSK
4.	Complexity	Lower than DPSK	Higher than BPSK
5.	Detection method	Synchronous	Synchronous
6.	Effect of noise	Low	Higher than BPSK
7.	Need of synchronous carrier	Needed	Not needed
8.	Bit determination at the receiver	Based on single bit interval	Based on signal received in two successive bit intervals

6.7 Coherent Binary Frequency Shift Keying (BFSK) : SPPU : May 08, May 12

University Questions

Q. 1 With a neat diagram, explain how a coherent binary FSK wave can be generated and detected. **(May 08, 8 Marks)**

Q. 2 Explain BFSK transmitter and receiver with a proper sketch. What are the salient features of BFSK signal ? **(May 12, 8 Marks)**

- In "binary frequency shift keying (BFSK)", the frequency of a sinusoidal carrier is shifted between two discrete values.
- One of these frequencies (f_H) represents a binary "1" and the other value (f_L) represents a binary "0". Note that frequency f_H is higher than f_L.
- The representation of digital data using FSK is as shown in Fig. 6.7.1. Note that there is no change in the amplitude and phase of the sinusoidal carrier.
- In BFSK the binary data waveform d (t) is used to generate a binary signal,

$$V_{BFSK}(t) = \sqrt{2\,P_s}\cos[\,\omega_c t + d(t)\,\Omega t\,] \quad ...(6.7.1)$$

Where d (t) = ± 1 corresponding to the logic levels 1 or 0 at the input respectively.

- The transmitted BFSK signal is of constant amplitude $\sqrt{2\,P_s}$ and it is given by,

$$s_H(t) = V_{BFSK}(t) = \sqrt{2\,P_s}\cos(\omega_c + \Omega)\,t$$

... for logic "1" level input

$$\text{and} \quad s_L(t) = V_{BFSK}(t) = \sqrt{2\,P_s}\cos(\omega_c - \Omega)\,t$$

... for logic "0" level input.

If we call $(\omega_c + \Omega) = \omega_H$ and $(\omega_c - \Omega) = \omega_L$ then the above equations will be modified as,

$$s_H(t) = \sqrt{2\,P_s}\cos\omega_H t \quad ...(6.7.2)$$

$$\text{and} \quad s_L(t) = \sqrt{2\,P_s}\cos\omega_L t \quad ...(6.7.3)$$

- As already mentioned, $P_s = E_b / T_b$

$$\therefore \quad s_H(t) = \sqrt{\frac{2\,E_b}{T_b}}\cos\omega_H t \quad ...(6.7.4)$$

$$\text{and } s_L(t) = \sqrt{\frac{2\,E_b}{T_b}}\cos\omega_L t \quad ...(6.7.5)$$

6.7.1 Generation of BFSK :

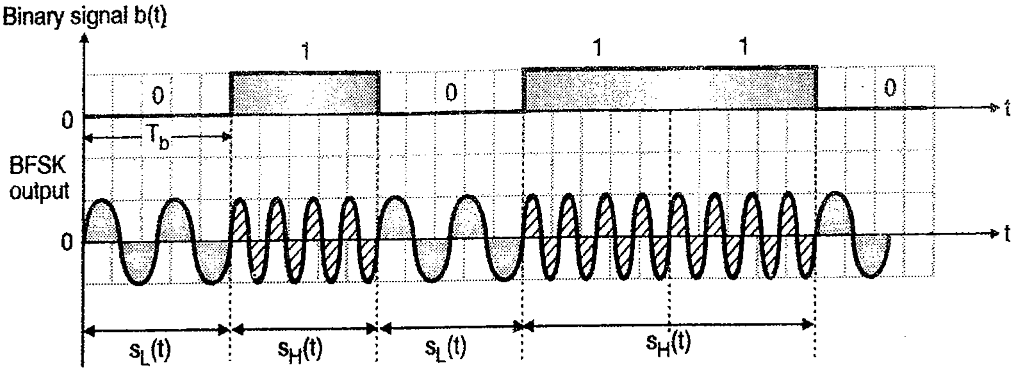
SPPU : May 08. May 11. May 12

University Questions

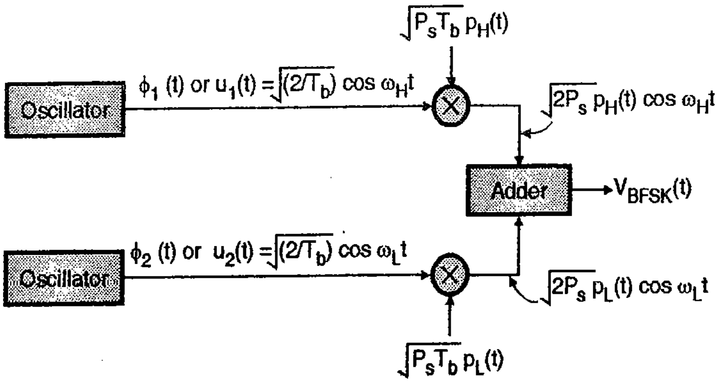
Q. 1 With a neat diagram, explain how a coherent binary FSK wave can be generated and detected. **(May 08, 8 Marks, May 11, 6 Marks)**

Q. 2 Explain BFSK transmitter and receiver with a proper sketch. What are the salient features of BFSK signal ? **(May 12, 8 Marks)**

- The block diagram of a BFSK generator is shown in Fig. 6.7.2. It consists of two oscillators which produce carriers at frequencies f_H and f_L respectively.



(E-408) **Fig. 6.7.1 : Representation of digital signal using BFSK**



(E-409) **Fig. 6.7.2 : BFSK generation**

- The oscillator outputs are applied to the inputs of the multipliers (balance modulator). The other input to the two multipliers are the signals $p_H(t)$ and $p_L(t)$.
- These signals are derived from the data bits d (t) as follows :

Data bit to be transmitted	Value of d (t)	$p_H(t)$	$p_L(t)$
Binary 0	– 1V	0	1
Binary 1	+ 1V	1	0

- The other inputs to the two multipliers are the reference signals $u_1(t)$ or $\phi_1(t)$ and $u_2(t)$ or $\phi_2(t)$ which are generated by the two oscillators.

$$u_1(t) = \phi_1(t) = \sqrt{2/T_b}\cos\omega_H t \quad ...(6.7.6)$$

$$\text{and } u_2(t) = \phi_2(t) = \sqrt{2/T_b}\cos\omega_L t \quad ...(6.7.7)$$

- The multiplier outputs are then added together to get the BFSK signal, given by Equation (6.7.1).
- Thus when a binary "0" is to be transmitted, $p_L(t) = 1$ and $p_H(t) = 0$ and for a binary "1" to be transmitted, $p_H(t) = 1$ and $p_L(t) = 0$.
- Hence the transmitted signal will have a frequency of either f_H or f_L.

6.7.2 Spectrum of BFSK :

SPPU : May 06, May 09, May 12

University Questions

Q. 1 Derive the expression for frequency spectrum of binary FSK signal and plot it.

(May 06, 8 Marks, May 09, 6 Marks)

Q. 2 Derive and draw the spectrum of BPSK, QPSK and BFSK signal and compare their bandwidths.

(May 12, 8 Marks)

- The BFSK output in terms of the variables $p_H(t)$ and $p_L(t)$ is given by,

$$V_{BFSK}(t)$$

$$= \sqrt{2P_s}\,p_H\cos(\omega_H t+\theta_H) + \sqrt{2P_s}\,p_L\cos(\omega_L t+\theta_L) \quad ...(6.7.8)$$

- Each term in the Equation (6.7.8) looks like the signal $\sqrt{2P_s}\,b(t)\cos\omega_c t$ which we have used in BPSK Equation (6.4.7). We have already obtained the spectrum for BPSK in section 6.4.7, but we cannot repeat that spectrum here because of an important difference between BPSK and BFSK.
- The difference is that in BPSK, the signal b (t) is bipolar which can have two values, either + 1 or – 1. In BFSK the two signals $p_H(t)$ and $p_L(t)$ are polar, and they change their values between 0 and + 1 only.
- So now let us write p_H and p_L in the form of sums of a constant and a bipolar variable as follows :

$$p_H(t) = \frac{1}{2} + \frac{1}{2}p_H'(t) \quad ...(6.7.9)$$

$$p_L(t) = \frac{1}{2} + \frac{1}{2}p_L'(t) \quad ...(6.7.10)$$

where $p'_H(t)$ and $p'_L(t)$ are bipolar variables. which alternate between + 1 and – 1. Also they are complementary signals i.e. when $p'_H = 1$, $p'_L = -1$ and vice versa.

- Substituting the values of $p_H(t)$ and $p_L(t)$ from Equations (6.7.9) and (6.7.10) into the expression for BFSK Equation (6.7.8) we get,

$$V_{BFSK}(t)$$

$$= \sqrt{\frac{P_s}{2}} \cos [\omega_H (t) + \theta_H] + \sqrt{\frac{P_s}{2}} \cos [\omega_L (t) + \theta_L]$$

$$+ \sqrt{\frac{P_s}{2}} \omega'_H \cos [\omega_H t + \theta_H] + \sqrt{\frac{P_s}{2}} \omega'_L \cos [\omega_L t + \theta_L] \quad ...(6.7.11)$$

This expression will help us to draw the spectrum of BFSK.

- The first term in Equation (6.7.11) will produce a power spectral density which will be centered about and consists of an impulse at f_H.
- Similarly the second term produces a power spectral density which is centered about and consists of an impulse at f_L.
- The third and the fourth terms in Equation (6.7.11), together will produce the spectrum of a BPSK (refer to Fig. 6.4.7(c) of section 6.4.7).
- The individual power spectral density patterns of the last two terms in Equation (6.7.11) are as shown in Fig. 6.7.3.
- Note that these patterns have been drawn with an assumption that $f_H - f_L = 2 f_b$.
- With this separation, the bandwidth of BFSK is given by,

$$\text{BW (BFSK)} = (f_H + f_b) - (f_L - f_b) = (f_H - f_L) + 2 f_b$$

$$\therefore \text{BW (BFSK)} = 4 f_b$$

Note that this bandwidth is twice the bandwidth of BPSK.

6.7.3 BFSK Receiver (Coherent Receiver) :

SPPU : May 08, May 11, May 12

University Questions

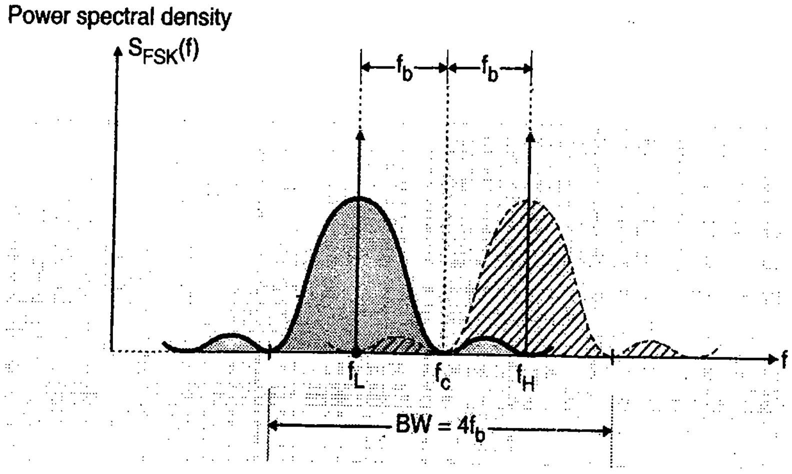
Q. 1 With a neat diagram, explain how a coherent binary FSK wave can be generated and detected. **(May 08, 8 Marks, May 11, 6 Marks)**

Q. 2 Explain BFSK transmitter and receiver with a proper sketch. What are the salient features of BFSK signal ? **(May 12, 8 Marks)**

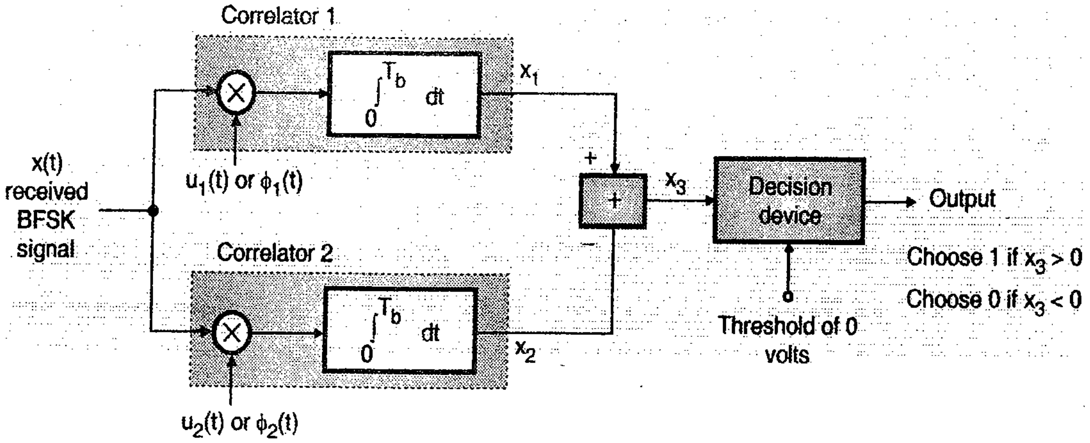
- The BFSK receiver block diagram is as shown in Fig. 6.7.4. It is supposed to regenerate the original digital data signal from the BFSK signal at its input.
- The received BFSK signal is denoted by x (t). It is applied to the two correlators.
- These correlators are supplied with locally generated coherent reference signals ϕ_1 (or u_1) and ϕ_2 (or u_2). These are the basis functions discussed earlier.

$$\phi_1(t) \text{ or } u_1(t) = \sqrt{2/T_b} \cos \omega_H t \quad ...(6.7.12)$$

$$\text{and } \phi_2(t) = u_2(t) = \sqrt{2/T_b} \cos \omega_L t \quad ...(6.7.13)$$



(E-410) **Fig. 6.7.3 : Spectrum of BFSK**



(E-411) **Fig. 6.7.4 : Coherent BFSK receiver**

- The outputs of the two correlator are then subtracted to get a signal x_3 such that,

$$x_3 = x_1 - x_2$$

- The signal x_3 is then applied to a decision device which compares it with a threshold level of zero volts.
- If $x_3 > 0$, then the receiver decides that a 1 was transmitted whereas if $x_3 < 0$ then the decision is that a 0 was transmitted.

6.7.4 Geometric Representation of Orthogonal BFSK :

SPPU : May 07, Dec. 08, May 11

University Questions

Q. 1 Draw the signal space representation for orthogonal and non-orthogonal BFSK signal. **(May 07, 6 Marks)**

Q. 2 Diagram the signal space representation of the following : Orthogonal BFSK, Non-Orthogonal BFSK, offset QPSK, Non-offset QPSK, 16-Ary PSK and 16 QAM. **(Dec. 08, 6 Marks)**

Q. 3 Diagram the geometric representation of :
(a) Orthogonal and non-orthogonal BFSK.
State the Euclidean distance of above mentioned systems by explaining the importance of Euclidean distance. **(May 11, 3 Marks)**

- From Fig. 6.7.5 it is clear that we are using two carriers $u_1(t)$ and $u_2(t)$ having two different frequencies f_H and f_L. These carriers are being used for modulation.
- For orthogonal BFSK, these two carriers should be orthogonal. In order to make $u_1(t)$ and $u_2(t)$ orthogonal, the frequencies f_H and f_L should be integer multiple of frequency f_b (where $f_b = 1/T_b$).

That means,
$$f_H = m f_b \quad ...(6.7.14)$$
and
$$f_L = n f_b \quad ...(6.7.15)$$

- Therefore the two carriers $u_1(t)$ and $u_2(t)$ are given by,

$$u_1(t) = \sqrt{2/T_b}\,\cos(2\pi m f_b t) \quad ...(6.7.16)$$

and
$$u_2(t) = \sqrt{2/T_b}\,\sin(2\pi n f_b t) \quad ...(6.7.17)$$

where m and n are integers.

- Therefore Equations (6.7.2) and (6.7.3) can be modified as :

$$s_H(t) = \sqrt{P_s T_b}\,\sqrt{2/T_b}\,\cos(2\pi m f_b t) \quad ...(6.7.18)$$

and
$$s_L(t) = \sqrt{P_s T_b}\,\sqrt{2/T_b}\,\sin(2\pi n f_b t) \quad ...(6.7.19)$$

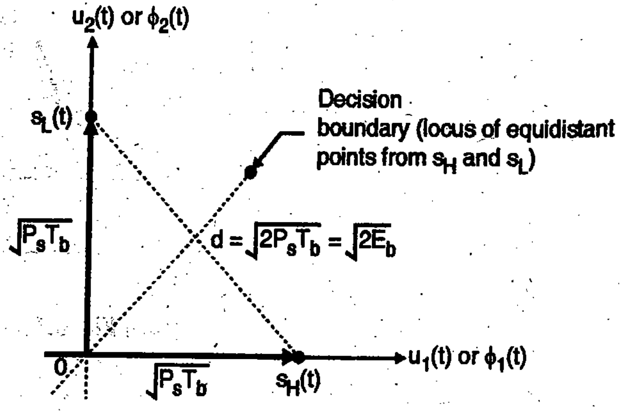
where $2\pi m f_b = \omega_H$ and $2\pi n f_b = \omega_L$

- Equations (6.7.16) and (6.7.17) can be represented in the form of $u_1(t)$ or $\phi_1(t)$ and $u_2(t)$ or $\phi_2(t)$ as follows :

$$s_H(t) = \sqrt{P_s T_b} \times u_1(t) \quad ...(6.7.20)$$

and
$$s_L(t) = \sqrt{P_s T_b} \times u_2(t) \quad ...(6.7.21)$$

- Using these equations, we can plot the signal space representation of the orthogonal BFSK as shown in Fig. 6.7.5.



(E-412) **Fig. 6.7.5 : Signal space representation of orthogonal BFSK signals**

- The Eucledean distance for the orthogonal BFSK is $d = \sqrt{2E_b}$.

6.7.5 Geometric Representation of Non-orthogonal BFSK Signals :

SPPU : May 07, Dec. 08, May 11

University Questions

Q. 1 Draw the signal space representation for orthogonal and non-orthogonal BFSK signal. **(May 07, 6 Marks)**

Q. 2 Diagram the signal space representation of the following : Orthogonal BFSK, Non-Orthogonal BFSK, offset QPSK, Non-offset QPSK, 16-Ary PSK and 16 QAM. **(Dec. 08, 6 Marks)**

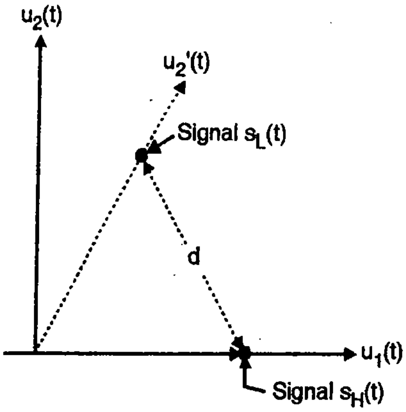
Q. 3 Diagram the geometric representation of :

(a) Orthogonal and non-orthogonal BFSK.

State the Euclidean distance of above mentioned systems by explaining the importance of Euclidean distance. **(May 11, 3 Marks)**

- If the two carriers $u_1(t)$ and $u_2(t)$ expressed by Equations (6.7.12) and (6.7.13) are non-orthogonal, then the signal points $s_H(t)$ and $s_L(t)$ will not lie on the axes $u_1(t)$ and $u_2(t)$. Therefore the signal space representation of non-orthogonal BFSK is as shown in Fig. 6.7.6.
- The expression for the Euclidean distance "d" is given by,

$$d^2 \approx 2E_b \left[1 - \frac{\sin(\omega_H - \omega_L) T_b}{(\omega_H - \omega_L) T_b} \right] \qquad ...(6.7.22)$$



(E-413) **Fig. 6.7.6 : Geometric representation of non-orthogonal BPSK**

Ex. 6.7.1 : In a coherent FSK system, the signals $s_1(t)$ and $s_2(t)$ representing symbols 1 and 0, respectively, are defined by

$$s_1(t), s_2(t) = A_c \cos\left[2\pi \left(f_c \pm \frac{\Delta f}{2}\right) t\right]$$

$0 \le t \le T_b$.

Assuming that $f_c > \Delta f$, show that the correlation coefficient of the signals $s_1(t)$ and $s_2(t)$, is approximately given by,

$$\frac{\int_0^{T_b} s_1(t)\, s_2(t)\, dt}{\int_0^{T_b} s_1^2(t)\, dt} = \text{sinc}(2\Delta f T_b)$$

Soln. :

The correlation coefficient of $s_1(t)$ and $s_2(t)$ is given by

$$\rho = \frac{\int_0^{T_b} s_1(t) \times s_2(t)\, dt}{\left[\int_0^{T_b} s_1^2(t)\, dt\right]^{1/2} \left[\int_0^{T_b} s_2^2(t)\, dt\right]^{1/2}}$$

Substituting the values of $s_1(t)$ and $s_2(t)$ we get

$$\rho = \frac{A_c^2 \int_0^{T_b} \cos\left[2\pi(f_c + \frac{\Delta f}{2}) t\right] \cos\left[2\pi(f_c - \frac{\Delta f}{2}) t\right] dt}{\left[\int_0^{T_b} A_c^2 \cos^2[2\pi(f_c + \frac{\Delta f}{2}) t\, dt\right]^{1/2} \left[\int_0^{T_b} A_c^2 \cos^2[2\pi(f_c - \frac{\Delta f}{2}) t\, dt\right]^{1/2}}$$

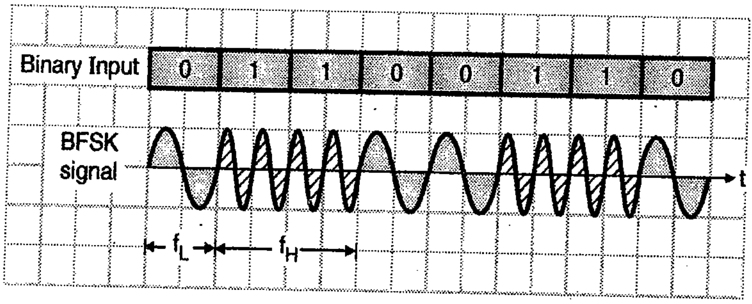
Using the standard identity

$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$ ... for the numerator

and $\cos^2\theta = \frac{\cos 2\theta + 1}{2}$ ... for the denominator

$$\text{We get } \rho = \frac{\frac{A_c^2}{2}\int_0^{T_b}[\cos(4\pi f_c)t + \cos(2\pi\Delta f)t]\,dt}{\left\{\frac{A_c^2}{2}\int_0^{T_b}[\cos 4\pi(f_c + \frac{\Delta f}{2})t + 1]\,dt\right\}^{1/2}\left\{\frac{A_c^2}{2}\int_0^{T_b}[\cos 4\pi(f_c - \frac{\Delta f}{2})t + 1]\,dt\right\}^{1/2}}$$

$$= \frac{A_c^2}{2}\,\frac{\int_0^{T_b}[\cos(4\pi f_c t) + \cos(2\pi\Delta f t)]\,dt}{\left[\frac{A_c^2 T_b}{2}\right]^{1/2}\left[\frac{A_c^2 T_b}{2}\right]^{1/2}} = \frac{1}{T_b}\int_0^{T_b}[\cos(2\pi\Delta f_c t) + \cos(4\pi f_c t)]\,dt$$



(E-414) **Fig. P. 6.7.2 : BFSK waveform**

Integrating we get,

$$\rho = \frac{1}{T_b}\left\{\frac{\sin(2\pi\Delta f_c T_b)}{2\pi\Delta f_o} + \frac{\sin(4\pi f_c T_b)}{4\pi f_c t}\right\}$$

As $f_c >> \Delta f$, the second term in the expression above can be neglected.

$$\therefore \quad \rho = 1 \times \frac{\sin(2\pi\Delta f_c T_b)}{(2\pi\Delta f_c T_b)}$$

$$\therefore \quad \rho = \text{sinc}(2\Delta f_c T_b) \qquad \textbf{...Proved.}$$

Ex. 6.7.2 : For the following binary sequence draw the BFSK waveform :

0 1 1 0 0 1 1 0

Soln. : The BFSK waveform is as shown in Fig. P. 6.7.2. Note that there is no change in amplitude as well as in phase.

6.7.6 Difference between Orthogonal and Non-orthogonal BFSK :

Sr. No.	Orthogonal BFSK	Non-orthogonal BFSK
1.	The frequencies f_H and f_L are the integral multiples of f_b.	f_H and f_L are not the integral multiples of f_b.
2.	The signals $s_H(t)$ and $s_L(t)$ lie on the axes $u_1(t)$ and $u_2(t)$ in the signal space representation.	The signal $s_H(t)$ and $s_L(t)$ do not lie on the axes $u_1(t)$ and $u_2(t)$.
3.	The Euclidean distance "d" is more than that for non-orthogonal BFSK.	The Euclidean distance "d" is less than that for orthogonal BFSK.
4.	Error probability P_e is less than that for the non-orthogonal BFSK system.	Error probability P_e is higher than that for the non-orthogonal BFSK.

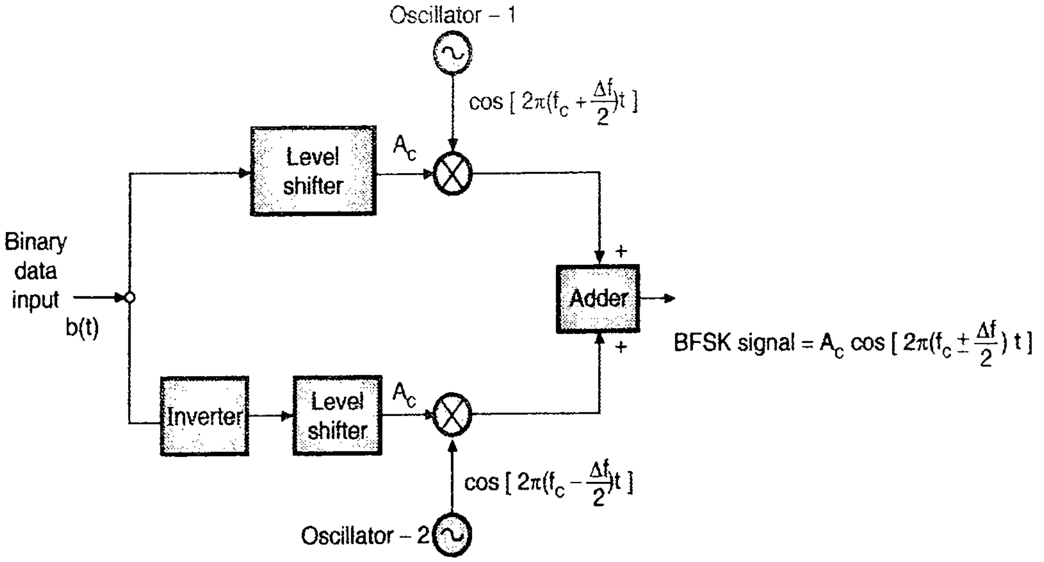
Ex. 6.7.3 : Realise a system (transmitter and receiver) with signals of the form :

$$s_1(t)s_2(t) = A_c \cos\left[2\pi\left(f_c \pm \frac{\Delta f}{2}\right)t\right] \quad 0 \le t \le T_b.$$

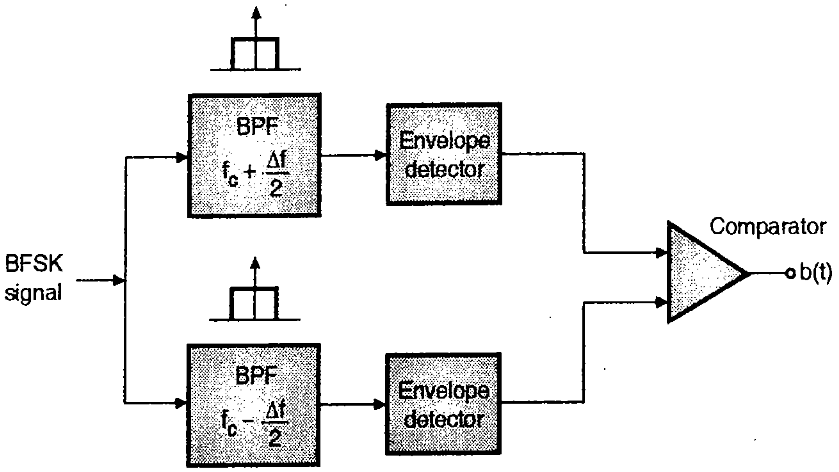
Is it a coherent system ? Justify.

Soln. :

The system represented by the given equation is a BFSK system. The transmitted of this system is shown in ? Fig. P. 6.7.3(a) and its receiver is shown in Fig. P. 6.7.3(b).



(E-858) **Fig. P. 6.7.3(a) : Transmitter**



(E-859) **Fig. P. 6.7.3(b) : Receiver**

Transmitter :

- The level shifter shifts the binary voltage into A_c. Due to the use of inverter at any instant output of only one multiplier will be present. Hence at the adder output we will get output of only one oscillator at a time. Thus we obtain the BFSK at the transmitter output.

Receiver :

- The receiver for BFSK is shown in Fig. P. 6.7.3(b). It is a non-coherent type receiver.

The operation of the receiver is as explained in section 6.7.1.

6.7.7 Bit Error Probability of Coherently Detected BFSK :

SPPU : Dec. 07. May 08. May 09. May 11

University Questions

Q. 1 Derive the error probability expression for BPSK and BFSK.

(Dec. 07, 10 Marks, May 08, May 09, May 11, 8 Marks)

- Here also we are going to assume **that a matched filter is being used for the detection of BFSK signal.** In BFSK (binary frequency shift keying, the received signal is as follows :

Binary 1 : $x_1(t) = A \cos(\omega_c + \Omega) t$

Binary 0 : $x_2(t) = A \cos(\omega_c - \Omega) t$

- We know that it is possible to synthesizing a matched filter with the help of a correlation receiver system, because the correlation receiver will give exactly the same performance as a matched filter provided that the locally generated waveform is $[x_1(t) - x_2(t)]$.

∴ Local signal : $x_1(t) - x_2(t)$

$$= A \cos(\omega_c + \Omega) t - A \cos(\omega_c - \Omega) t \quad ...(6.7.23)$$

But $A = \sqrt{2 P_s}$

$$\therefore \; x_1(t) - x_2(t) = \sqrt{2 P_s} \, [\cos(\omega_c + \Omega) t - \cos(\omega_c - \Omega) t] \quad ...(6.7.24)$$

- We know that the error probability of a matched filter is given by,

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{x_{o1}(T) - x_{o2}(T)}{2\sqrt{2}\,\sigma}\right] \quad ...(6.7.25)$$

- And for a matched filter detection, the maximum signal to noise ratio is given by,

$$\left[\frac{x_{o1}(T) - x_{o2}(T)}{\sigma}\right]^2_{\max} = \frac{2}{N_0}\int_0^T x^2(t)\,dt \quad ...(6.7.26)$$

But $x(t) = x_1(t) - x_2(t)$

$$= \sqrt{2P_s}[\cos(\omega_c + \Omega)\,t - \cos(\omega_c - \Omega)\,t]$$

...from Equation (6.7.24)

- Substituting this value of x(t) into Equation (6.7.26) we get,

$$\left[\frac{x_{o1}(T) - x_{o2}(T)}{\sigma}\right]^2_{\max}$$

$$= \frac{2}{N_0}\int_0^T \left\{\sqrt{2P_s}\,[\cos(\omega_c + \Omega)\,t - \cos(\omega_c - \Omega)\,t]\right\}^2 dt \quad ...(6.7.27)$$

But we do not have the value of $x^2(t)$. i.e. the RHS of the above equation.

- Therefore we will obtain the value of $x^2(t)$, as follows :

$$x^2(t) = \left\{\sqrt{2P_s}\,[\cos(\omega_c + \Omega)\,t - \cos(\omega_c - \Omega)\,t]\right\}^2$$

$$= 2P_s[\cos(\omega_c + \Omega)\,t - \cos(\omega_c - \Omega)\,t]^2 \quad ...(6.7.28)$$

- Let us use the following standard trigonometric identity which states that,

$$\cos(A + B) - \cos(A - B) = -2\sin A \sin B$$

- Apply this identity to Equation (6.7.28) to get,

$$x^2(t) = 2P_s[-2\sin(\omega_c t)\sin(\Omega t)]^2$$

$$= 2P_s[4\sin^2(\omega_c t)\sin^2(\Omega t)]$$

$$\therefore\ x^2(t) = 2P_s[2\sin^2(\omega_c t) \times 2\sin^2(\Omega t)] \quad ...(6.7.29)$$

We know that $2\sin^2\theta = 1 - \cos 2\theta$

$$\therefore\ x^2(t) = 2P_s[(1 - \cos 2\omega_c t)(1 - \cos 2\Omega t)]$$

$$= 2P_s[1 - \cos 2\Omega t - \cos 2\omega_c t + \cos 2\omega_c t \cos 2\Omega t)] \quad ...(6.7.30)$$

- In this Equation use : $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$ to get,

$$x^2(t) = 2P_s\left[1 - \cos 2\Omega t - \cos 2\omega_c t + \frac{1}{2}\cos 2(\omega_c - \Omega)\,t + \frac{1}{2}\cos 2(\omega_c + \Omega)\,t\right] \quad ...(6.7.31)$$

Thus we have obtained the expression for $x^2(t)$.

Obtain the value of $\int_0^T x^2(t)\,dt$:

Take integration of both the sides of Equation (6.7.31) to get,

$$\int_0^T x^2(t)\,dt = \int_0^T 2P_s\left\{1 - \cos 2\Omega t - \cos 2\omega_c t + \frac{1}{2}[\cos 2(\omega_c - \Omega)\,t + \cos 2(\omega_c + \Omega)\,t]\right\} dt$$

$$= 2P_s\left\{\int_0^T 1\,dt - \int_0^T \cos 2\Omega t\,dt - \int_0^T \cos 2\omega_c t\,dt + \frac{1}{2}\int_0^T \cos 2(\omega_c - \Omega)\,t\,dt + \frac{1}{2}\int_0^T \cos 2(\omega_c + \Omega)\,t\,dt\right\}$$

$$= 2P_s \quad ...(6.7.32)$$

$$\therefore\ \int_0^T x^2(t)\,dt = 2P_s T\left\{1 - \frac{\sin 2\Omega T}{2\Omega T} - \frac{\sin 2\omega_c T}{2\omega_c T} + \frac{1}{2}\,\frac{\sin 2(\omega_c - \Omega)\,T}{2(\omega_c - \Omega)\,T} + \frac{1}{2}\,\frac{\sin 2(\omega_c + \Omega)\,T}{2(\omega_c + \Omega)T}\right\} \quad ...(6.7.33)$$

- If we assume that the offset angular frequency Ω is very small as compared to the carrier angular frequency ω_c, then the last three terms in Equation (6.7.33) each will have a form $(\sin 2\,\omega_c\,T)/2\,\omega_c\,T$. This ratio approaches zero as the value of $\omega_c\,T$ increases.
- Generally $\omega_c\,T >> 1$ therefore the last three terms in RHS of Equation (6.7.33) can be neglected. Therefore Equation (6.7.33) gets modified as follows :

$$\int_0^T x^2(t)\,dt = 2\,P_s\,T\left[1-\frac{\sin 2\,\Omega\,T}{2\,\Omega\,T}\right] \quad ...(6.7.34)$$

- Thus we have obtained the expression for $\int_0^T x^2(t)\,dt$.

  Now substitute it into Equation (6.7.26) to get,

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{\max} = \frac{2}{N_0}\times 2\,P_s\,T\left[1-\frac{\sin 2\,\Omega\,T}{2\,\Omega\,T}\right]$$

$$= \frac{4\,P_s\,T}{N_0}\left[1-\frac{\sin 2\,\Omega\,T}{2\,\Omega\,T}\right] \quad ...(6.7.35)$$

- The value of the quantity $\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{\max}$ in Equation (6.7.35) attains its largest value when Ω is selected so that $2\,\Omega\,T = \frac{3\,\pi}{2}$. Substitute this value of Ω into Equation (6.7.35) to get,

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{\max} = \frac{4\,P_s\,T}{N_0}\left[1-\frac{\sin(3\,\pi/2)}{(3\,\pi/2)}\right]$$

$$= \frac{4\,P_s\,T}{N_0}\left[1-\frac{(-1)}{(3\,\pi/2)}\right]$$

$$= 4.84\,\frac{P_s\,T}{N_0} \quad ...(6.7.36)$$

- Taking square root of both the sides we get,

$$\therefore \left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]_{\max} = \sqrt{\frac{4.84\,P_s\,T}{N_0}} \quad ...(6.7.37)$$

- Substitute this value of $\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]_{\max}$ into Equation (6.7.37) to obtain the error probability as :

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{1}{2\sqrt{2}}\cdot\sqrt{\frac{4.84\,P_s\,T}{N_0}}\right]$$

$$\therefore\ P_e = \frac{1}{2}\,\text{erfc}\left[\sqrt{\frac{0.6\,P_s\,T}{N_0}}\right]$$

- But $P_s\,T$ = Energy E of one bit.

$$\therefore\ P_e = \frac{1}{2}\,\text{erfc}\left[\sqrt{\frac{0.6\,E}{N_0}}\right] \quad ...(6.7.38)$$

- This is the required expression for bit error probability of BFSK when matched filter is used for detection (P_B).
- Compare this equation with the probability of error obtained for BPSK. As the "erfc" is a monotonic decreasing function, the error probability for BFSK system is higher than that of a BPSK system. This happens because in BPSK, $x_2(t) = -\,x_1(t)$ and in BFSK this condition is not satisfied. Thus the noise performance of BFSK system is inferior to that of the BPSK system.

6.7.8 Comparison of BASK, BFSK, BPSK :

Table 6.7.1

Sr. No.	Parameter	Binary ASK	Binary FSK	Binary PSK
1.	Variable characteristic	Amplitude	Frequency	Phase
2.	Bandwidth (Hz) (spectral efficiency)	$2\,f_b$	$4\,f_b$	$2\,f_b$
3.	Noise immunity	low	high	high
4.	Error probability	high	low	low
5.	Performance in presence of noise	Bad	Better than ASK	Better than FSK
6.	Complexity	Simple	Moderately complex	Very complex
7.	Bit rate	Suitable upto 100 bits/sec.	Suitable upto about 1200 bits/sec.	Suitable for high bit rates
8.	Detection method	Envelope	Envelope	Coherent

6.8 Coherent Quadrature Modulation Techniques :

- There are three main aims of a digital communication system as follows :
  1. To provide a reliable performance.
  2. To reduce the probability of error.
  3. To utilize the channel bandwidth efficiently.
- All these requirements are satisfied by the coherent quadrature modulation techniques.

- Two such techniques will be discussed in this section. The techniques are :
  1. Quadrature phase shift keying (QPSK)
  2. Minimum shift keying (MSK).

6.9 Quadrature Phase Shift Keying (QPSK) :

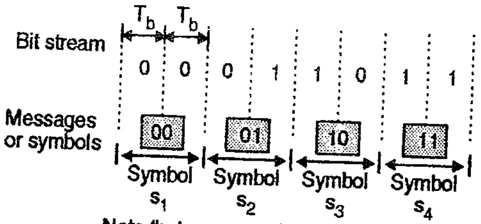
SPPU : May 06, Dec. 07, May 08, Dec. 16

University Questions

Q. 1 Give mathematical representation of QPSK signal. Draw the signal space diagram of QPSK signal. Write the expression of all the message points in the diagram. **(May 06, May 08, Dec. 16, 8 Marks)**

Q. 2 Diagram the geometric representation of orthogonal QPSK, non- orthogonal QPSK, M-ary PSK, M-ary FSK and QASK. What is the importance of Euclidean distance ? Write its expression for above representation and compare them. **(Dec. 07, 10 Marks)**

- The modulation schemes discussed so far are all two level modulation. (ASK and BPSK), because they can represent only two states of the digital data (0 or 1).
- Therefore the bit rate and the baud rate are same for these systems. The maximum bit rate which can be achieved using ASK, BFSK or BPSK systems does not meet the requirements of data communication systems.
- This happens due to the limited bandwidth of the telephone voice channel.
- We can keep the baud rate same and increase the bit rate by using multilevel modulation techniques.
- In this type of systems, the data groups are divided into groups of two or more bits and each group of bits is represented by a specific value of amplitude, frequency or phase the carrier.
- QPSK (Quadrature PSK) is an example of such multilevel phase modulation.
- In QPSK system two successive bits in a bit stream are grouped together to form a message and each message is represented by a distinct value of phase shift of the carrier.
- The symbols and corresponding phase shifts are shown in Fig. 6.9.1.



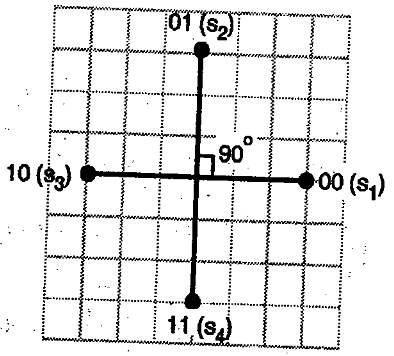
Note that every symbol consists of two bits

(E-381) **Fig. 6.9.1 : Grouping of bits in QPSK**

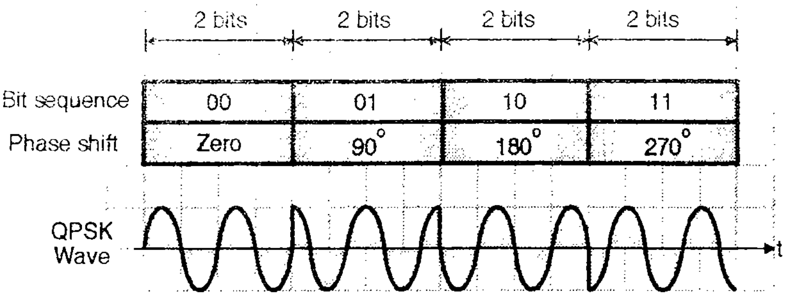
- Each symbol or message contains two bits. So the symbol duration $T_s = 2\ T_b$.
- These symbols are transmitted by transmitting the same carrier frequency at four different phase shifts as shown in Table 6.9.1 and Fig. 6.9.2.
- Since there are four phase shifts involved, this system is called as quadrature PSK or 4-PSK system.
- If the symbol 00 is to be transmitted then we have to transmit a carrier at 0° phase shift. If 01 is to be transmitted, then the same carrier is transmitted with a phase shift of 90°.
- Similarly the message 10 and 11 are transmitted by transmitting the carrier at 180° and 270° respectively.
- This concept will be clear after referring to the QPSK waveform of Fig. 6.9.3.

Table 6.9.1 : Phase shift in QPSK

Symbol	Phase
00 (s_1)	0
01 (s_2)	90
10 (s_3)	180
11 (s_4)	270

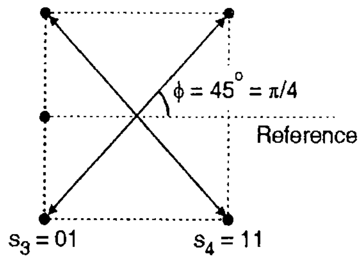


(E-382) **Fig. 6.9.2 : Constellation diagram of QPSK**



(E-383) Fig. 6.9.3 : Waveforms of QPSK

Table 6.9.2

Sr. No.	Symbol	Input successive bits		Phase shift in the carrier	
		A	B		
1.	s_1	1	0	$\pi/4$ rad	
2.	s_2	0	0	$3\pi/4$ rad	
3.	s_3	0	1	$5\pi/4$ rad	
4.	s_4	1	1	$7\pi/4$ rad.	

(E-384) Fig. 6.9.4

Alternative representation :

An alternate representation of QPSK symbols is shown in Table 6.9.2 and Fig. 6.9.4.

Mathematical representation of QPSK :

- A QPSK signal can be represented mathematically as :

$$V_{QPSK}(t) = \sqrt{2P_s}\cos\left[\omega_c t + (2m+1)\frac{\pi}{4}\right], m = 0, 1, 2, 3$$

- By substituting the values of m from 0 to 3 we get the four messages listed in Table 6.9.2.

$$\text{i.e. } V_{QPSK} = s_1 = \sqrt{2P_s}\cos\left[\omega_c t + \frac{\pi}{4}\right] \quad \text{... for } m = 0$$

$$V_{QPSK} = s_2 = \sqrt{2P_s}\cos\left[\omega_c t + \frac{3\pi}{4}\right] \quad \text{... for } m = 1$$

- Similarly we can obtain the QPSK output for m = 2 and m = 3.
- As explained earlier we can substitute P_s in terms of symbol energy and symbol time duration as:

$$P_s = E/T$$

- The QPSK system of modulation is also called as four state PSK (or simply 4 PSK).

6.9.1 Offset QPSK (OQPSK) or Staggered QPSK Transmitter :

SPPU : Dec. 06, Dec. 08, Dec. 10, May 11

University Questions

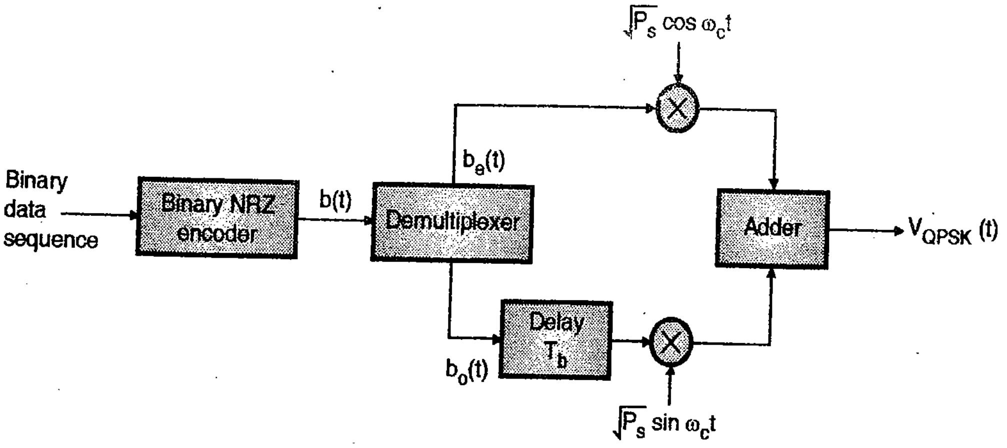
Q. 1 With mathematical expression and block diagram, explain the operation of offset QPSK. Also express the bandwidth requirement. **(Dec. 06, 10 Marks)**

Q. 2 Diagram the signal space representation of the following : Orthogonal BFSK, Non-Orthogonal BFSK, offset QPSK, Non-offset QPSK, 16-Ary PSK and 16 QAM. **(Dec. 08, 6 Marks)**

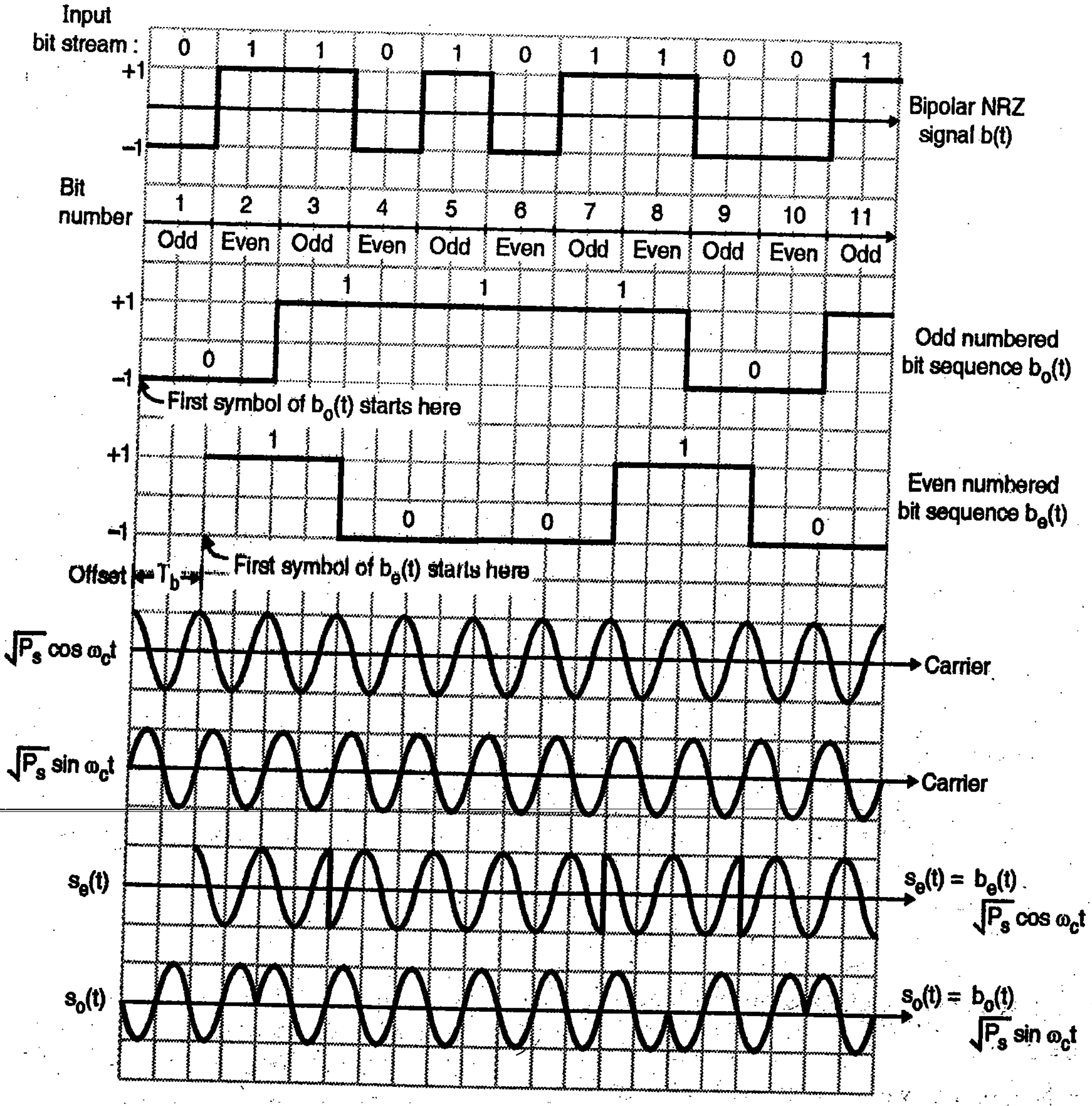
Q. 3 Write signal expression for QPSK. Draw the block diagram of QPSK transmitter and receiver and explain the working. **(Dec. 10, 8 Marks)**

Q. 4 Draw the block diagram and with the help of mathematical expression explain in detail the QPSK transmitter and receiver. Diagram the geometric representation and draw its power spectral density, along with its expression thereby comment on its Euclidean distance and bandwidth. **(May 11, 10 Marks)**

- The block diagram of offset QPSK (OQPSK) transmitter is shown in Fig. 6.9.5. This shows the mechanism by which a bit stream b (t) generates a QPSK signal for transmission.



(E-385) **Fig. 6.9.5 : An offset QPSK transmitter**



(E-386) **Fig. 6.9.6 : Waveforms for OQPSK transmitter**

Operation of the OQPSK transmitter :

- The input binary data sequence is first converted into a bipolar NRZ signal, b (t). The value of b (t) = +1V for a logic 1 input and b (t) = –1V when the binary input is equal to 0.

Binary input	NRZ signal b (t)
0	–1V
1	+1V

- The demultiplexer will divide b (t) into two separate bit streams named $b_o(t)$ and $b_e(t)$. The bit stream $b_e(t)$ consists of only the even numbered bits i.e. bit numbers 2, 4, 6, ... as shown in Fig. 6.9.6, whereas the $b_o(t)$ bit stream consists of only the odd numbered bits i.e. bit numbers 1, 3, 5, ... as shown in Fig. 6.9.6.
- Each bit in the even or odd bit stream will be held for a period of 2 T_b seconds. This duration is called as "symbol duration" T_s . This is because every symbol contains two bits.

$$\therefore T_s = 2T_b$$

- As shown in Fig. 6.9.6, the first odd bit (bit 1) will occur before the first even bit (bit 2). Therefore the even bit stream $b_e(t)$ will start with a delay of one bit period after the first odd bit. This delay is equal to one bit period T_b as shown in Fig. 6.9.6. This delay is called as "offset" and therefore the name of this system is offset QPSK. This offset has been introduced so that, the bit streams $b_o(t)$ and $b_e(t)$ cannot change their levels at the same instant of time.
- The bit streams $b_e(t) = \pm 1$ V is superimposed on a carrier $\sqrt{P_s}\cos\omega_c t$ and the bit stream $b_o(t) = \pm 1$ V is superimposed on a carrier $\sqrt{P_s}$ $\sin\omega_c t$, by means of using two multipliers (i.e. balanced modulators), to generate two signal $s_e(t)$ and $s_o(t)$ respectively. These two signals are basically BPSK signals.
- The multiplier outputs are then added together to generate the QPSK output signal $V_{QPSK}(t)$, as shown in Fig. 6.9.5 and it is expressed mathematically as follows :

$$\therefore V_{QPSK}(t) = \sqrt{P_s}\, b_o(t) \sin\omega_c(t) + \sqrt{P_s}\, b_e(t) \cos\omega_c t \quad ...(6.9.1)$$

Important conclusions :

The important conclusions from the discussion till now are as follows :

1. When $b_o(t) = 1$ i.e. representing a logic 1 then $s_o(t) = \sqrt{P_s}\sin\omega_c t$ and $s_o(t) = -\sqrt{P_s}\sin\omega_c t$ when $b_o(t) = -1$ which represents a logic 0.
2. Similarly for $b_e(t) = \pm 1$, $s_e(t) = \pm\sqrt{P_s}\cos\omega_c t$. These four signals are represented by phasors in Fig. 6.9.7 and as seen, they are in phase quadrature.
3. In Fig. 6.9.7, we have drawn the phasors corresponding to the $V_{QPSK}(t)$ where :

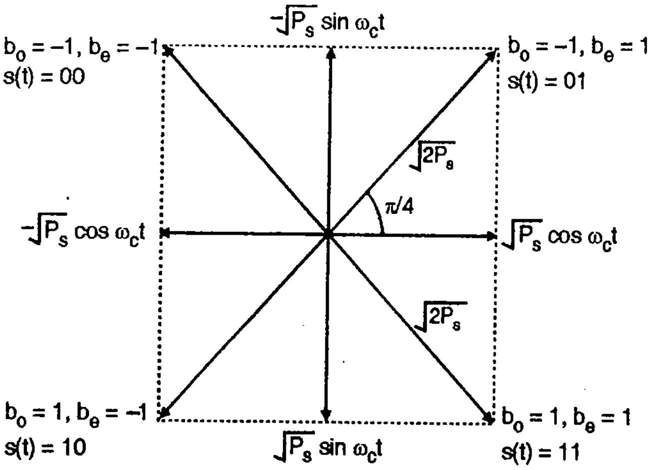
   $V_{QPSK}(t) = s_o(t) + s_e(t)$

   These four possible output signals are of equal amplitude $\sqrt{2P_s}$ and are in phase quadrature. We can identify them by their corresponding values of $b_o(t)$ and $b_e(t)$, as shown in Table 6.9.3.

4. At the end of each bit interval i.e. after a period T_b, either b_o or b_e will change but both of them will not change simultaneously due to the presence of offset. Therefore the QPSK system shown in Fig. 6.9.3 is called as offset QPSK or staggered QPSK. Therefore after every time period T_b, the transmitted signal, if it changes, changes phase by 90° rather than by 180° in BPSK.

Table 6.9.3

Symbol $s(t)$	$b_o(t)$	$b_e(t)$	V_{QPSK}	Quadrant
01	–1	1	$-\sqrt{P_s}\sin\omega_c t$ $+\sqrt{P_s}\cos\omega_c t$	I
00	–1	–1	$-\sqrt{P_s}\sin\omega_c t$ $-\sqrt{P_s}\cos\omega_c t$	II
10	+1	–1	$\sqrt{P_s}\sin\omega_c t -$ $\sqrt{P_s}\cos\omega_c t$	III
11	+1	+1	$\sqrt{P_s}\sin\omega_c t +$ $\sqrt{P_s}\cos\omega_c t$	IV



(E-387) **Fig. 6.9.7 : Phasor diagram**

Symbol transmission rate :

In QPSK two bits are grouped together to form a symbol. Therefore when the symbols are transmitted, the signal changes occur at the symbol rate which is half the bit rate.

$$\therefore \quad \text{The symbol time } T_s = 2T_b \quad ...(6.9.2)$$

6.9.2 Non-offset QPSK :

SPPU : Dec. 08. Dec. 10

University Questions

Q. 1 Diagram the signal space representation of the following : Orthogonal BFSK, Non-Orthogonal BFSK, offset QPSK, Non-offset QPSK, 16-Ary PSK and 16 QAM. **(Dec. 08, 6 Marks)**

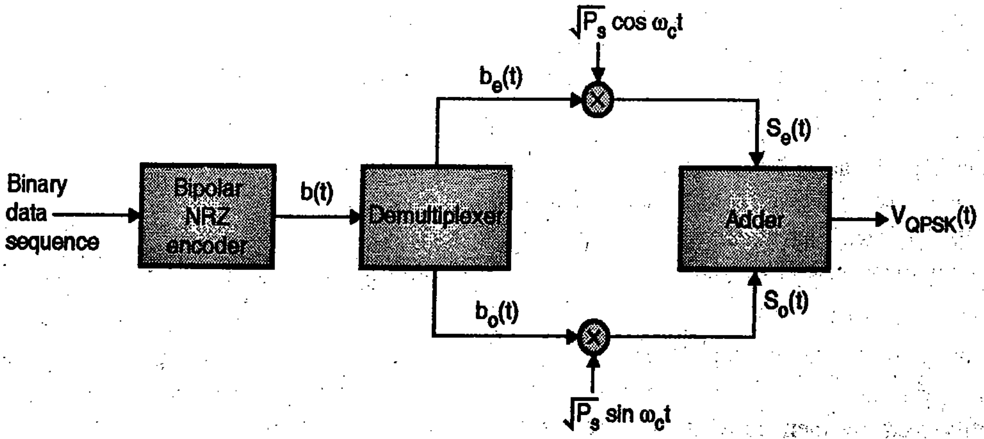
Q. 2 Write signal expression for QPSK. Draw the block diagram of QPSK transmitter and receiver and explain the working. **(Dec. 10, 8 Marks)**

- The block diagram of non-offset QPSK transmitter is shown in Fig. 6.9.8.
- The transmitter of the non-offset QPSK is exactly same as that of the offset QPSK except for one change. The odd bit stream $b_o(t)$ has been delayed by one bit period T_b, by adding a delay block as shown in Fig. 6.9.8. The non-offset QPSK is simply called as QPSK.
- As a result of this additional time delay, we shall find that two bits which occur in time sequence (i.e. serially) in the input bit stream b (t), will appear at the same time (i.e. in parallel) in the outputs of the even and odd flip-flops.
- Therefore $b_o(t)$ and $b_e(t)$ can change at the same time, after every 2 T_b seconds and there can be a phase change of 180° in the output signal. Therefore this system is called as non-offset QPSK system. Otherwise offset QPSK and non-offset QPSK operate on the same principle.
- In QPSK as well as in OQPSK there will be brief variations in the amplitude of the transmitted waveform. These variations take place because the bits $b_o(t)$ and $b_e(t)$ cannot change instantaneously when the phase of the transmitted signal is being changed.
- These amplitude changes are more evident in the QPSK system than the OQPSK system, because in case of QPSK, both $b_o(t)$ and $b_e(t)$ can be zero simultaneously and the signal amplitude may be reduced to zero temporarily.
- The amplitude variations in the transmitted signal can be due to one more reason. In QPSK similar to BPSK a filter is used to suppress the sidebands. When the waveforms which exhibit abrupt phase changes are applied to a filter, the amplitude variations in the waveforms is observed at the instants of abrupt phase changes. Thus larger amplitude variations are observed for the QPSK signal as the phase shift involved is 180°.
- These amplitude variations can pose problems in QPSK communication systems which use repeaters. The repeaters generally make use of output power stages which operate nonlinearly.
- Due to this nonlinearity, the amplitude variations will result into generation of spectral components outside the range of the main lobe. This will negate away the effect of band limiting filter and results in the interchannel interference.
- Further filtering to suppress the effect of amplitude variation has an adverse effect on the phase of the signal. **And do not forget that it is the phase which carries information.**

Symbol transmission rate :

- In QPSK two bits are grouped together to form a symbol. Therefore when the symbols are transmitted, the signal changes occur at the symbol rate which is half the bit rate.

$\therefore$ The symbol time $T_s = 2\,T_b$ ...(6.9.3)



(E-388) **Fig. 6.9.8 : Block diagram of non-offset QPSK transmitter**

6.9.3 The QPSK Receiver :

SPPU - Dec. 10, May 11, Dec. 12, Dec. 15

University Questions

Q. 1 Write signal expression for QPSK. Draw the block diagram of QPSK transmitter and receiver and explain the working. **(Dec. 10, 8 Marks)**

Q. 2 Draw the block diagram and with the help of mathematical expression explain in detail the QPSK transmitter and receiver. Diagram the geometric representation and draw its power spectral density, along with its expression thereby comment on its Euclidean distance and bandwidth. **(May 11, 10 Marks)**

Q. 3 Draw the block diagram of QPSK receiver and explain the working in detail with mathematical expressions. **(Dec. 12, 8 Marks)**

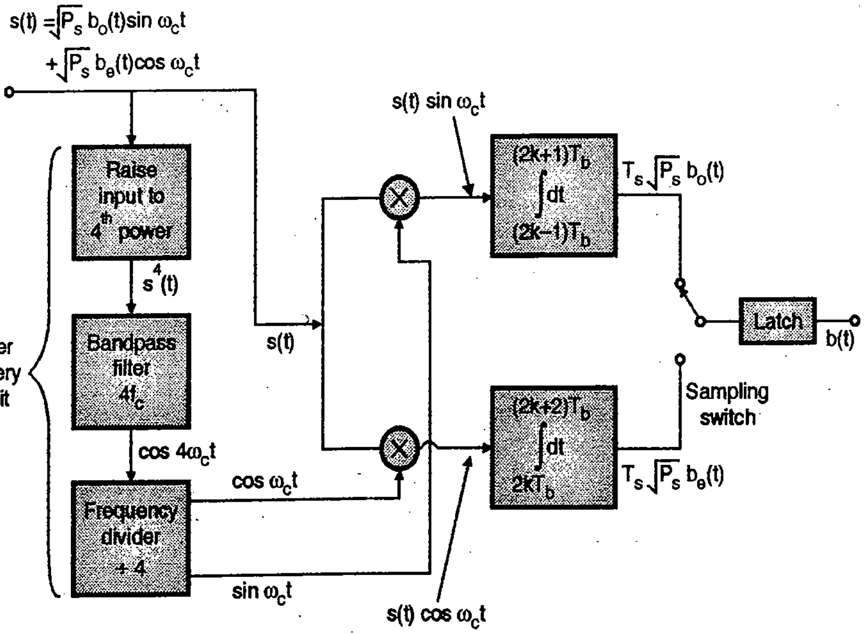
Q. 4 Explain with block diagram QPSK receiver. Write an expression for its error probability. **(Dec. 15, 8 Marks)**

- The block diagram of a QPSK receiver is shown in Fig. 6.9.9. As shown, we use the synchronous detection technique. Hence it is necessary to locally generate the carriers cos $\omega_c t$ and sin $\omega_c t$. The technique for carrier regeneration is similar to the one employed for BPSK system.

Operation :

- Let us represent the received QPSK signal by s (t) instead of V_{QPSK} (t). The received QPSK signal s (t) is raised to fourth power i.e. s^4 (t). This signal is then filtered by using a bandpass filter with a center frequency of 4 ω_ct. The output of bandpass filter is cos 4 ω_ct.
- A frequency divider which divides the frequency at the filter output by 4 generates the two carrier signals sin ω_ct and cos ω_ct.
- The incoming signal s (t) is applied to two synchronous demodulators which are made of a multiplier (balanced modulator) followed by an integrator. Each integrator integrates over a two-bit interval of duration $T_s = 2\,T_b$.
- One synchronous demodulator uses cos ω_ct as carrier signal and the other one uses sin ω_ct as a carrier signal.
- The input to the upper integrator is given by,

$$s(t) \times \sin \omega_c t = b_o(t)\sqrt{P_s}\,\sin^2 \omega_c t + b_e(t)\sqrt{P_s}\,\sin \omega_c t \cos \omega_c t \quad ...(6.9.4)$$



(E-389) **Fig. 6.9.9 : A QPSK receiver**

- This signal is applied to the upper integrator. This integrator will integrate its input signal over a symbol period of $T_s = 2\,T_b$. The upper integrator output is given by :

$$\text{Integrator output} = \int_{(2k-1)\,T_b}^{(2k+1)\,T_b} s(t) \times \sin \omega_c t \, dt$$

$$= b_o(t)\sqrt{P_s} \int_{(2k-1)\,T_b}^{(2k+1)\,T_b} \sin^2 \omega_c t \, dt + b_e(t)\sqrt{P_s} \int_{(2k-1)\,T_b}^{(2k+1)\,T_b} \sin \omega_c t \cos \omega_c t \, dt \qquad ...(6.9.5)$$

We know that $\sin^2 \omega_c t = \frac{1}{2}[1 - \cos 2\,\omega_c t\,]$

and $\sin \omega_c t \cos \omega_c t = \frac{1}{2} \sin 2\omega_c t$

- Substituting these values into Equation (6.9.5) we get,

$$\text{Integrator output} = \frac{1}{2} b_o(t)\sqrt{P_s} \int_{(2k-1)\,T_b}^{(2k+1)\,T_b} [1 - \cos 2\omega_c t\,]dt + \frac{1}{2} b_e(t)\sqrt{P_s} \int_{(2k-1)\,T_b}^{(2k+1)\,T_b} \sin 2\omega_c t \, dt$$

$$= \frac{1}{2} b_o(t)\sqrt{P_s} \int_{(2k-1)\,T_b}^{(2k+1)\,T_b} 1 \times dt - \frac{1}{2} b_o(t)\sqrt{P_s} \int_{(2k-1)\,T_b}^{(2k+1)\,T_b} \cos 2\omega_c t \, dt + \frac{1}{2} b_e(t)\sqrt{P_s} \int_{(2k-1)\,T_b}^{(2k+1)\,T_b} \sin 2\omega_c t \, dt \qquad ...(6.9.6)$$

- In the above equation, the value of second and the third term is zero because integration of a sinusoidal signal over a period corresponding to its integral number of cycles is zero.

$$\therefore \quad \text{Integrator output} = \frac{1}{2} b_o(t)\sqrt{P_s} \int_{(2k-1)\,T_b}^{(2k+1)\,T_b} 1 \times dt$$

$$= \frac{1}{2} b_o(t)\sqrt{P_s}\,[t]_{(2k-1)\,T_b}^{(2k+1)\,T_b}$$

$$= \frac{1}{2} b_o(t)\sqrt{P_s}\,[(2k+1)\,T_b - (2k-1)\,T_b\,]$$

$$= \frac{1}{2} b_o(t)\sqrt{P_s}\,[2k\,T_b + T_b - 2k\,T_b + T_b\,] = b_o(t)\sqrt{P_s}\,T_b \qquad ...(6.9.7)$$

- Similarly we can prove that output of the lower integrator is given by $b_e(t)\sqrt{P_s}\,T_b$.
- Thus at the output of the two integrators we obtain the bit streams b_o (t) and b_e (t).
- As explained for the BPSK detection, we need to use bit synchronizers in QPSK receiver as well. It is used to establish the beginning and end of the bit intervals of each bit stream. The bit synchronizer is also used to operate the sampling switch.
- At the end of each integration time for each integrator, the integrator output is sampled as discussed in section 6.4.5 for BPSK.
- The samples are taken alternately from the two integrator outputs, at the end of each bit time T_b and these samples are then held in the latch for the bit time T_b. Each individual integrator output is thus sampled at intervals of 2 T_b. At the output of the latch we get the signal b (t).

6.9.4 Signal Space Representation of QPSK Signal :

SPPU : May 06, May 08, May 11, Dec. 12, Dec. 14

University Questions

Q. 1 Give mathematical representation of QPSK signal. Draw the signal space diagram of QPSK signal. Write the expression of all the message points in the diagram. **(May 06, May 08, 8 Marks)**

Q. 2 Draw the block diagram and with the help of mathematical expression explain in detail the QPSK transmitter and receiver. Diagram the geometric representation and draw its power spectral density, along with its expression thereby comment on its Euclidean distance and bandwidth. **(May 11, 10 Marks)**

Q. 3 Draw the block diagram of QPSK receiver and explain the working in detail with mathematical expressions. **(Dec. 12, 8 Marks)**

Q. 4 Compare BPSK, QPSK and M'ary PSK with the help of equations, signal space representation, symbol rate and bandwidth. **(Dec. 14, 9 Marks)**

- We know that a QPSK signal is mathematically represented as :

$$V_{QPSK}(t) = \sqrt{2P_s}\cos\left[\omega_c t + (2m+1)\frac{\pi}{4}\right]$$

$$m = 0, 1, 2, 3 \qquad \ldots(6.9.8)$$

By substituting different values of "m", we can get the sinusoidal signals with different phase shifts, as listed in Table 6.9.2.

- Let us expand Equation (6.9.8) using the standard trigonometric rule.

i.e. cos (A + B) = cos A cos B – sin A sin B

- Applying this to Equation (6.9.8) we get,

$$V_{QPSK}(t) = \sqrt{2P_s}$$

$$\left\{\left[\cos\omega_c t \times \cos(2m+1)\frac{\pi}{4}\right] - \left[\sin\omega_c t \sin(2m+1)\frac{\pi}{4}\right]\right\} \qquad \ldots(6.9.9)$$

- Rearrange Equation (6.9.8) to get,

$$V_{QPSK}(t) =$$

$$\left\{\sqrt{P_s T_s}\cos\left[(2m+1)\frac{\pi}{4}\right] \times \sqrt{2/T_s}\cos\omega_c t\right\}$$

$$-\left\{\sqrt{P_s T_s}\sin\left[(2m+1)\frac{\pi}{4}\right] \times \sqrt{2/T_s}\sin\omega_c t\right\} \qquad \ldots(6.9.10)$$

- In Equation (6.9.10) let us assume that,

$$b_e(t) = \sqrt{2}\cos\left[(2m+1)\frac{\pi}{4}\right] \text{ and } b_o(t)$$

$$= -\sqrt{2}\sin\left[(2m+1)\frac{\pi}{4}\right] \qquad \ldots(6.9.11)$$

- Also assume that,

$$\phi_1(t) = u_1(t) = \sqrt{2/T_s}\cos\omega_c t \qquad \ldots(6.9.12)$$

$$\text{and} \quad \phi_2(t) = u_2(t) = \sqrt{2/T_s}\sin\omega_c t \qquad \ldots(6.9.13)$$

- Substitute Equations (6.9.11), (6.9.12) and (6.9.13) in Equation (6.9.10) to get,

$$V_{QPSK}(t) = \sqrt{P_s T_s} \times \frac{1}{\sqrt{2}} b_e(t) \times u_1(t)$$

$$+ \sqrt{P_s T_s} \times \frac{1}{\sqrt{2}} b_o(t) \times u_2(t) \qquad \ldots(6.9.14)$$

Where $u_1(t)$ and $u_2(t)$ are the two orthonormal basis functions. In the generation of QPSK signal, these orthonormal signals are used as carriers.

- In Equation (6.9.14), T_s is the symbol duration. In QPSK, two successive bits will form one symbol. Therefore $T_s = 2\,T_b$. Substituting in Equation (6.9.14) we get,

$$V_{QPSK}(t) = \sqrt{2P_s T_b} \times \frac{1}{\sqrt{2}} b_e(t) \times u_1(t)$$

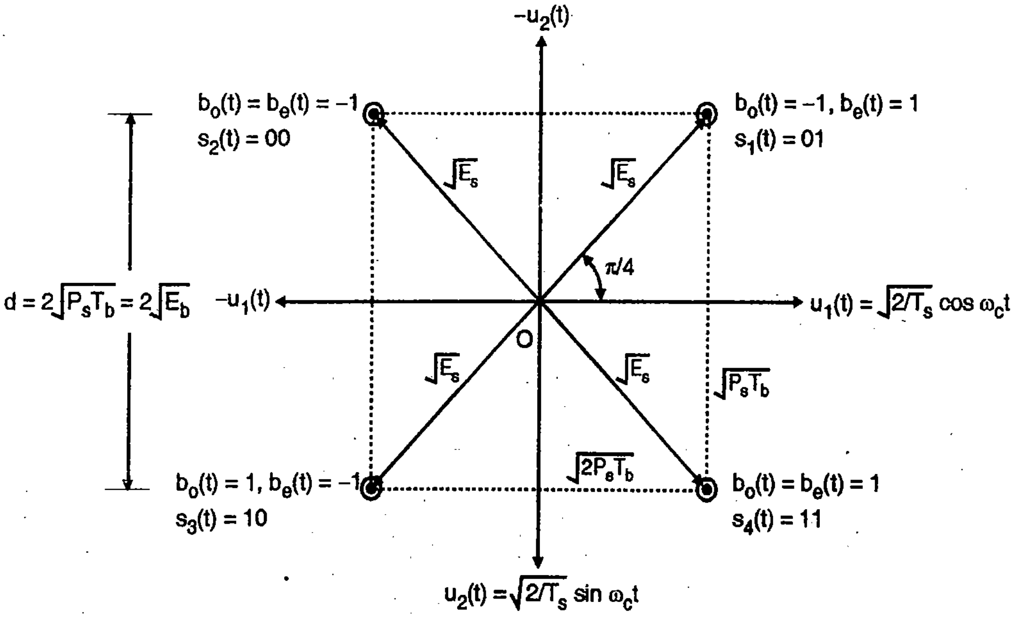
$$+ \sqrt{2P_s T_b} \times \frac{1}{\sqrt{2}} b_o(t) \times u_2(t)$$

- But $P_s T_b = E_b$ i.e. energy per bit.

$$\therefore \quad V_{QPSK}(t) = \sqrt{E_b} \times b_e(t)$$

$$\times u_1(t) + \sqrt{E_b} \times b_o(t) \times u_2(t) \qquad \ldots(6.9.15)$$

- Equation (6.9.15) can be used to plot the signal space representation of a QPSK signal, as shown in Fig. 6.9.10. The two signals $u_1(t)$ and $u_2(t)$ are used as reference axes.



(E-860) **Fig. 6.9.10 : Signal space representation of QPSK signal**

Conclusions :

Some of the important points about the signal space representation are :

1. The distance of a signal point from the origin is given by,

$$[\text{ Distance } (0 - s_4)\,]^2 = \sqrt{(P_s T_b)^2} + \sqrt{(P_s T_b)^2}$$

$$\therefore \quad (0 - s_4)^2 = T_s T_b + P_s T_b = 2 P_s T_b$$

$$= 2 E_b = E_s$$

$$\therefore \quad (0 - s_4) = \sqrt{E_s} \qquad ...(6.9.16)$$

As shown in Fig. 6.9.10 all the signal points s_1 (t) to s_4 (t) are at a distance $\sqrt{E_s}$ from the origin.

2. The ability of the receiver to make a correct decision without error is measured by the distance in signal space between the points corresponding to different values of the bit. From Fig. 6.9.10 it is clear that the distance between the points which differ in a single bit is given by,

$$d = 2\sqrt{P_s T_b} = 2\sqrt{E_b} \qquad ...(6.9.17)$$

where E_b = Energy contained in a bit transmitted for time T_b.

3. The distance "d" for QPSK (shown by Equation (6.9.17)) is exactly same as that for a BPSK system. Thus the noise immunity of the QPSK system is exactly same as that of the BPSK system.

6.9.5 Spectrum of QPSK :

SPPU : Dec. 07, May 11, May 12, May 13

University Questions

Q. 1 For rectangular data pulses calculate the second null-to-null bandwidth of BPSK, QPSK, MSK, 16 PSK and 16 QAM. Discuss advantages and disadvantages of using each of these methods.
(Dec. 07, 6 Marks)

Q. 2 Draw the block diagram and with the help of mathematical expression explain in detail the QPSK transmitter and receiver. Diagram the geometric representation and draw its power spectral density, along with its expression thereby comment on its Euclidean distance and bandwidth.
(May 11, 10 Marks)

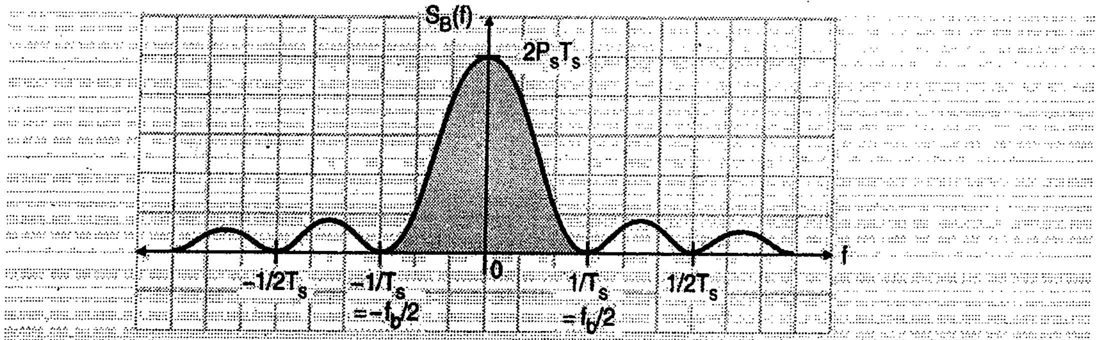
Q. 3 Derive and draw the spectrum of BPSK, QPSK and BFSK signal and compare their bandwidths.
(May 12, 8 Marks)

Q. 4 Draw signal space and spectral diagram of following digital CW modulation and state only the bandwidth requirement : 16 QAM, 16-ary PSK, QPSK and MSK.
(May 13, 6 Marks)

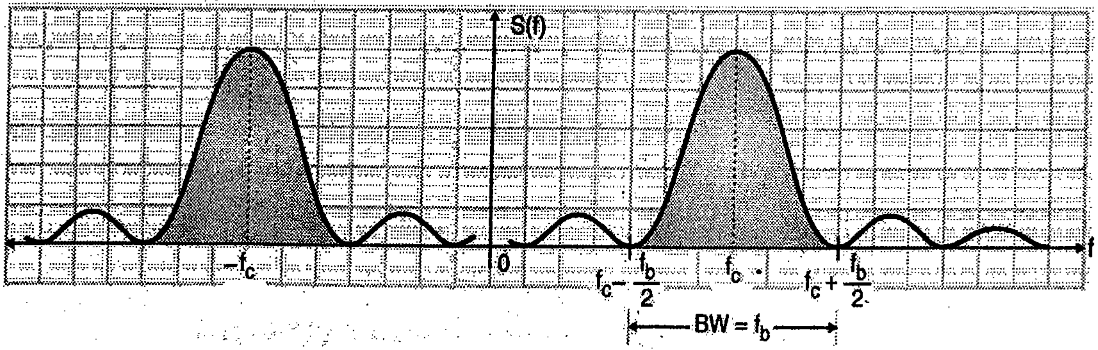
- The power spectral density of an NRZ bipolar signal with each bit extending over a period T_b as :

$$S(f) = P_s T_b \left[\frac{\sin(\pi f T_b)}{(\pi f T_b)} \right]^2 \qquad ...(6.9.18)$$

This is the PSD of signal b (t).



(a) Power spectral density of a baseband QPSK signal



(b) Power spectral density of a QPSK signal

(E-390) Fig. 6.9.11

- In QPSK, this signal b (t) is divided into even and odd bit streams i.e. b_e (t) and b_o (t) respectively. Each symbol in these bit streams has a period of $T_s = 2\ T_b$ seconds. Therefore their power spectral densities are given by,

$$S_e(f) = P_s T_s \left[\frac{\sin(\pi f T_s)}{\pi f T_s} \right]^2 \qquad ...(6.9.19)$$

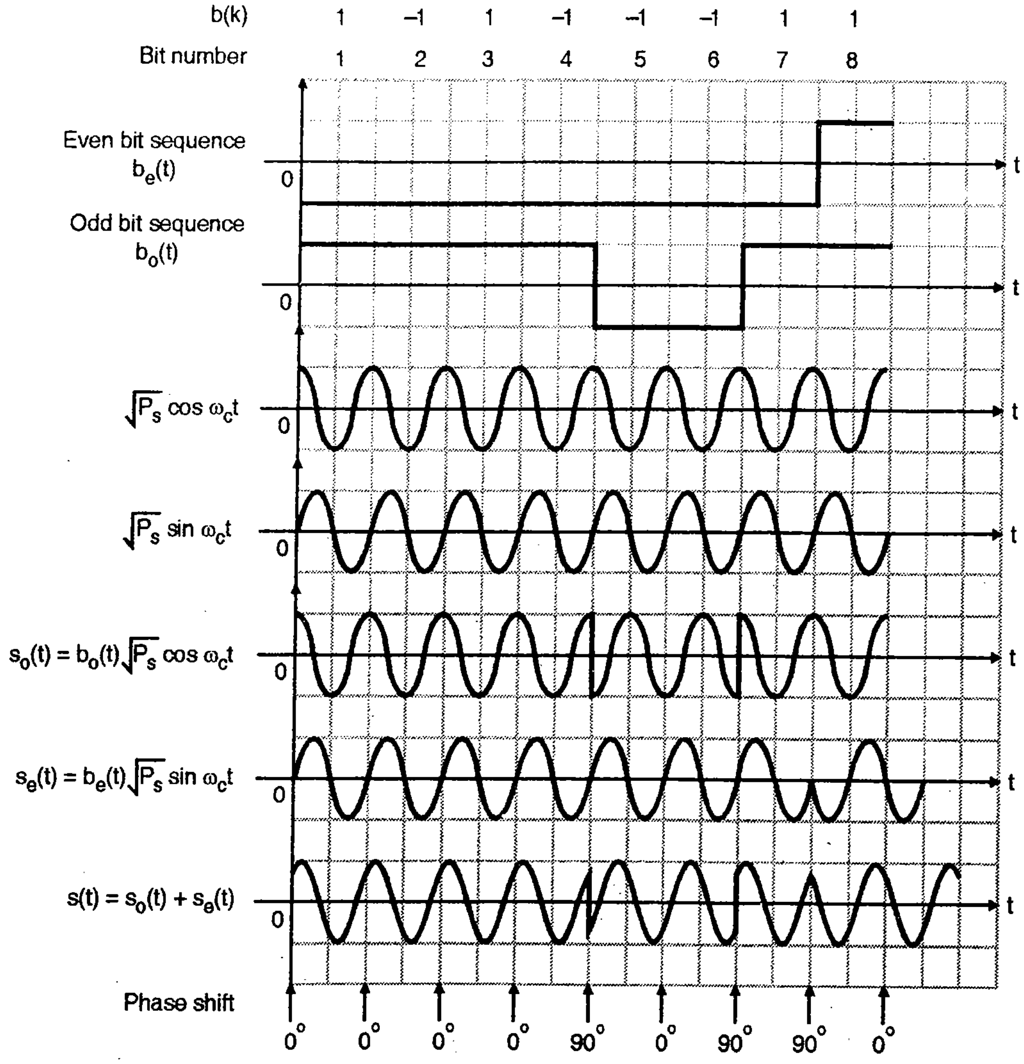
and $$S_o(f) = P_s T_s \left[\frac{\sin(\pi f T_s)}{\pi f T_s} \right]^2 \qquad ...(6.9.19(a))$$

- Comparing these Equations with Equation (6.4.23), you will find that T_b has been replaced by T_s.
- As b_o (t) and b_e (t) are statistically independent, the power spectral density of a QPSK signal is given by,

$$S_B(f) = S_e(f) + S_o(f)$$

$$= 2 P_s T_s \left[\frac{\sin(\pi f T_s)}{\pi f T_s} \right]^2 \qquad ...(6.9.20)$$

- This equation gives us the power spectral density of a QPSK signal, which is plotted in Fig. 6.9.11(a).
- When we use this signal to modulate the sinusoidal carrier of frequency f_c, the PSD given by Equation (6.9.20) will be shifted to $\pm f_c$. The graphs of PSD for QPSK are as shown in Fig. 6.9.11(b). These are very similar to the PSD plots of BPSK shown in Fig. 6.4.7.



(E-391) **Fig. P. 6.9.1**

6.9.6 Bandwidth of QPSK :

SPPU : Dec. 06, May 11, May 12, May 13

University Questions

Q. 1 With mathematical expression and block diagram, explain the operation of offset QPSK. Also express the bandwidth requirement. **(Dec. 06, 10 Marks)**

Q. 2 Draw the block diagram and with the help of mathematical expression explain in detail the QPSK transmitter and receiver. Diagram the geometric representation and draw its power spectral density, along with its expression thereby comment on its Euclidean distance and bandwidth. **(May 11, 10 Marks)**

Q. 3 Derive and draw the spectrum of BPSK, QPSK and BFSK signal and compare their bandwidths. **(May 12, 8 Marks)**

Q. 4 Draw signal space and spectral diagram of following digital CW modulation and state only the bandwidth requirement : 16 QAM, 16-ary PSK, QPSK and MSK. **(May 13, 6 Marks)**

- The bandwidth of QPSK system is one half of the bandwidth of BPSK system.

$$\therefore \quad \text{BW} = \frac{2 f_b}{2} = f_b$$

This is as shown in Fig. 6.9.11(b).

- Thus the advantage of multilevel modulation is reduction in required bandwidth.

Ex. 6.9.1 : For the following input binary sequence. $b(k) = \{1, -1, 1, -1, -1, -1, 1, 1\}$. Find the transmitted phase sequence and sketch the transmitted waveform for QPSK.

Soln. :

The required waveforms are as shown in Fig. P. 6.9.1.

6.9.7 Error Probability for QPSK System :

SPPU : Dec. 10, May 11, Dec. 15

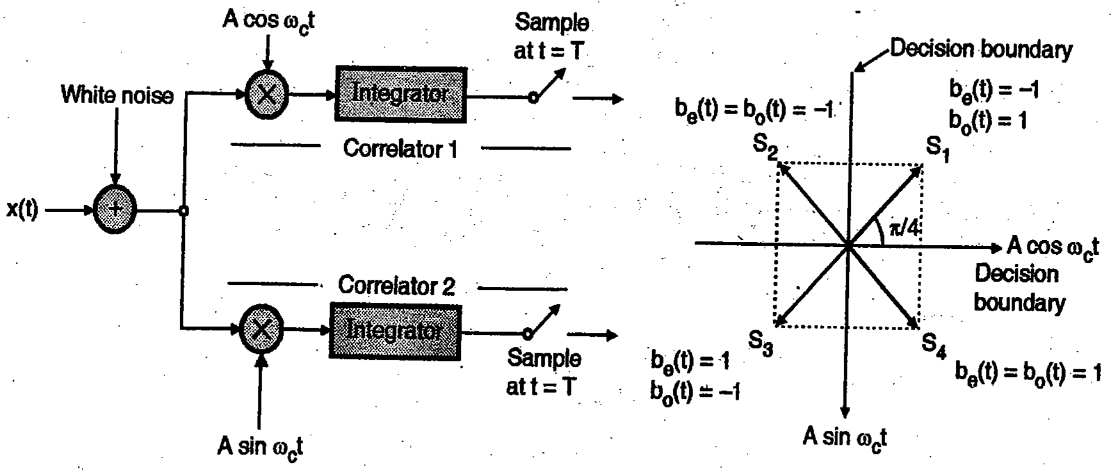
University Questions

Q. 1 Calculate the symbol error probability of QPSK receiver. **(Dec. 10, 8 Marks)**

Q. 2 Show that the probability of error of QPSK is same as that of BPSK for 1 bit duration. **(May 11, 8 Marks)**

Q. 3 Explain with block diagram QPSK receiver. Write an expression for its error probability. **(Dec. 15, 8 Marks)**

- In order to understand the error probability let us reproduce the QPSK receiver and the phasor diagram of QPSK, as shown in Figs. 6.9.12(a) and (b) respectively.
- From Fig. 6.9.12(a) it is clear that two correlators are required and the locally generated reference waveforms are $A \cos \omega_c t$ and $A \sin \omega_c t$.
- The received signal plus noise is passed through these correlators to generate the even and odd bit sequences $b_e(t)$ and $b_o(t)$ as explained earlier. These bit sequences are then added together to obtain the required message signal as shown in Fig. 6.9.12(b).
- Observe Fig. 6.9.12(b) carefully. The reference waveform of correlator 1 i.e. $A \cos \omega_c t$ is at an angle $\phi = 45°$ to the axes of orientation of all the four possible signals. Therefore the axis $A \cos \omega_c t$ can be treated as the decision boundary.



(a) A correlation receiver for QPSK (b) Signal space representation of QPSK

(E-466) Fig. 6.9.12

- Note that this decision boundary is at $\phi = 45°$ therefore any correlator of the two present can make a mistake if a phase shift of 45° or $\pi/4$ radians occurs in the corresponding carrier. Therefore the probability that correlator 1 or correlator 2 make an error is given by,

$$P_1'(e) = P_2'(e) = \frac{1}{2}\,\text{erfc}\sqrt{\frac{E}{N_0}\cos^2\phi} \qquad ...(6.9.21)$$

- Substitute $\phi = 45°$ in Equation (6.9.21) to get,

$$P_1'(e) = P_2'(e) = \frac{1}{2}\,\text{erfc}\sqrt{\frac{E}{N_0}(\cos 45)^2}$$

But $(\cos 45)^2 = 1/2$

$$\therefore \; P_1'(e) = P_2'(e) = \frac{1}{2}\,\text{erfc}\sqrt{E/2\,N_0} \qquad ...(6.9.22)$$

- The probability "P(c)" that the QPSK receiver will correctly identify the transmitted signal is equal to the product of the individual probabilities of correct identification of the two correlators.

$\therefore$ Probability of correct reception $P(c) = P_1'(c) \times P_2'(c)$ ...(6.9.23)

Where, $P_1'(c)$ = Probability that correlator 1 receives signal correctly.

$P_2'(c)$ = Probability that correlator 2 receives signal correctly.

- But $P_1'(c) = 1 - P_1'(e)$ and $P_2'(e) = 1 - P_2'(e)$. Substituting these values into Equation (6.9.23) we get,

$$P(c) = [1 - P_1'(e)] \times [1 - P_2'(e)]$$

$$\therefore \; P(c) = 1 - P_1'(e) - P_2'(e) + P_1'(e)\,P_2'(e) \qquad ...(6.9.24)$$

Substitute $P_1'(e) = P_2'(e) = P'(e)$ in Equation (6.9.24)

to get, $$P(c) = 1 - 2\,P'(e) + (P'(e))^2 \qquad ...(6.9.25)$$

- Normally $P'(e)$ is very very small. Therefore $(P'(e))^2$ is still smaller. Hence we can neglect the third term on RHS of Equation (6.9.25) to get,

$$P(c) = 1 - 2\,P'(e) \qquad ...(6.9.26)$$

- Hence the error probability of a QPSK system is given by,

$$P(e) = 1 - P(c) = 1 - [1 - 2\,P'(e)]$$

$$\therefore \; P(e) = 2\,P'(e) \qquad ...(6.9.27)$$

But $$P'(e) = P_1'(e) = P_2'(e) = \frac{1}{2}\,\text{erfc}\sqrt{E/2\,N_0}$$

...from Equation (6.9.22)

$\therefore$ Error probability of QPSK system

$$= P(e) = 2 \times \frac{1}{2}\,\text{erfc}\sqrt{E/2\,N_0}$$

$$\therefore \; P(e) = \text{erfc}\sqrt{E/2\,N_0} \qquad ...(6.9.28)$$

- This is the required expression. In Equation (6.9.28), E corresponds to the energy of each symbol (s_1, s_2, ... etc.). As each symbol is two bit duration long.

$$E = 2\,E_b$$

Substituting this into Equation (6.9.28) we get,

$$P(e) = \text{erfc}\sqrt{E_b/N_0} \qquad ...(6.9.29)$$

6.9.8 Advantages of QPSK :

1. Very good noise immunity.
2. Baud rate is half the bit rate therefore more effective utilization of the available bandwidth of the transmission channel.
3. Low error probability.

- Due to these advantages the QPSK is used for very high bit rate data transmission.

6.9.9 Disadvantage :

The generation and detection of QPSK is complex.

6.9.10 QPSK is Better than PSK :

The QPSK is better than PSK because :

1. Due to multilevel modulation used in QPSK, it is possible to increase the bit rate to double the bit rate of PSK without increasing the bandwidth.
2. The noise immunity of QPSK is same as that of PSK system.
3. Available channel bandwidth is utilized in a better way by the QPSK system than PSK system.

Ex. 6.9.2 : **What are the phase states of the carrier when the bit stream,**

1 0 0 1 1 0 1 1 0 0

is applied to a QPSK modulator.

Soln. :

Step 1 : Divide the input data stream into groups of two bits i.e. dibits :

10 | 01 | 10 | 11 | 00|

Step 2 : Phase shift to each dibit are as follows :

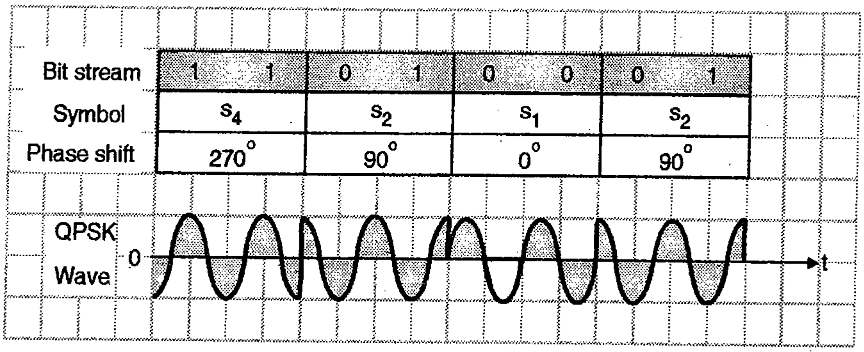
Modulator input	10	01	10	11	00
Phase state	180	90°	180°	270°	0°

Ex. 6.9.3 : For the following data stream draw the QPSK (4 PSK) signal.

11010001

Soln. :

The required waveform is shown in Fig. P. 6.9.3. Note that the carrier frequency has not been changed at all. Only the phase shift is being changed according to the symbol.



(E-392) **Fig. P. 6.9.3**

6.9.11 Comparison of BPSK and QPSK :

Sr. No.	Parameter/characteristics	BPSK	QPSK
1.	Variable characteristics of the carrier.	Phase	Phase
2.	Type of modulation	Two level (binary)	Four level
3.	Type of representation.	A binary bit is represented by one phase state.	A group of two binary bits is represented by one phase state.
4.	Bite rate/baud rate	Bit rate = baud rate	Bit rate = 2 baud rate
5.	Detection method	Coherent	Coherent
6.	Complexity	Complex	Very complex
7.	Applications	Suitable for applications that need high bit rate	Suitable for applications needing very high bit rates.

6.9.12 Difference between OQPSK and QPSK :

Sr. No.	OQPSK	QPSK
1.	There is an offset of T_b seconds between $b_o(t)$ and $b_e(t)$.	There is no offset between $b_o(t)$ and $b_e(t)$.
2.	$b_o(t)$ and $b_e(t)$ will never change simultaneously.	$b_o(t)$ and $b_e(t)$ can change simultaneously.

Sr. No.	OQPSK	QPSK
3.	Maximum phase change in the output signal is 90° or (π/2)	Maximum phase change is 180° or π.
4.	Amplitude variations at the instant of abrupt phase changes are of smaller amplitudes as compared to QPSK.	Amplitude variations at the instants of abrupt phase changes are of larger amplitudes as compared to OQPSK.

6.10 Noncoherent Binary Modulation Techniques :

- Till now we have discussed the coherent modulation techniques in which the phase synchronous locally generated carrier was used for the purpose of detection.
- But practically sometimes it is not possible to have the knowledge of carrier phase.
- Then we have to use the non coherent binary systems.
- In this section we will consider the noncoherent FSK system and the differential PSK or DPSK system.

6.10.1 Noncoherent BFSK :

- In the binary FSK system (BFSK) we know that the transmitted signal can be either $s_H(t)$ or $s_L(t)$ with $s_H(t)$ and $s_L(t)$ are given by,

$$s_H(t) = \sqrt{2E_b/T_b}\cos(2\pi f_H t) \quad ...(6.10.1)$$

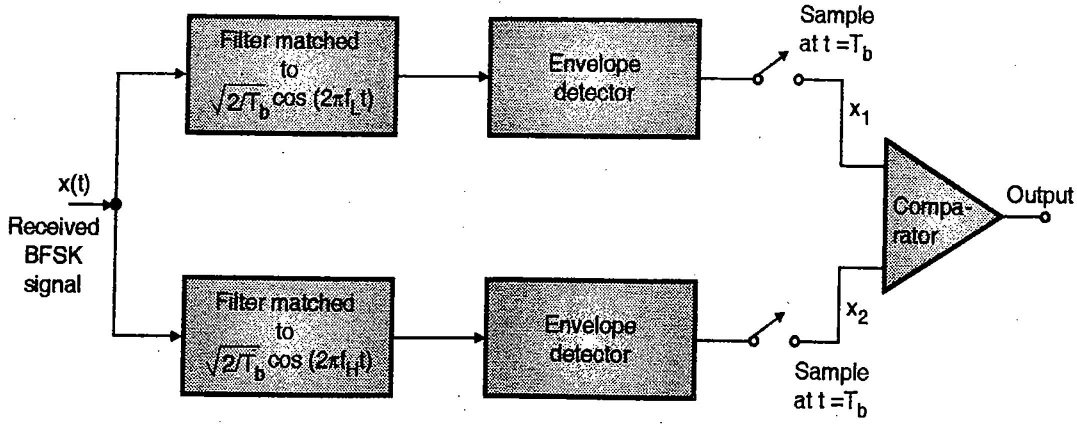
$$\text{and } s_L(t) = \sqrt{2E_b/T_b}\cos(2\pi f_L t) \quad ...(6.10.2)$$

- The transmission frequency f_H represents a logic 1 and f_L represents a logic 0.
- The noncoherent receiver of BFSK is shown in Fig. 6.10.1.
- A pair of matched filters alongwith envelope detectors is being used.
- The matched filter in the upper path is matched to $\sqrt{2/T_b}\cos(2\pi f_L t)$ whereas the filter in the lower path is matched to $\sqrt{2/T_b}\cos(2\pi f_H t)$.
- The outputs of envelope detector are sampled at $t = T_b$ with the help of electronic switches and a comparator is used for comparing their values i.e. x_1 and x_2.
- If $x_1 > x_2$ then the receiver decides that a logic 1 was transmitted whereas if $x_1 < x_2$ then receiver's decision is that a logic 0 was transmitted.
- The error probability of noncoherent BFSK is,

$$P_e = \frac{1}{2}e^{-(E_b/2N_0)} \quad ...(6.10.3)$$

6.10.2 BFSK Receiver :

- The BFSK receiver block diagram is as shown in Fig. 6.10.2. It is supposed to regenerate the original digital data signal from the BFSK signal applied at its input.
- The receiver consists of two band pass filters one with center frequency "f_L" and the other with a center frequency of "f_H". The envelope detectors are simple diode detectors which rectify and filter their inputs, to generate a dc voltage proportional to the ac input.



(E-415) **Fig. 6.10.1 : Noncoherent BFSK receiver**

Operation :

- Suppose a binary "1" is received. That means the received signal is as follows :

$$V_{BFSK}(t) = \sqrt{2P_s}\cos(\omega_H t)$$

Therefore the BPF_1 will pass this signal through to its envelope detector. The output of BPF_2 will be zero.

- The output of the upper envelope detector will be positive and that of the lower one will be zero. Therefore comparator output will be high representing that a logic 1 was received.
- Similarly if a binary "0" is received, the received BFSK signal will have a frequency "f_L". The output of BPF_1 will be zero. The BPF_2 will pass this signal to envelope detector D_2 to produce a proportional dc voltage. Output of D_1 is zero. Therefore comparator output will be zero which represents a logic "0". Thus the original data is recovered by the receiver.
- This type of detection is known as the asynchronous detection because the synchronized carrier and balanced modulator alongwith the integrators are not being used here.

6.10.3 Coherent BFSK Reception and Non-coherent BFSK Reception :

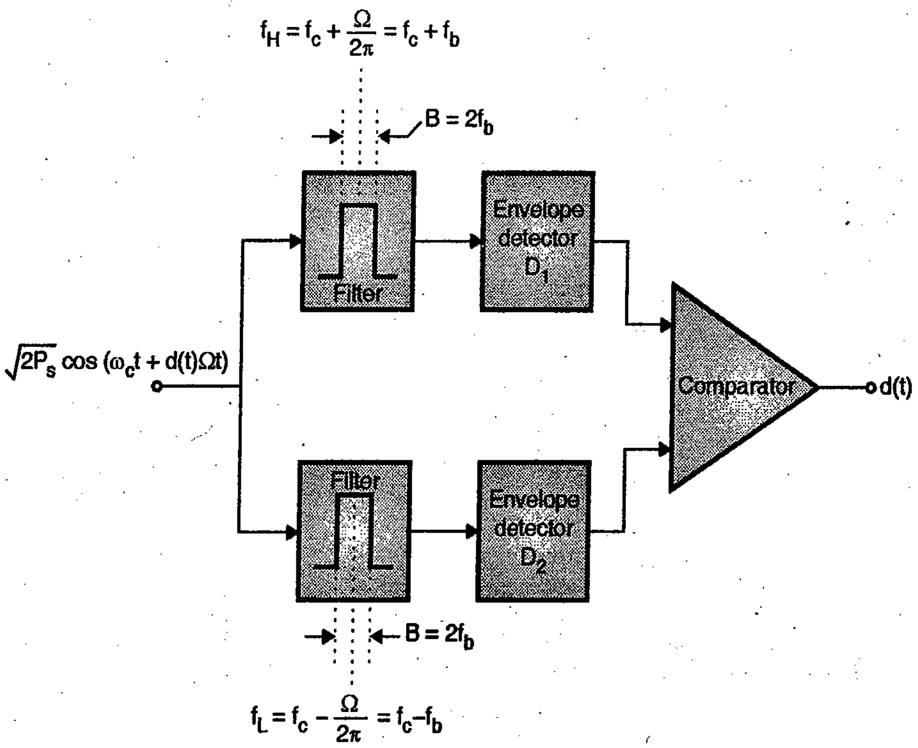
Sr. No.	Coherent BFSK reception	Non-coherent BFSK reception
1.	It uses a regenerated carrier, multiplier and integrator to recover the original binary signal.	It uses two bandpass filters, envelope detectors, and comparator to recover the original signal.
2.	The receiver is more complicated.	The receiver is less complicated.
3.	Reception process will not work unless the regenerated carrier of exactly the same frequency is generated at the receiver.	Regenerated carrier is not required for the reception.
4.	Error probability is low	Error probability is high.
5.	Less effect of noise.	Effect of noise is more.

6.10.4 Advantages of FSK :

1. FSK is relatively easy to implement.
2. It has better noise immunity than ASK. Therefore the probability of error free reception of data is high.

6.10.5 Disadvantages of FSK :

1. The major disadvantage is its high bankwidth requirement as discussed earlier.
2. Therefore FSK extensively used in low speed modems having bit rates below 1200 bits/sec.
3. The FSK is not preferred for the high speed modems because with increase in speed, the bit rate increases.
4. This increases the channel bandwidth required to transmit the FSK signal.



(E-416) **Fig. 6.10.2 : BFSK receiver**

5. As the telephone lines have a very low bandwidth, it is not possible to satisfy the bandwidth requirement of FSK at higher speed. Therefore FSK is preferred only for the low speed modems.

6.10.6 Bit Error Probability for a Noncoherently Detected Binary Orthogonal FSK :

- Now let us consider the BFSK system in which the noncoherent detection using the bandpass filters is done.
- The bit error probability of such a system is given by,

$$P_B = \frac{1}{2}\exp\left[\frac{-A^2}{4 N_0 W_f}\right] \qquad ...(6.10.4)$$

Where A = Peak signal amplitude

$N_0/2$ = PSD of white noise

W_f = Filter bandwidth

- Equation (6.10.4) tells us that the error performance of the noncoherent BFSK system is dependent on the bandwidth of the filter W_f.
- The error probability decreases with decrease in the filter bandwidth W_f. This expression is valid only when the intersymbol interference (ISI) is negligible.

6.11 M-ary Modulation Techniques :

- In an M-ary modulation scheme, there are M possible signals such as $s_1(t), s_2(t), ... s_M(t)$, and we can send any one of them during each signaling interval of duration T seconds.
- The number of possible signals i.e. M is given by,

$$M = 2^N \qquad ...(6.11.1)$$

where N is an integer.

6.11.1 Advantage :

- The M-ary schemes are preferred to the binary schemes because they need less channel bandwidth for their successful transmission.

6.11.2 Disadvantages :

- The bandwidth reduction is achieved at the cost of following disadvantages :
  1. Increase in transmitted power.
  2. Increase in error probability.

6.11.3 Types of M-ary systems :

In the following sections we will discuss the following M-ary systems in depth :

1. M-ary PSK
2. M-ary QAM
3. M-ary FSK.

6.12 M-ary PSK :

SPPU : May 07, Dec. 07, May 13, Dec. 14, Dec. 15

University Questions

Q. 1 Write the mathematical expression for M-ary PSK. Draw PSD and signal space representation and bandwidth for M-ary PSK. **(May 07, 8 Marks)**

Q. 2 Diagram the geometric representation of orthogonal QPSK, non- orthogonal QPSK, M-ary PSK, M-ary FSK and QASK. What is the importance of Euclidean distance ? Write its expression for above representation and compare them. **(Dec. 07, 10 Marks)**

Q. 3 Explain block diagrams for generation and reception of M-ary PSK signals. With suitable mathematical expressions, signal space representation, Bandwidth and PSD. **(May 13, 12 Marks)**

Q. 4 Compare BPSK, QPSK and M'ary PSK with the help of equations, signal space representation, symbol rate and bandwidth. **(Dec. 14, 9 Marks)**

Q. 5 Explain block diagrams for generation and reception of M-ary PSK signals. With suitable mathematical expressions, signal space representation Bandwidth and PSD. **(Dec. 15, 10 Marks)**

- In QPSK we grouped together two consecutive bits to form messages.
- Depending on the two bit message (00, 01, 10 or 11), a sinusoidal signal of duration equal to 2 T_b which has a particular phase shift is transmitted.
- The QPSK signals are displaced by 90° or $\pi/2$ radians in phase with respect to each other.
- The same principle can be further extended to obtain the M-ary PSK system. The M-ary PSK signals are obtained as follows :
  1. Group "N" successive bits together to form N-bit symbols.
  2. Each such symbol will extend over a period of N T_b where T_b is the duration of one bit.
  3. Due to the grouping of N bits per symbol, we can have $2^N = M$ possible symbols.
  4. These M symbols are represented by sinusoidal signals of duration $T_s = NT_b$ which differ from one another by a phase $2\pi / M$ radians. Thus the M-ary PSK waveform can be mathematically represented as,

$$V_{M-ary\,PSK} = \sqrt{2 P_s}\cos(\omega_c t + \phi_m) \qquad ...(6.12.1)$$

where $m = 0, 1, ... (M-1)$

5. In Equation (6.12.1), ϕ_m represents the symbol phase angle, which is given by,

$$\phi_m = (2m+1)\frac{\pi}{M} \qquad ...(6.12.2)$$

6. The M-ary PSK waveforms corresponding to Equation (6.12.1) are represented by dots in Fig. 6.12.1(a). This is a signal space diagram in which the co-ordinate axes are formed by the orthonormal basis functions $u_1(t)$ and $u_2(t)$ given by the following equations :

$$u_1(t) = \sqrt{(2/T_s)}\cos\omega_c t$$

and $$u_2(t) = \sqrt{(2/T_s)}\sin\omega_c t$$

These are same orthonormal signals used for the signal space representation of a QPSK signal.

7. The Euclidean distance of each M-ary PSK waveform from the origin is,

$$\sqrt{E_s} = \sqrt{P_s T_s}$$

6.12.1 Distance between Signal Points (d) :

SPPU : May 07, May 13

University Questions

Q. 1 Write the mathematical expression for M-ary PSK. Draw PSD and signal space representation and bandwidth for M-ary PSK. **(May 07, 8 Marks)**

Q. 2 Draw signal space and spectral diagram of following digital CW modulation and state only the bandwidth requirement : 16 QAM, 16-ary PSK, QPSK and MSK. **(May 13, 6 Marks)**

- The distance between signal points (d) is also called as the Euclidean distance.
- This distance is important in deciding the error probability of a communication system.
- The error probability goes on decreasing with increase in the Euclidean distance "d". This is a desired result.
- In order to obtain the expression for the distance "d" refer Fig. 6.12.1(b).

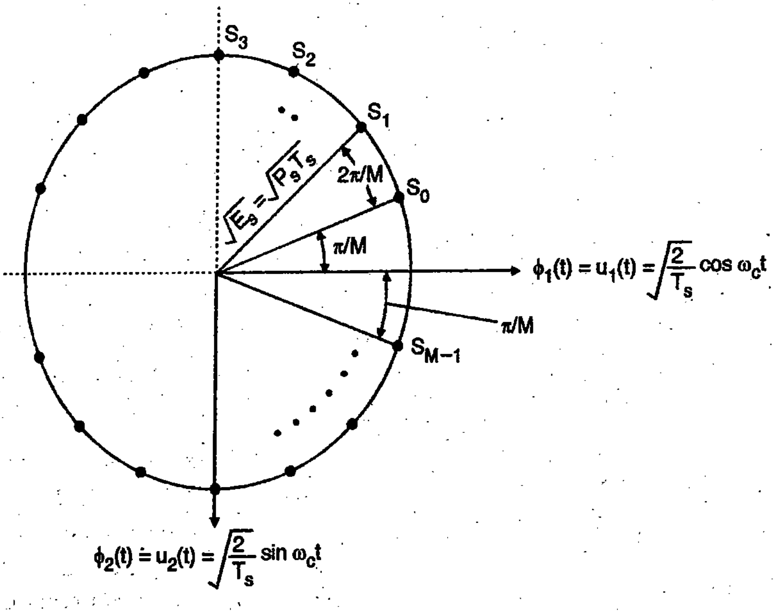
1. As shown in Fig. 6.12.1(b), the angle θ is equal to (2 π / M) because angle 2 π corresponds to the entire circle and M is the number of messages, which are equidistantly placed on the circle as shown.

2. Now consider the right angle triangle $0 - S_1 - X$. Using the definition of sin θ we get,

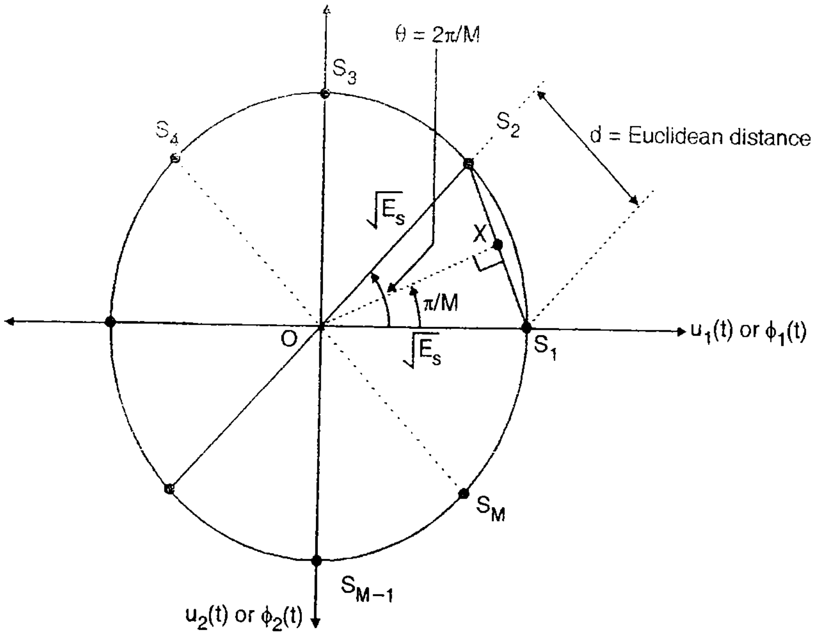
$$\sin(\pi/M) = \frac{\text{Distance } S_1 - X}{\text{Distance } 0 - S_1} \qquad ...(6.12.3)$$

$$= \frac{(d/2)}{\sqrt{E_s}}$$

$$\therefore \quad d = 2\sqrt{E_s}\,\sin(\pi/M) \qquad ...(6.12.4)$$



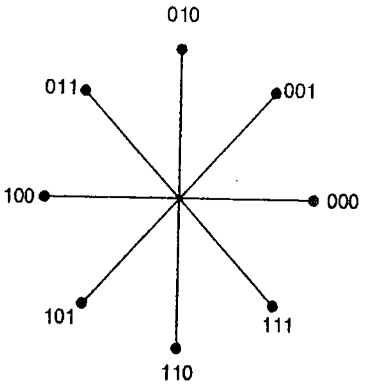
(E-393) **Fig. 6.12.1(a) : Geometric representation of M-ary PSK signals**



(E-394) **Fig. 6.12.1(b) : Signal space diagram to calculate distance d**

Symbol	Phase
000	0
001	45
010	90
011	135
100	180
101	225
110	270
111	315

(a)



(b) Constellation diagram

(E-395) **Fig. 6.12.2 : 8-PSK system**

Equation (6.12.4) is a general expression. We can substitute any value of M to get the Euclidean distance d.

6.12.2 8 PSK System :

SPPU : Dec. 15

University Questions

Q. 1 Draw and explain signal space representation of following signal. 8 Ary PSK. **(Dec. 15, 4 Marks)**

- If three consecutive bits are grouped together to form a message or symbol, then there will be $2^3 = 8$ messages, in a PSK system.
- A PSK system that uses eight different phase shifts to transmit 8 symbols is known as 8 PSK system.
- The messages, corresponding phases and the constellation diagram of 8-PSK is shown in Fig. 6.12.2.
- Baud rate is one third of bit rate for 8-PSK system. We can transmit thrice the number of symbols as compared to BPSK with the same bandwidth.

6.12.3 Power Spectral Density and Bandwidth of M-ary PSK :

SPPU : May 13, Dec. 15

University Questions

Q. 1 Explain block diagrams for generation and reception of M-ary PSK signals. With suitable mathematical expressions, signal space representation, Bandwidth and PSD. **(May 13, 12 Marks)**

Q. 2 Draw signal space and spectral diagram of following digital CW modulation and state only the bandwidth requirement : 16 QAM, 16-ary PSK, QPSK and MSK. **(May 13, 6 Marks)**

Q. 3 Explain block diagrams for generation and reception of M-ary PSK signals. With suitable mathematical expressions, signal space representation Bandwidth and PSD. **(Dec. 15, 10 Marks)**

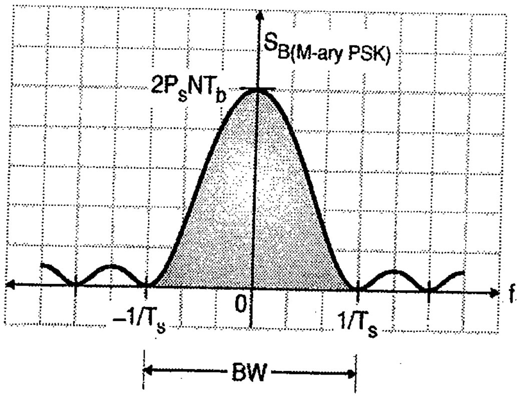
- In the preceding section 6.9.5 we have obtained the expression for power spectral density of the baseband QPSK signal as :

$$S_{B\,(QPSK)}(f) = 2\,P_s\,T_s\left[\frac{\sin(\pi f\,T_s)}{\pi f\,T_s}\right]^2$$

- However QPSK is a special case of M-ary PSK with M = 4. Therefore we can use the above expression to write the expression for PSD of the baseband M-ary PSK system. The only modification required is, that we have to substitute $T_s = NT_b$ in the above equation.

$$S_{B\,(M\text{-ary PSK})}(f) = 2\,P_s\,N\,T_b\left[\frac{\sin(\pi f\,N\,T_b)}{\pi f\,N\,T_b}\right]^2 \quad ...(6.12.5)$$

- Using Equation (6.12.5) we can plot the PSD of the baseband M-ary PSK signal as shown in Fig. 6.12.2(c).



(E-396) **Fig. 6.12.2(c) : PSD of baseband M-ary PSK signal**

Bandwidth of M-ary PSK :

- From the plot of PSD in Fig. 6.12.2(c).

$$BW = \frac{1}{T_s} - \left(-\frac{1}{T_s}\right) = \frac{2}{T_s}$$

But $T_s = N\,T_b$

$$\therefore \; BW = \frac{2}{NT_b}$$

But $\frac{1}{T_b} = f_b$

$$\therefore BW = \frac{2\,f_b}{N} \quad ...(6.12.6)$$

- We know that the bandwidth of a BPSK system is $2\,f_b$. The above expression tells us that with increase in number of bits per message, the bandwidth reduces.

6.12.4 M-ary PSK Transmitter :

SPPU : May 12, May 13, Dec. 15, Dec. 16

University Questions

Q. 1 Explain M-ary PSK transmitter and receiver with suitable block diagram and waveforms. What are the advantages of M-ary PSK over M-ary FSK ? **(May 12, 8 Marks)**

Q. 2 Explain block diagrams for generation and reception of M-ary PSK signals. With suitable mathematical expressions, signal space representation, Bandwidth and PSD. **(May 13, 12 Marks)**

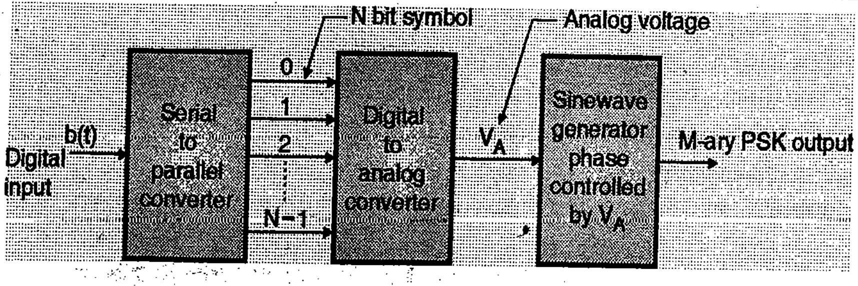
Q. 3 Explain block diagrams for generation and reception of M-ary PSK signals. With suitable mathematical expressions, signal space representation Bandwidth and PSD. **(Dec. 15, 10 Marks)**

Q. 4 Explain M-ary PSK transmitter with suitable block diagram. What are the advantages of M-ary over M-ary FSK ? **(Dec. 16, 8 Marks)**

- The block diagram of an M-ary PSK system is as shown in Fig. 6.12.3.

Operation :

- The bit stream b (t) is applied to a serial to parallel converter. This block can store the N bits of a symbol. These N bits per symbol appear serially, in the form of a sequence one after the other.
- The N bits per symbol are first assembled by the serial to parallel converter block. Then all these bits are presented simultaneously (in the parallel form) on the N output lines of the converter. Thus the N-bit message appears in the parallel form at the output of the serial to parallel converter.
- The output of the serial to parallel converter remains unchanged for a duration of NT_b of a symbol. This time duration is used by the converter to assemble a new group of N bits. After every NT_b seconds, the converter output changes to a new N bit symbol.



(E-397) **Fig. 6.12.3 : M-ary PSK transmitter**

- The N bit output of the converter is then applied to a D/A converter. The N bit digital input, is converted into an analog output V_A. The N bit digital input can have $2^N = M$ number of possible combinations. Therefore the D/A converter output V_A will have M number of distinct values, corresponding to the M symbols.
- Finally, this analog voltage is applied to a sinusoidal signal generator, which produces a constant amplitude sinusoidal output voltage, the phase ϕ_m of which is proportional to the D/A converter output V_A.
- Thus at the output of the transmitter, we get a fixed amplitude sinusoidal waveform, the phase of which has a one to one correspondence to the N bit symbols. The phase will change only once per symbol time $T_s = N\ T_b$. Thus the M-ary PSK is generated.

6.12.5 M-ary PSK Receiver :

SPPU : May 12, Dec. 15

University Questions

Q. 1 Explain M-ary PSK transmitter and receiver with suitable block diagram and waveforms. What are the advantages of M-ary PSK over M-ary FSK ?

(May 12, 8 Marks)

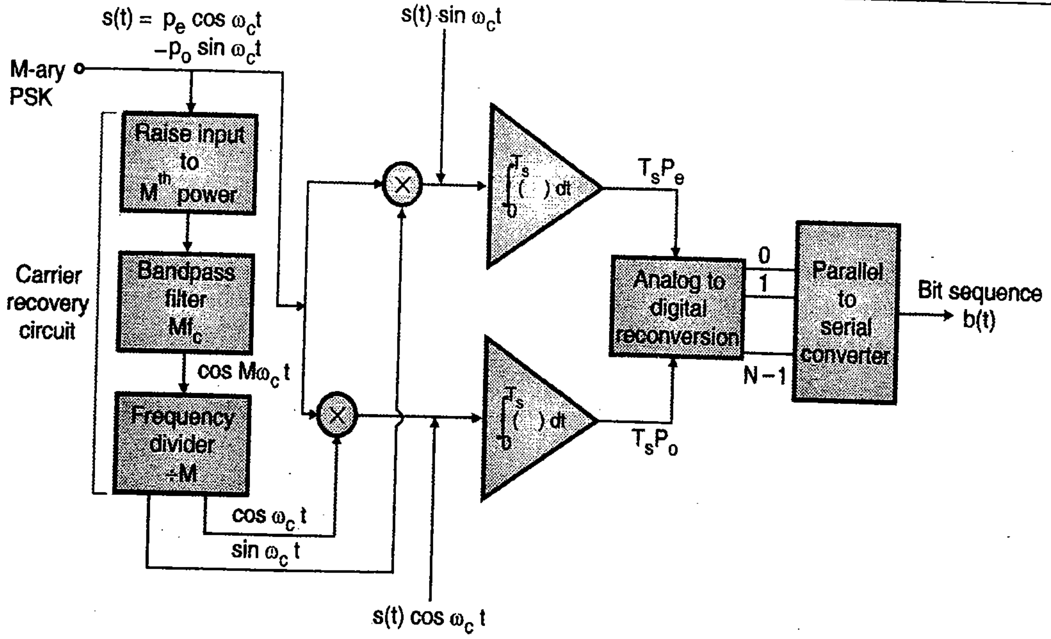
Q. 2 Explain block diagrams for generation and reception of M-ary PSK signals. With suitable mathematical expressions, signal space representation Bandwidth and PSD.

(Dec. 15, 10 Marks)

- The block diagram of M-ary PSK receiver is as shown in Fig. 6.12.4. This is same as the non-offset QPSK receiver.

Operation :

- The M-ary receiver operates on the principle of synchronous demodulation which we had discussed for BPSK and QPSK.
- The carrier recovery system will require a device which can raise the received signal to M^{th} power. This signal is then applied to a bandpass filter, which has been designed to have the center frequency to be equal to Mf_c.
- At the filter output we get a sinusoidal signal at frequency Mf_c that is M^{th} harmonic of the carrier frequency f_c. This frequency is then divided by M to obtain the carrier at frequency f_c. The two carriers produced at the filter output are $\cos \omega_c t$ and $\sin \omega_c t$.
- These recovered carriers are then applied to two multipliers (balanced modulators). The other input of each multiplier block is connected to the received M-ary PSK signal.
- The outputs of the balanced modulators are applied to the integrators. Since the M-ary PSK system is a non-offset or non-staggered type of system, the integrators will extend their integration over the same time interval. These integrators will work alongwith a bit synchronizer, which has not been shown in Fig. 6.12.4.
- The outputs of the integrators are proportional to $T_s\ P_e$ and $T_s\ P_o$ respectively and they (outputs) change at the symbol rate. These outputs are then applied to an A to D converter which yields the N bit transmitted signal. This signal is converted into b (t) by using a parallel to serial converter.



(E-864) **Fig. 6.12.4 : M-ary PSK receiver**

- Now the operating systems with N = 4 bits and $M = 2^4 = 16$ are common. The bandwidth of such a system will be given by,

$$BW = \frac{2 f_b}{N} = \frac{2 f_b}{4} = \frac{f_b}{2} \qquad ...(6.12.7)$$

6.12.6 Advantages of M-ary PSK :

SPPU : May 12, Dec. 16

University Questions

Q. 1 Explain M-ary PSK transmitter and receiver with suitable block diagram and waveforms. What are the advantages of M-ary PSK over M-ary FSK ?

(May 12, 8 Marks)

Q. 2 Explain M-ary PSK transmitter with suitable block diagram. What are the advantages of M-ary over M-ary FSK ? **(Dec. 16, 8 Marks)**

1. Bandwidth reduces with increase in the number of bits per symbol (N).
2. As the information is transmitted through signal phase and not through signal amplitude, this system operates very well in the situations where the signal amplitude varies due to the characteristics of the transmission medium.

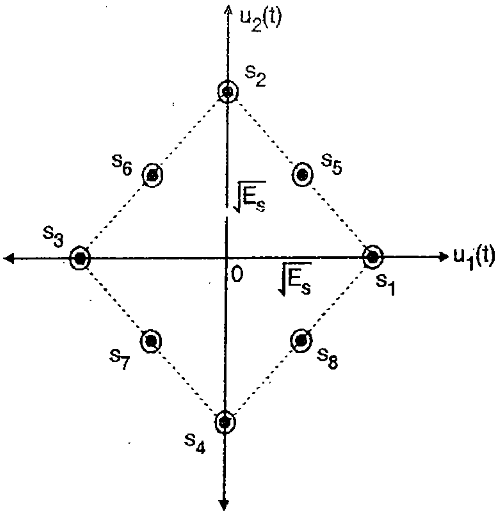
6.12.7 Disadvantages of M-ary PSK :

1. The probability of error increases with increase in the number of bits N per symbol, (i.e. increase in number of messages M) as the distance d decreases with increase in N.
2. The transmitter and receiver of M-ary PSK is very complex.

Ex. 6.12.1 : Consider an 8-level PSK signal that employs two different amplitudes as shown in Fig. P. 6.12.1. Compute the distance between the nearest neighbours in terms of signal energy. Compare the minimum distance for standard 8-level PSK. Which scheme will result in the smallest probability of error ?

Soln. :

- Looking at Fig. P. 6.12.1 it is clear that the signal points s_1 , s_2 , s_3 and s_4 are at distance $\sqrt{E_s}$ from the origin. As the given 8-level PSK employs two amplitudes, let us assume that half amplitude is being used for the remaining four signals i.e. s_5 , s_6 , s_7 and s_8.



(E-399) **Fig. P. 6.12.1**

- Then s_5 will be located exactly midway between s_1 and s_2 , s_6 will be exactly midway between s_2 and s_3 and so on as shown in Fig. P. 6.12.1. Therefore the distance between the nearest neighbours will be the distance between s_1 and s_5 or between s_2 and s_5 etc.

$$\text{Distance between } s_1 \text{ and } s_2 = \sqrt{(\sqrt{E_s})^2 + (\sqrt{E_s})^2}$$

$$d = \sqrt{2 E_s}$$

$$\therefore \text{ Distance between } s_2 \text{ and } s_5 = d / 2 = \frac{\sqrt{2 E_s}}{2}$$

$$= \sqrt{\frac{E_s}{2}} = 0.707 \sqrt{E_s} \qquad ...(1)$$

- Now consider the standard 8-level PSK system (M = 8). The expression for minimum distance is given by Equation (6.12.4) as,

$$d_{min} = 2 \sqrt{E_s} \sin (\pi / M)$$

Substitute M = 8 in the above equation to get,

$$d_{min} = 2 \sqrt{E_s} \sin (\pi / 8)$$

$$\therefore \quad d_{min} = 0.7653 \sqrt{E_s} \qquad ...(2)$$

- Comparing Equations (1) and (2) it is clear that the distance is more for the standard 8-level PSK system. Therefore error probability will be less for the standard 8-level PSK system.

Ex. 6.12.2 : In a digital CW communication system, the bit rate of NRZ data stream is 1 Mbps and carrier frequency of transmission is 100 MHz. Find the symbol rate of transmission and bandwidth requirement of the channel in following cases of different techniques used :

1. BPSK system 2. QPSK system
3. 16-ary PSK system.

Dec. 06, 10 Marks; Dec. 08, Dec. 11, Dec. 13, 8 Marks

Soln. : The bit rate of NRZ data stream is 1 MHz. Therefore the bit period

$$T_b = \frac{1}{f_b} = \frac{1}{1\ \text{MHz}} = 1\ \mu S \quad ...(1)$$

1. BPSK system :

1. Symbol rate of transmission :

In BPSK each binary bit is a symbol. Therefore symbol duration is equal to bit duration.

$$\therefore \quad T_s = T_b = 1\ \mu S$$

$$\therefore \text{ Symbol rate of transmission} = \frac{1}{T_s} = \frac{1}{1\ \mu S}$$

$$= 10^6 \text{ symbols/sec} \quad \textbf{...Ans.}$$

2. Bandwidth $BW = 2 f_b = \frac{2}{T_b} = \frac{2}{1 \times 10^{-6}} = 2\ \text{MHz}$ **...Ans.**

2. QPSK system :

1. In QPSK we group two successive bits to form one symbol.

$\therefore$ Symbol duration $T_s = 2\ T_b = 2\ \mu S$.

$\therefore$ Symbol transmission rate $= \frac{1}{2\ \mu S} = 500 \times 10^3$ symbols/sec

...Ans.

2. Bandwidth $BW = f_b = \frac{1}{T_b}$

$$= \frac{1}{1\ \mu S} = 1\ \text{MHz} \quad \textbf{...Ans.}$$

3. 16-Ary PSK system :

1. Here $M = 16$ $\quad \therefore N = 4$

$\therefore$ Symbol duration $T_s = N\ T_b = 4\ T_b$

$$\therefore \quad T_s = 4\ \mu S$$

$\therefore$ Symbol transmission rate $= \frac{1}{4\ \mu S}$

$$= 250 \times 10^3 \text{ symbols/sec} \quad \textbf{...Ans.}$$

2. Bandwidth $BW = \frac{2 f_b}{N} = \frac{2 \times 1\ \text{MHz}}{4} = 500\ \text{kHz}$ **...Ans.**

Ex. 6.12.3 : Assume you are required to transmit $f_b = 90$ Mb/s data in an authorised bandwidth of 20 MHz. Which modulation techniques would you consider ? Explain why ?

Soln. :

Given that $f_b = 90$ Mb/s and BW = 20 MHz. So the technique that can be used here is M-ary PSK. The bandwidth of an M-ary PSK system is given by,

$$BW = \frac{2 f_b}{N}$$

where N = Number of bits/symbol

$$\therefore 20\ \text{Mb/s} = \frac{2 \times 90\ \text{MHz}}{N}$$

$$\therefore N = \frac{180}{20} = 9$$

If the number of bits N = 9 then the value of M is 2^N i.e. $2^9 = 512$. Hence we will have to use a 512-ary PSK to transmit the 90 MHz signal in the available bandwidth of 20 MHz.

Ex. 6.12.4 : A 4-ary PSK has the transmitted waveforms given by,

$$s_i(t) = 4 \cos \left[2 \pi f_c t + i \frac{\pi}{2} \right], \ i = 0, 1, 2, 3 \quad ...(1)$$

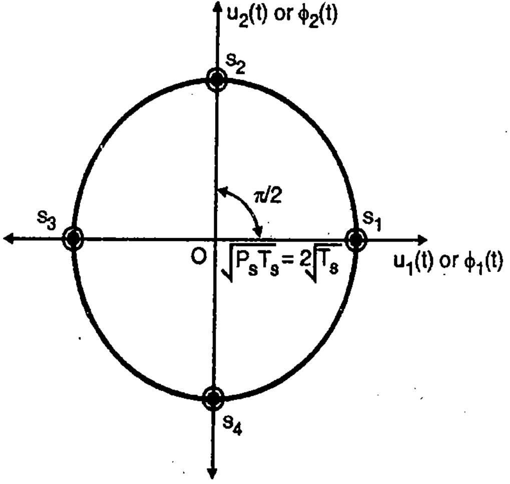
and $0 \le t \le T_s$ or $0 \le t \le 2\ T_s$ Draw its signal space diagram.

Soln. :

We know that the M-ary PSK signal is represented as :

$$V_{MPSK}(t) = \sqrt{2 P_s} \cos [\omega_c t + \phi_m] \quad ...(2)$$

Where $\phi_m = (2m + 1) \frac{\pi}{M}$ ... where m = 0, 1, ...



(E-400) **Fig. P. 6.12.4**

Compare Equations (1) and (2) to write that,

$$\sqrt{2 P_s} = 4 \qquad \therefore 2 P_s = 16$$

$$\therefore P_s = 8 \quad ...(3)$$

$$\phi_i = \phi_m = i \frac{\pi}{2} \quad \text{with} \quad i = 0, 1, 2, 3$$

$$\therefore \quad \phi_0 = 0, \quad \phi_1 = \frac{\pi}{2}, \quad \phi_2 = \pi, \quad \phi_3 = \frac{3\pi}{4}$$

For an M-ary PSK system, the signal space diagram is a circle with signal points placed on the circumference of the circle.

Each point is at $\sqrt{E_s}$ distance from the origin, where $E_s = P_s T_s$.

$$\text{As} \quad P_s = 8$$

$$\therefore \quad E_s = 8\,T_s \quad \text{and} \quad \sqrt{E_s} = 2\sqrt{2\,T_s}.$$

The signal space diagram of the 4-ary PSK system is as shown in Fig. P. 6.12.4.

Ex. 6.12.5 : A 4-ary PSK has the transmitted waveforms

$$s_i(t) = A\cos\left[2\pi f_c t + i\frac{\pi}{2}\right]$$

$i = 0, 1, 2, 3$ and $0 \le t \le T_s$ or $0 \le t \le 2\,T_b$

Draw its signal space diagram and find the Euclidean distance. What is the bandwidth requirement of the system ?

Soln. : This is a QPSK system. Substituting different values of i in the given expression we get the four symbols as follows :

$$s_1 = A\cos(\omega_c t + 0)$$

$$s_2 = A\cos\left(\omega_c t + \frac{\pi}{2}\right)$$

$$s_3 = A\cos(\omega_c t + \pi)$$

$$s_4 = \cos\left(\omega_c t + \frac{3\pi}{2}\right)$$

- Therefore the constellation diagram is as shown in Fig. P. 6.12.5.

The Euclidean distance is given by,

$$d = \left[(\sqrt{P_s T_s})^2 + (\sqrt{P_s T_s})^2\right]^{1/2}$$

$$\therefore \quad d = \sqrt{[2 P_s T_s]}$$

$$\text{But} \quad T_s = 2\,T_b$$

$$\therefore \quad d = \sqrt{4 P_s T_b} = 2\sqrt{P_s T_b}$$

$$\text{But} \quad P_s T_b = E_b$$

$$\therefore \quad d = 2\sqrt{E_b}$$ **...Ans.**

The bandwidth is same as that of QPSK i.e.

$$BW = f_b \text{ Hz}$$

Ex. 6.12.6 : Derive an expression for the spectral spread 16-ary PSK system.

Soln. :

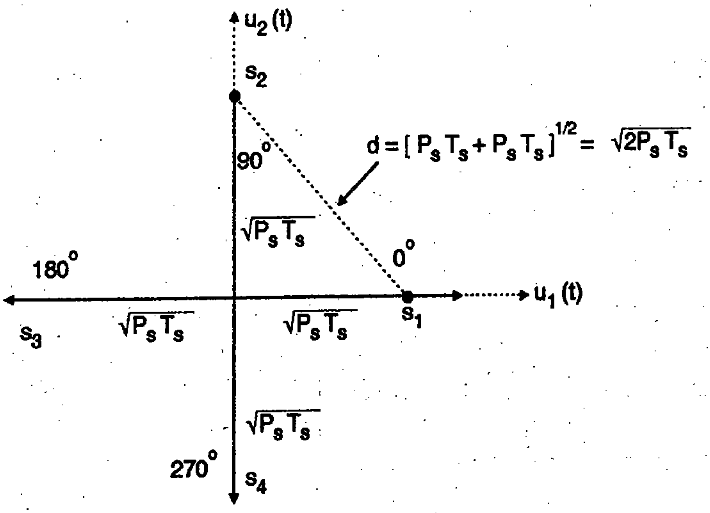
This is nothing but derivation of the power spectral density of 16-ary PSK. The expression for the PSD of a baseband signal of M-ary PSK.

$$S_{B\,(M-\text{ary PSK})}(f) = 2\,P_s N T_b \left[\frac{\sin(\pi f N T_b)}{\pi f N T_b}\right]^2$$

In this equation, substitute the value of M = 16 i.e. N = 4 to obtain the PSD of the baseband signal as follows :

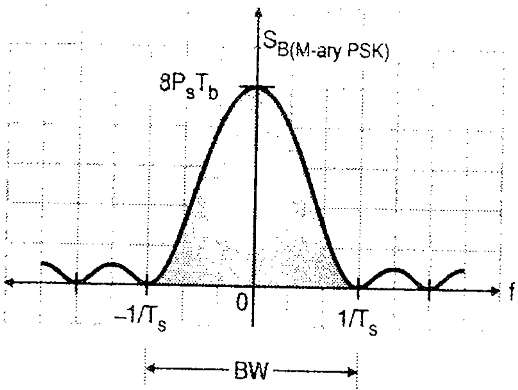
$$\therefore \quad S_{B\,(16-\text{ary PSK})} = 2\,P_s \times 4 \times T_b \left[\frac{\sin(\pi f \times 4\,T_b)}{\pi f \times 4\,T_b}\right]^2$$

$$= 8\,P_s T_b \left[\frac{\sin(4\pi f T_b)}{4\pi f T_b}\right]^2$$



(E-401) **Fig. P. 6.12.5 : Constellation diagram and minimum distance**

The corresponding spectrum is shown in Fig. P. 6.12.6.



(E-865) **Fig. P. 6.12.6 : Spectrum of the baseband signal of 16-ary PSK**

Ex. 6.12.7 : Show that, in an AWGN channel, the detection performance of equal energy signals depends only on the 'distance' between the two pertinent message points in the signal space.

Soln. :

- Even if the energy of the two transmitted signals is same, the receiver will make a correct decision if and only if the two signals are at a sufficient distance from each other in the constellation diagram.
- If the distance between them is small, then the possibility of the receiver making mistake in making a decision about the received signal increases.
- This fact becomes evident from the expressions of error probabilities and the distance d.

For the M-ary PSK system :

$$d = 2\sqrt{E_s}\,\sin(\pi/M) \qquad ...(1)$$

$$\text{and } P_e = \text{erf}_c\sqrt{E_s/2N_0} \qquad ...(2)$$

- As we increase the number of symbols M in Equation (1), the value of sin (π / M) decreases and the value of d also decreases hence the value of E_s will decrease.
- The complementary error function of Equation (2) is a monotonically decreasing function. Hence as E_s decreases, the value of P_e increases.
- This shows that the error probability increases with decrease in the distance d between the message points.

Ex. 6.12.8 : In a digital communication system, the bit rate of NRZ data stream is 5 Mbps and carrier frequency of transmission is 100 MHz. Find :

(a) Mathematical equation

(b) Symbol rate (c) Bandwidth

for the following modulation schemes :

1. BPSK 2. QPSK 3. 16-ary PSK.

Dec. 14. 9 Marks

Soln. :

Given : Bit rate $= 5\text{ Mbps} = 5 \times 10^6$

$$\therefore\ T_b = \frac{1}{f_b} = \frac{1}{5 \times 10^6} = 0.2 \times 10^{-6} = 0.2\ \mu S$$

Carrier frequency = 100 MHz

1. BPSK system :

(a) Mathematical equation :

Refer section 6.4.1.

(b) Symbol rate of transmission :

In BPSK each binary bit is a symbol. Therefore symbol duration is equal to bit duration.

$$\therefore\ T_s = T_b = 0.2\ \mu S$$

$$\therefore\ \text{Symbol rate of transmission} = \frac{1}{T_s}$$

$$= \frac{1}{0.2\ \mu S} = 5 \times 10^6 \text{ symbols/sec.}$$

(c) Bandwidth : $BW = 2f_b = 2 \times 5 \times 10^6 = 10\text{ MHz}$

2. QPSK system :

(a) Mathematical equation :

Refer section 6.9.

(b) Symbol rate of transmission :

In QPSK we group two successive bits to form one symbol.

$$\therefore\ \text{Symbol duration } T_s = 2T_b = 2 \times 0.2\ \mu S = 0.4\ \mu S$$

$$\therefore\ \text{Symbol rate of transmission} = \frac{1}{0.4\ \mu S}$$

$$= 2.5 \times 10^6 \text{ symbols/sec.}$$

(c) Bandwidth : $BW = f_b = 5\text{ MHz.}$

3. 16-ary PSK system :

(a) Mathematical equation :

Refer section 6.12.

(b) Symbol rate of transmission :

$$\text{Here } M = 16,$$

$$N = \log_2 M = \frac{\log_{10} 16}{\log_{10} 2} = 4$$

$$\therefore\ \text{Symbol duration } T_s = N\,T_b = 4\,T_b$$

$$= 4 \times 0.2\ \mu S = 0.8\ \mu S$$

$\therefore$ Symbol rate of transmission $= \frac{1}{0.8\ \mu S} = 1.25 \times 10^6$ symbols /sec.

(c) **Bandwidth :** $BW = \frac{2f_b}{N} = \frac{2 \times 5 \times 10^6}{4} = 2.5$ MHz

6.12.8 Probability of Symbol Error for MPSK :

- Now we will learn about the error performance of M-ary systems. The symbol error probability is denoted by P_e (M).
- If the energy to noise ratio of an M-ary PSK signal is large then the symbol error probability using the coherent psk is given by,

$$P_e(M) = 2Q\left[\sqrt{\frac{2E_s}{N_0}} \sin\frac{\pi}{M}\right] \quad ...(6.12.8)$$

$$\text{But } Q(x) = \frac{1}{2}\text{erfc}(x/\sqrt{2})$$

$$\therefore P_e(M) = 2 \times \frac{1}{2}\text{erfc}\left[\frac{1}{\sqrt{2}} \times \sqrt{\frac{2E_s}{N_0}} \sin\frac{\pi}{M}\right]$$

$$\therefore P_e(M) = \text{erfc}\left[\sqrt{E_s/N_0} \sin\frac{\pi}{M}\right] \quad ...(6.12.9)$$

- In Equations (6.12.8) and (6.12.9),

P_e (M) = Probability of symbol error

E_s = Energy per symbol = E_b ($\log_2$ M)

M = Number of messages being transmitted $= 2^N$

N = Number of bits per symbol.

- Equations (6.12.8) and (6.12.9) indicates that the probability of symbol error depends on the ratio $\sqrt{E_s/N_0}$ and M.
- The error probability decreases i.e. error performance improves with increases in the value of $\sqrt{E_s/N_0}$ but error probability increases with increase in the value of M.
- Fig. 6.12.5 shows the graph of symbol error probability versus (E_s/N_0) with coherent detection.

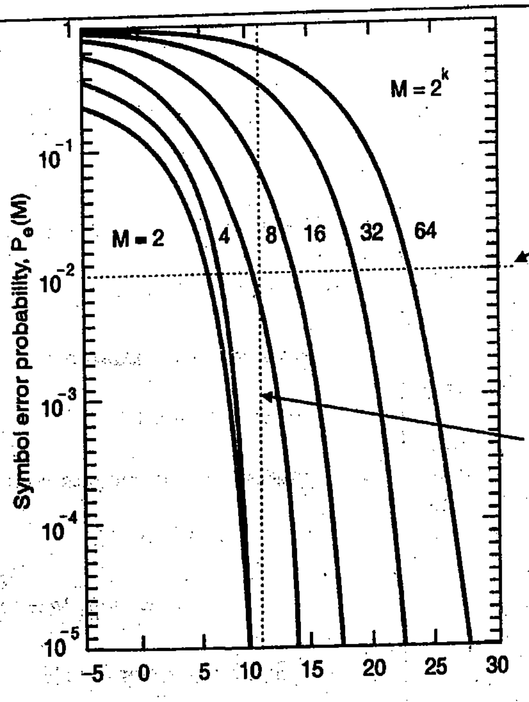
Comments :

- Fig. 6.12.5 indicates that in order to obtain the same symbol error, we have to increase the value of signal energy to noise ratio $\sqrt{E_s/N_0}$ as the value of M goes on increasing.
- If the ratio $\sqrt{E_s/N_0}$ is maintained constant then with increase in the value of M, the error probability goes on increasing.

Thus the error performance of BPSK is superior to the M-ary PSK systems with **M > 2**.

6.13 Quadrature Amplitude Shift Keying (QASK) or QAM :

- In all the PSK methods discussed till now, one symbol is distinguished from the other in phase, but all the symbols transmitted using BPSK, QPSK or M-ary PSK are of "same amplitude".
- The ability of a receiver to distinguish between one signal vector from another in presence of noise, depends on the distance between the vector end points.



For a constant value of P_e(M), the value of E_s/N_0 has to be increased with increase in M

For a constant value of E_s/N_0, the error probability increases with increase in M

(E-467) **Fig. 6.12.5 : Symbol error performance for M-ary PSK system using coherent detection**

- This suggests that the noise immunity will improve if the signal vectors differ not only in phase, but also in amplitude.
- Such a system is called as amplitude and phase shift keying system.
- In this system the direct modulation of carriers in quadrature (i.e. cos $\omega_c t$ and sin $\omega_c t$) is involved, therefore this system is called as the quadrature amplitude phase shift keying i.e. QAPSK or simply QASK.
- It is also known as quadrature amplitude modulation (QAM).

6.13.1 Geometrical Representation of QASK :

SPPU : Dec. 07, May 13, Dec. 16

University Questions

Q. 1 Diagram the geometric representation of orthogonal QPSK, non- orthogonal QPSK, M-ary PSK, M-ary FSK and QASK. What is the importance of Euclidean distance ? Write its expression for above representation and compare them. **(Dec. 07, 10 Marks)**

Q. 2 Write a notes on : 16-ary QAM. **(May 13, 2 Marks)**

Q. 3 Draw signal space of 16-QAM system and comment on Euclidean distance and probability of error for 16-QAM signals. **(Dec. 16, 8 Marks)**

- The geometrical representation is also called as signal space representation.
- Let us assume that using QASK we want to transmit a symbol consisting of 4 bits. That means N = 4 and there are 2^4 = 16 different possible symbols. Hence the QASK system should be able to generate 16 different distinguishable signals.
- A possible geometric representation of 16 signals is shown in Fig. 6.13.1.
- In the geometric representation of Fig. 6.13.1, each signal point is equally distant from its nearest neighbours. This distance is d = 2a.
- Now let us assume that all the 16 signals are equally spaced. As these signal are placed symmetrically, we can determine the energy associated with a signal, by considering the four signals in the first quadrant.
- The average normalized energy of each signal is given by the average of the energy associated with signals in the first quadrant.

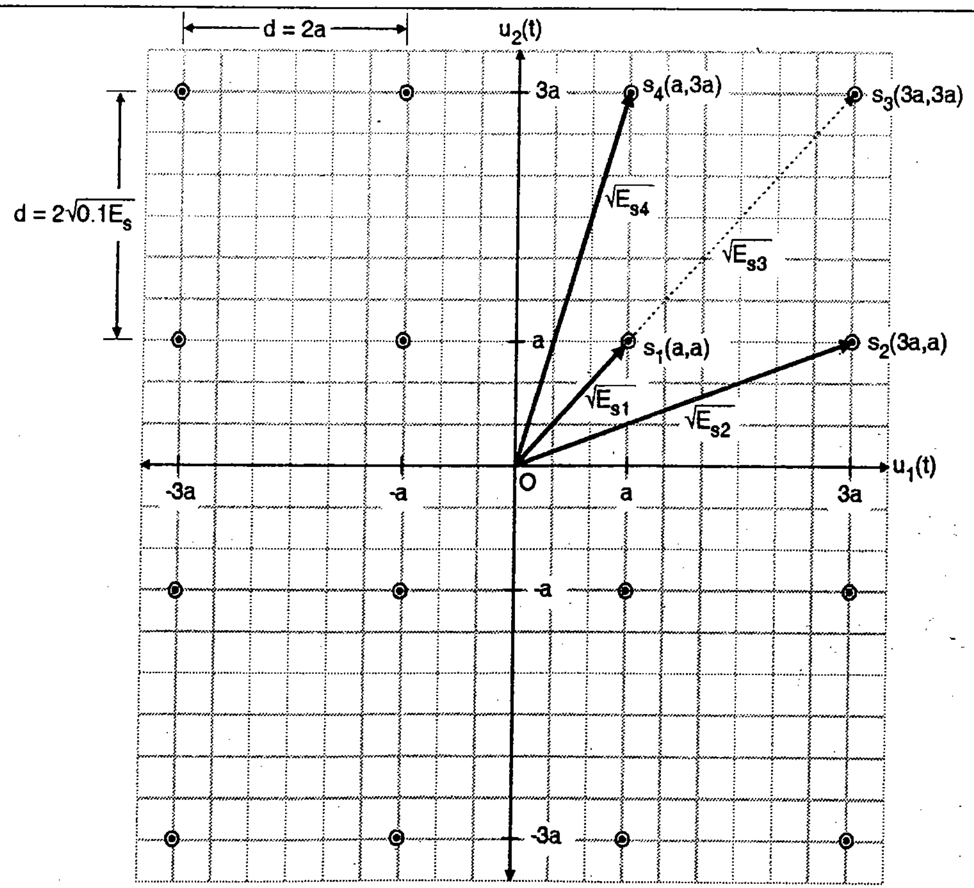
$$\therefore \quad E_s = \frac{E_{s1} + E_{s2} + E_{s3} + E_{s4}}{4}$$

- Looking at Fig. 6.13.1 we can write that,

$$E_{s1} = (a^2 + a^2), \quad E_{s2} = (9a^2 + a^2)$$

$$E_{s3} = (9a^2 + 9a^2) \text{ and } E_{s4} = (a^2 + 9a^2)$$

- Substituting these values into expression for E_s we get,



(E-402) **Fig. 6.13.1 : Geometric representation of 16 signals in a QASK system (16 - QAM)**

$$E_s = \frac{1}{4}[(a^2+a^2)+(9a^2+a^2)+(a^2+9a^2)+(9a^2+9a^2)]$$

$$= 10\,a^2 \qquad ...(6.13.1)$$

- Therefore, $a = \sqrt{0.1\,E_s}$ ...(6.13.2)

  and $d = 2a = 2\sqrt{0.1\,E_s}$ ...(6.13.3)

  where E_s = Normalized symbol energy.
  In this system, because each symbol consists of 4 bits, the normalized symbol energy is given by,

  $$E_s = 4\,E_b \qquad ...(6.13.4)$$

  where E_b is the normalized energy per bit.

- Substituting Equation (6.13.4) into Equations (6.13.2) and (6.13.3) we get,

  $$a = \sqrt{0.4\,E_b} \quad \text{and} \quad d = 2\sqrt{0.4\,E_b} \qquad ...(6.13.5)$$

- Equation (6.13.5) shows that the Euclidean distance "d" for QASK system is much less than the Euclidean distance between the adjacent QPSK signals where $d = 2\sqrt{E_b}$. But this distance "d" is greater than the 16-ary PSK where,

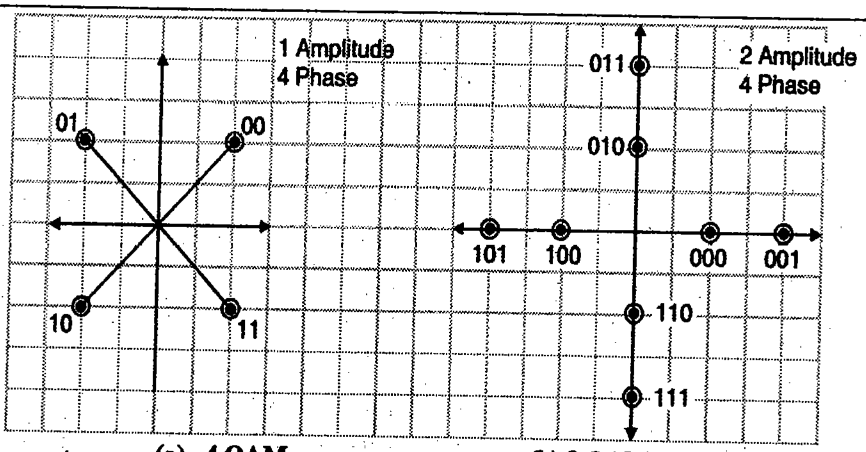
  $$d = \sqrt{16\,E_b \times \sin^2\frac{\pi}{16}} = 2\sqrt{0.15\,E_b} = \sqrt{0.6\,E_b} \qquad ...(6.13.6)$$

- Thus the 16 QASK system will have a low error rate as compared to 16-ary PSK but a higher error rate as compared to a QPSK system.

6.13.2 Types of QAM :

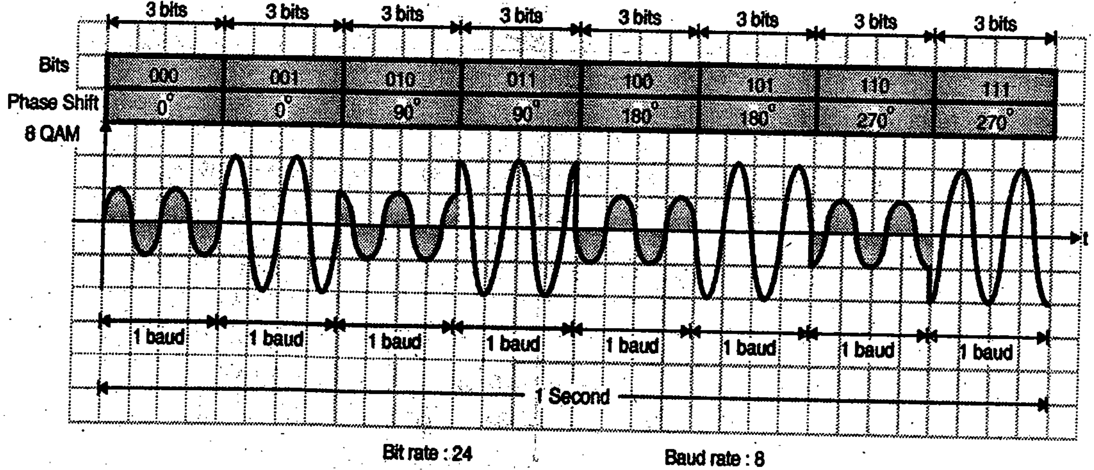
Depending on the number of bits per message the QAM signals are classified as follows :

Name	Bits per symbol	Number of symbols
4 QAM	2	$2^2 = 4$
8 QAM	3	$2^3 = 8$
16 QAM	4	$2^4 = 16$
32 QAM	5	$2^5 = 32$
64 QAM	6	$2^6 = 64$



(a) 4 QAM (b) 8 QAM

(E-403) Fig. 6.13.2 : Constellation diagrams



(E-404) Fig. 6.13.3 : Time domain display of 8 QAM

6.13.3 4 QAM and 8 QAM Systems :

- The constellation of 4 QAM system is shown in Fig. 6.13.2(a). All the symbols have same amplitude but different phases.
- Fig. 6.13.2(b) shows the constellation diagram of 8-QAM. Note that there are two amplitude levels and four phases involved.
- The time domain display of 8 QAM is shown in Fig. 6.13.3.

6.13.4 QASK Transmitter :

SPPU : Dec. 05, May 08, May 09

University Questions

Q. 1 Describe with the help of block diagram 16 point QAM system transmitter and receiver. Explain the working with the help of mathematical expressions. Also express the bandwidth requirement.

(Dec. 05, 6 Marks)

Q. 2 Describe with the help of block diagram 16-point QAM system transmitter and receiver. Explain the working with the help of mathematical expressions and expression for the bandwidth requirement.

(May 08, May 09, 10 Marks)

- The block diagram of a QASK transmitter is shown in Fig. 6.13.4.

Operation :

- The QASK signal shown in Fig. 6.13.4 can be mathematically represented as,

$$V_{QASK} = k_1 \, a \, u_1 (t) + k_2 \, a \, u_2 (t) \qquad ...(6.13.7)$$

where k_1 and k_2 are each equal to ± 1 or ± 3.

- We know that, the basis functions are represented by,

$$u_1 (t) = \sqrt{2 / T_s} \cos \omega_c t$$

$$\text{and} \quad u_2 (t) = \sqrt{2 / T_s} \sin \omega_c t \text{ and } a = \sqrt{0.1 E_s}$$

- We can substitute these expressions into Equation (6.13.7) to get,

$$V_{QASK}$$

$$= k_1 \times \sqrt{(0.2 E_s / T_s)} \cos \omega_c t + k_2 \times \sqrt{(0.2 E_s / T_s)} \sin \omega_c t \qquad ...(6.13.8)$$

- But $E_s / T_s = P_s$ hence equation for QASK is given by,

$$V_{QASK} = k_1 \sqrt{0.2 P_s} \cos \omega_c t + k_2 \times \sqrt{0.2 P_s} \sin \omega_c t \qquad ...(6.13.9)$$

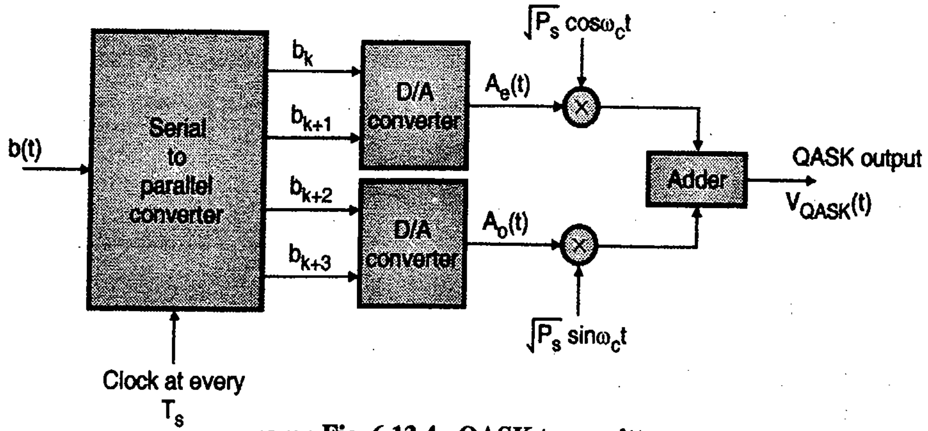
- The QASK generator is as shown in Fig. 6.13.4. The bit stream b (t) is applied to a serial to parallel converter operating on a clock which has a period of T_s sec. which is equal to the symbol duration. The bits b (t) are stored by the converter and then converted to the parallel form. The four bit symbol is $b_{k+3} \, b_{k+2} \, b_{k+1} \, b_k$.
- Out of these four bits, the first two bits are applied to the first D to A converter and the other two bits are applied to the second D to A converter.
- The output of the first D/A converter is $A_e (t)$ which is used to modulate the carrier $\sqrt{P_s} \cos \omega_c t$, whereas the output of the second D/A converter i.e. $A_o (t)$ is used to modulate the carrier $\sqrt{P_s} \sin \omega_c t$ with the help of the balanced modulators.
- The balance modulator outputs are added together to get the QASK output signal, which is expressed as follows :

$$V_{QASK} (t) = A_e (t) \times \sqrt{P_s} \cos \omega_c t + A_o (t) \times \sqrt{P_s} \sin \omega_c t \qquad ...(6.13.10)$$

- Comparing Equations (6.13.10) and (6.13.9) we get,

$$A_e (t) \text{ and } A_o (t) = \pm \sqrt{0.2} \text{ or } \pm 3 \sqrt{0.2} \qquad ...(6.13.11)$$

depending on the input to D/A converter.



(E-405) **Fig. 6.13.4 : QASK transmitter**

6.13.5 QASK Receiver :

SPPU : Dec. 05, May 08, May 09

University Questions

Q. 1 Describe with the help of block diagram 16 point QAM system transmitter and receiver. Explain the working with the help of mathematical expressions. Also express the bandwidth requirement.

(Dec. 05, 6 Marks)

Q. 2 Describe with the help of block diagram 16-point QAM system transmitter and receiver. Explain the working with the help of mathematical expressions and expression for the bandwidth requirement.

(May 08, May 09, 10 Marks)

The block diagram of QASK receiver is as shown in Fig. 6.13.5, which is very much similar to the QPSK receiver discussed earlier.

Operation :

- Like QPSK this is also a synchronous demodulation which requires a locally generated set of quadrature (90° phase shifted) carriers i.e. cos $\omega_c t$ and sin $\omega_c t$.
- These quadrature carriers are recovered from the received QASK signal. The input QASK signal is first raised to the fourth power and using a bandpass filter with a centre frequency of 4 f_c alongwith a frequency divider (÷ 4), we can recover these quadrature carriers.
- Remember that the values of A_e and A_o are not constant and equal in the QASK system. Therefore it is not sure if we can really recover the quadrature carriers or not. Hence let us check whether we can really recover the carriers correctly.
- The input signal $V_{QASK}(t)$ is raised to fourth power as,

$$V^4_{QASK}(t) = P^2_s [A_e(t) \cos \omega_c t + A_o(t) \sin \omega_c t]^4$$

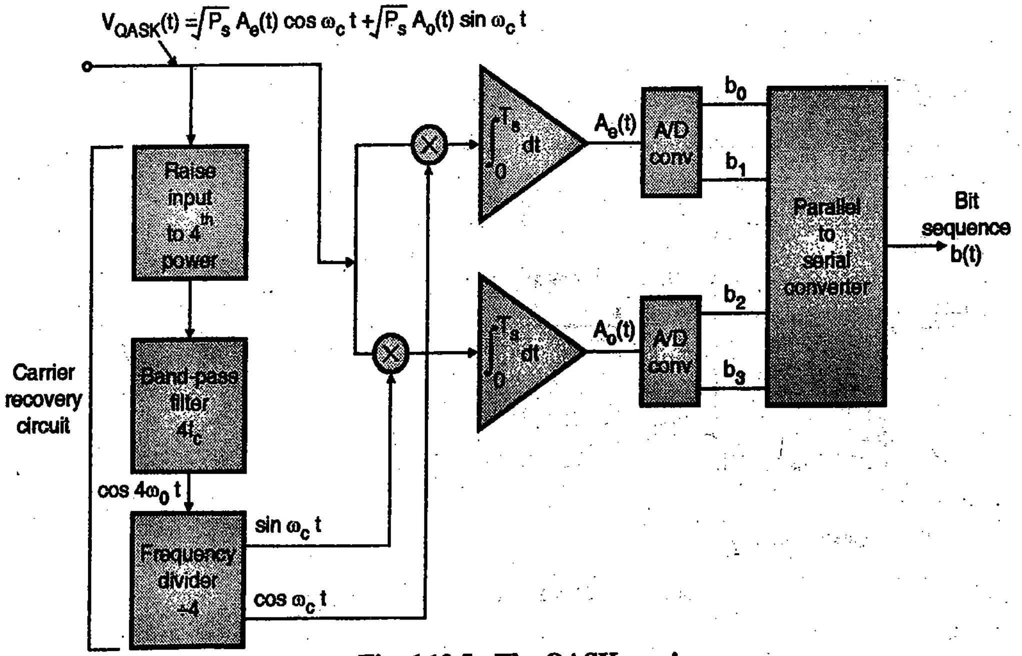
- This signal is then passed through a bandpass filter with a centre frequency 4 f_c, therefore we neglect all the other terms except for those which have a frequency of 4 f_c.

$$\therefore V^4_{QASK}(t)$$

$$= \frac{P_s}{8} \left[A^4_e(t) + A^4_o(t) - 6 A^2_e(t) A^2_o(t)\right] \cos 4 \omega_c t$$

$$+ \frac{P_s}{2} \left[A_e(t) + A_o(t) [A^2_e(t) - A^2_o(t)]\right] \sin 4 \omega_c t \quad \text{...(6.13.12)}$$

- In Equation (6.13.12), the average value of the coefficient of cos 4 $\omega_c t$ is not zero but the average value of the coefficient of sin 4 $\omega_c t$ will be zero. Thus at the output of the bandpass filter and frequency divider combination we get the quadrature carrier components cos $\omega_c t$ and sin $\omega_c t$.
- Then two balanced modulators (multipliers) are used alongwith two integrators to recover the signals $A_e(t)$ and $A_o(t)$. Both the integrators integrate over one symbol period i.e. T_s. The symbol time synchronizer which is not shown in Fig. 6.13.3 is actually used alongwith each integrator.



(E-406) Fig. 6.13.5 : The QASK receiver

- Finally the original bits are obtained from $A_e(t)$ and $A_o(t)$ by using two A to D converters. The outputs of the two A to D converters are then applied to a serial to parallel converter to obtain the sequence b(t).

6.13.6 Bandwidth of QASK System :

SPPU : Dec. 05, Dec. 07, May 08, May 09

University Questions

Q. 1 Describe with the help of block diagram 16 point QAM system transmitter and receiver. Explain the working with the help of mathematical expressions. Also express the bandwidth requirement.

(Dec. 05, May 08, May 09, 6 Marks)

Q. 2 For rectangular data pulses calculate the second null-to-null bandwidth of BPSK, QPSK, MSK, 16 PSK and 16 QAM. Discuss advantages and disadvantages of using each of these methods.

(Dec. 07, 6 Marks)

- The expression for the QASK output is very similar to that of a M-ary PSK. Therefore the plot of power spectral density also will be same as that for the M-ary PSK.
- The spectrum of QASK is shown in Fig. 6.13.6 which is quite similar to that of a M-ary PSK.
- From the Fig. 6.13.6, it is evident that main lobe of the frequency spectrum extends from $-f_s$ to $+f_s$. Therefore the bandwidth of QASK is given by,

$$BW = f_s - (-f_s) = 2f_s \qquad ...(6.13.13)$$

$$= \frac{2}{T_s} \qquad ... \text{As } f_s = \frac{1}{T_s}$$

$$= \frac{2}{N T_b} \qquad ... \text{As } T_s = N T_b$$

$$\therefore \quad BW = \frac{2 f_b}{N} \qquad ... \text{As } f_b = \frac{1}{T_b} \qquad ..(6.13.14)$$

Thus the bandwidth of QASK system is same as that of an M-ary PSK system.

6.13.7 Error Probability of 16 QAM (16 QASK) :

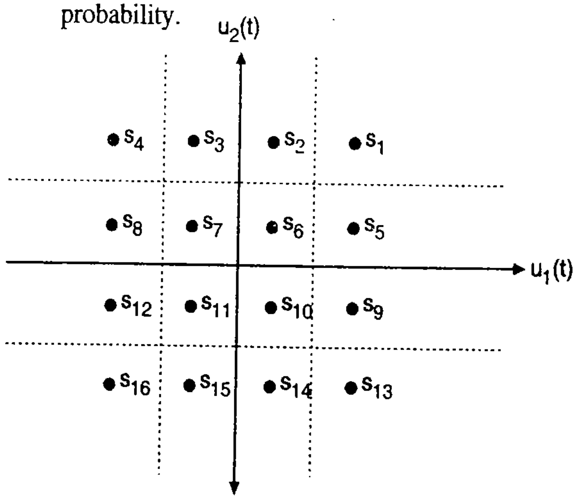
SPPU : May 13

University Questions

Q. 1 Draw signal space and spectral diagram of following digital CW modulation and state only the bandwidth requirement : 16 QAM, 16-ary PSK, QPSK and MSK.

(May 13, 6 Marks)

- The signal space diagram of 16 QAM is shown in Fig. 6.13.7.
- Let us calculate the error probability for the symbol such as s_6 in Fig. 6.13.7 which is located at (a, a).
- This signal has the largest probability of error. The signals s_7, s_{10} and s_{11} also will have the largest error probability.



(E-887) **Fig. 6.13.7 : Signal space of 16 QAM**

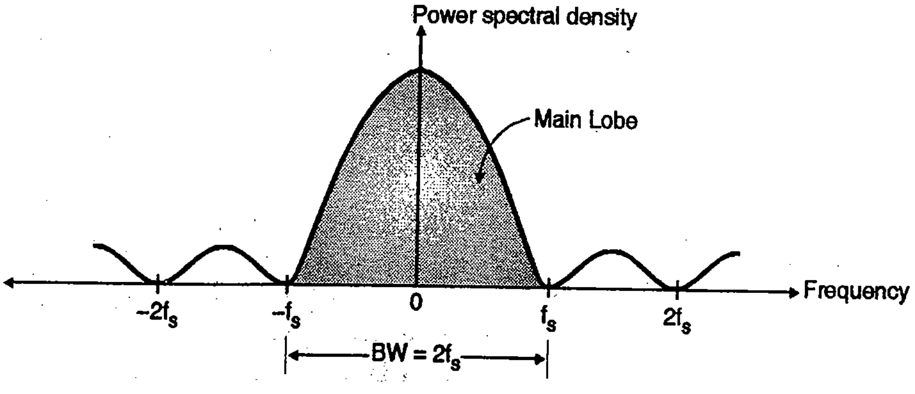
- The minimum distance is given by,

$$d = \sqrt{0.4 E_s} = \sqrt{1.6 E_b}$$

- And the error probability is given by,

$$P_e \leq 4 \times \frac{1}{2} \text{erfc} \left[\frac{1.6 E_b}{4 N_0} \right]^{1/2}$$

$$= 2 \,\text{erfc} \left[0.4 \frac{E_b}{N_0} \right]^{1/2}$$



(E-407) **Fig. 6.13.6 : Frequency spectrum of QASK**

6.13.8 Comparison of QASK and QPSK :

Table 6.13.1

Sr. No.	Parameter	QPSK	QASK
1.	Type of modulation	Quadrature phase modulation	Quadrature amplitude and phase modulation
2.	Location of signal points	On the circumference of a circle	Equally spaced and placed symmetrically about origin
3.	Distance between the signal points	$d = 2\sqrt{E_b}$ for N = 2	$d = 2\sqrt{0.4\,E_b}$ for N = 4 or M = 16
4.	Noise immunity	Better than QASK	Poorer than QPSK
5.	Probability of error	Less than QASK	More than QPSK
6.	Type of demodulation	Synchronous	Synchronous
7.	System complexity	Less complex than QASK	More complex than QPSK

6.13.9 Comparison of 16 PSK with 16 QASK :

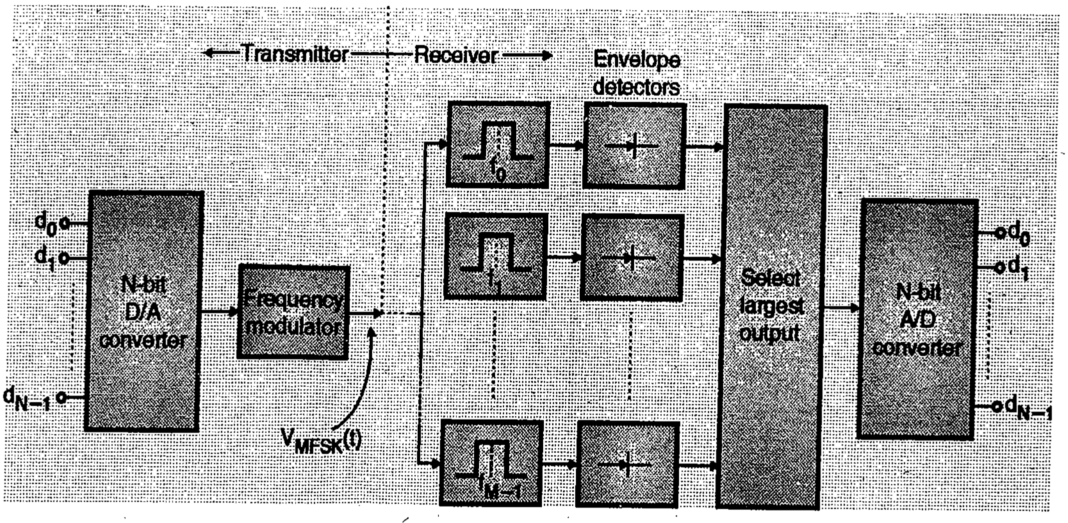
Sr. No.	Parameter	16 PSK	16 QASK
1.	Type of modulation	M-ary PSK with M = 16	M-ary QAM with M = 16
2.	Location of signal points	On the circumference of a circle	4 points in each quadrant
3.	Distance between signal points	$d = 2\sqrt{4\,E_b}\,\sin(\pi / M)$	$d = 2\sqrt{0.15\,E_b}$
4.	Noise immunity	Poorer than 16 QASK	Better than 16 PSK
5.	Number of symbols	M = 16	M = 16
6.	Number of bits per symbol	N = 4	N = 4
7.	Detection method	Coherent	Coherent
8.	Symbol duration	$T_s = 4\,T_b$	$T_s = 4\,T_b$
9.	Bandwidth	$B = \frac{2\,f_b}{N} = f_b / 2$	$f_b / 2$
10.	System complexity	Less than 16 QASK	More than 16 PSK

6.14 M-ary FSK System :

- An M-ary FSK system is the logical extension of a binary FSK system. In an M-ary FSK system the symbols are not one bit long like BFSK but each symbol is N bit long with a symbol duration of $T_s = N\,T_b$. The block diagram of an N bit M-ary FSK system is as shown in Fig. 6.14.1.

Operation :

- At the transmitter an N bit symbol is presented after a period of T_s to an N bit D/A converter.
- The output of the D/A converter is applied to a frequency modulator. The frequency modulator generates a carrier waveform whose frequency depends on the analog modulating signal at the output of D/A converter.



(E-417) Fig. 6.14.1 : An M-ary FSK communication system

- The transmitted signal, will be equal to the frequency f_0, f_1 .... or f_{M-1}, for a duration equal to the symbol interval i.e. N T_b. For example, if N = 2 then M = 2^2 = 4. Then there will be four different frequencies corresponding to the four different possible messages i.e. 00, 01, 11 and 10.
- The M-ary FSK signal is demodulated using envelope detectors as shown in Fig. 6.14.1. At the receiver the incoming signal is applied to M parallely connected bandpass filters each followed by an envelope detector.
- These bandpass filters have center frequencies f_0 , f_1, ..., f_{M-1}. Therefore these filters will pass only those signals which have frequencies equal to their center frequencies. Remember that at any given instant of time only one frequency is being received.
- The outputs of bandpass filters are then applied to envelope detectors. They produce dc voltages proportional to their ac input voltage.
- Outputs of all the envelope detectors are applied to a block named "select largest output" which determines which of the detector outputs has the largest value at the given time.
- The output of this device is applied to an N bit A to D converter which reproduces the originally transmitted N bit message.

6.14.1 Spectrum of M-ary FSK :

- As will be proved later, the probability of error is minimized if we select the frequencies f_0 , f_1, ..., f_{M-1} such that the M signals are mutually orthogonal.
- To achieve this, the carrier frequency is selected to be equal to successive even harmonics of the symbol frequency $f_s = 1/T_s$.

  That means $f_0 = k\, f_s$, $f_1 = (k + 2)\, f_s$, $f_2 = (k + 4)\, f_s$ etc.
- The spectrum of such an M-ary FSK is as shown in Fig. 6.14.2, which is just an extension of the BFSK spectrum shown in Fig. 6.7.3.

6.14.2 Bandwidth of M-ary FSK :

- From the spectrum of Fig. 6.14.2 it is clear that the M-ary FSK system needs a large bandwidth. To pass the spectrum shown in Fig. 6.14.2 the system bandwidth should be,

$$BW = 2\,M\,f_s \qquad ...(6.14.1)$$

$$\text{But} \quad f_s = f_b / N \text{ and } M = 2^N$$

$$\therefore \quad BW = 2 \times 2^N \times f_b / N$$

$$\therefore \; BW = 2^{N+1}\; f_b / N \qquad ...(6.14.2)$$

- Thus M-ary FSK needs a considerably large bandwidth as compared to M-ary PSK.

6.14.3 Advantage of M-ary FSK :

The major advantage of M-ary FSK is that its noise immunity improves and the probability of error reduces with increase in the value of M.

6.14.4 Disadvantages :

1. A number of correctly tuned filters are required for the reception of M-ary FSK signal.
2. Large bandwidth requirements.

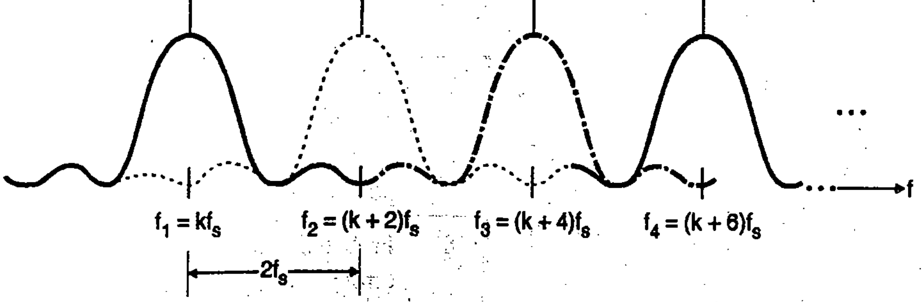
6.14.5 Geometric Representation of M-ary FSK :

SPPU : Dec. 07, May 11

University Questions

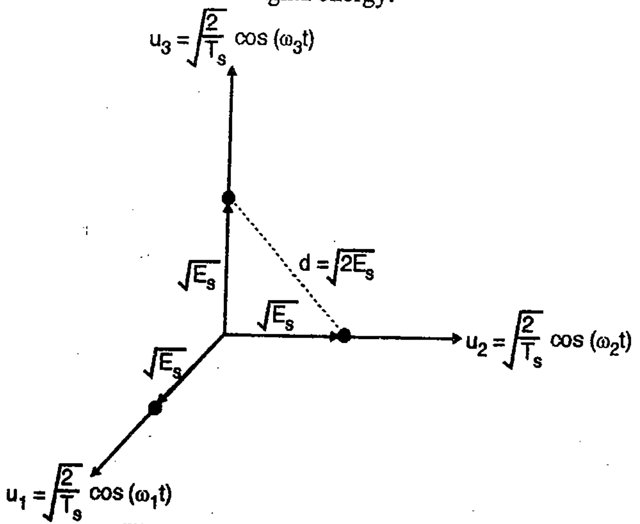
Q. 1 Diagram the geometric representation of orthogonal QPSK, non- orthogonal QPSK, M-ary PSK, M-ary FSK and QASK. What is the importance of Euclidean distance ? Write its expression for above representation and compare them. **(Dec. 07, 10 Marks)**

Q. 2 Diagram the geometric representation of : M-ary FSK. State the Euclidean distance of above mentioned systems by explaining the importance of Euclidean distance. **(May 11, 6 Marks)**



(E-418) **Fig. 6.14.2 : Spectrum of M-ary FSK**

- Fig. 6.7.5 shows a signal space representation for orthogonal binary FSK (BFSK) signal. The signal space representation for M-ary FSK is the logical extension of binary FSK as shown in Fig. 6.14.3.
- The signal space representation of M-ary FSK is simply a co-ordinate system with M mutually orthogonal co-ordinate axes. The signal vectors will then be parallel to these co-ordinate axes.
- The geometric representation of M-ary FSK for M = 3 is shown in Fig. 6.14.3. As shown in this figure, the square of the length of the signal vector is the normalized signal energy.



(E-419) **Fig. 6.14.3 : Geometric representation of orthogonal M-ary FSK (M = 3)**

- The distance between signal points is given by,

$$d = \sqrt{2E_s} = \sqrt{2NE_b} \qquad ...(6.14.3)$$

> **Note :** The value of "d" for M-ary FSK is greater than the values of "d" calculated for M-ary PSK with the exception of M = 2 and M = 4. This distance "d" is also greater than "d" in the 16-QASK system.

6.14.6 Probability of Symbol Error for MFSK (Coherent Detection) :

- In M-ary FSK signal, M orthogonal carrier waves are used to represent M different symbols or messages.
- The symbol error probability $P_e(M)$ for M equally likely orthogonal signals have an upper bound given by,

$$P_e(M) = (M-1)\,Q\left[\sqrt{E_s/N_0}\right] \qquad ...(6.14.4)$$

But $Q(x) = \frac{1}{2}\operatorname{erfc}(x/\sqrt{2})$

$$\therefore\ P_e(M) = (M-1) \times \frac{1}{2}\operatorname{erfc}\left[\sqrt{E_s/2N_0}\right] \qquad ...(6.14.5)$$

Where M = Number of symbols or messages

E_s = Energy per symbol = $E_b(\log_2 M)$

- Equations (6.14.4) and (6.14.5) indicates that the symbol error probability increases (error performance becomes poor) with increase in the value of M.
- But the error probability decreases (performance improves) with increase in the ratio $\sqrt{E_s/N_0}$.
- Fig. 6.14.4 shows a graph of symbol error probability versus (E_S/N_0) using the coherent detection.

Conclusions :

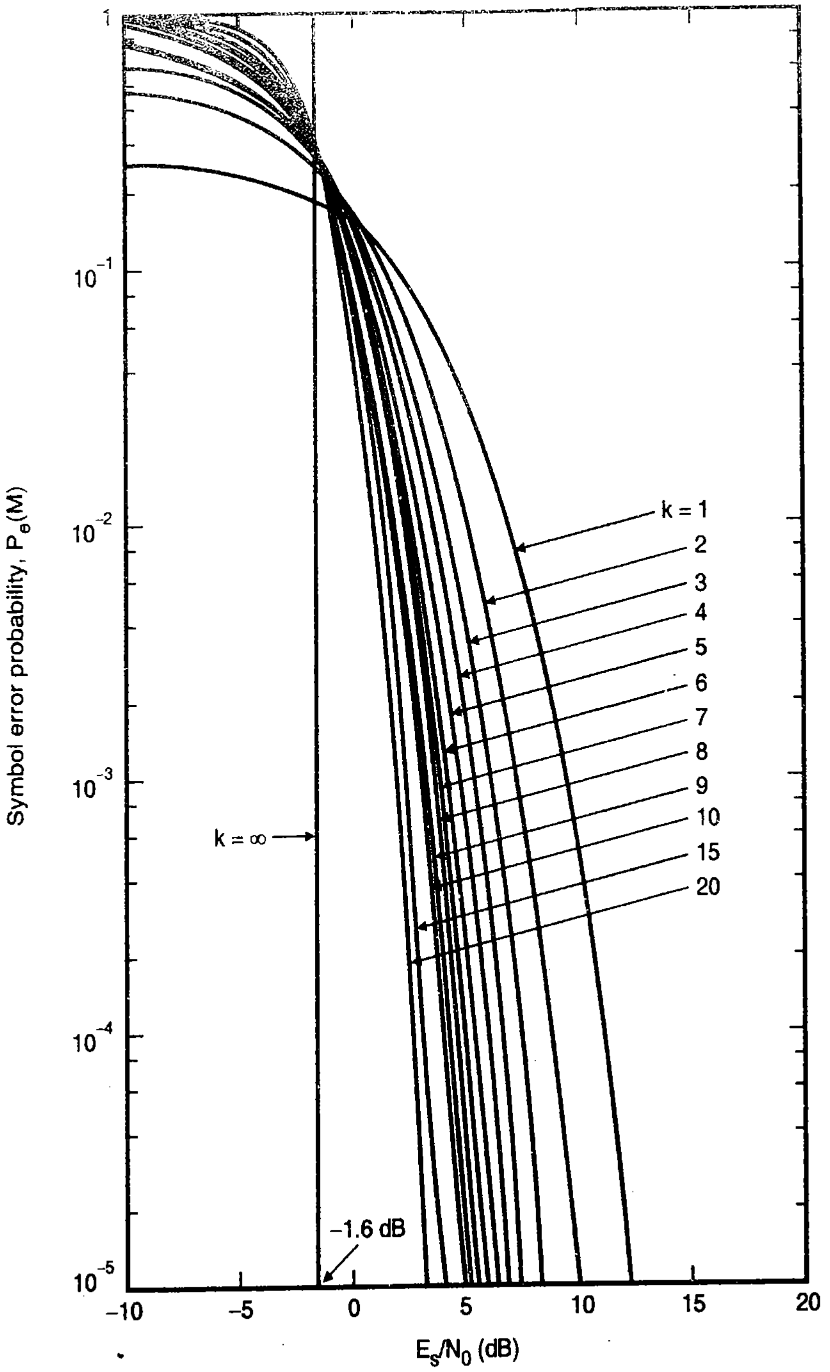
- Fig. 6.14.4 indicates that if we want to keep symbol error constant then we have to increase the value of signal energy to noise ratio E_s/N_0 as M is increased.
- If the ratio E_s/N_0 is maintained constant, then with increase in the value of M, the error probability goes on increasing.

Comparison of MPSK and MFSK systems :

- If the ratio E_s/N_0 and M are same for the MPSK and MFSK systems then the symbol error probability of MPSK system is less than that of MFSK system.
- Hence the error performance of MPSK system is better than that of MFSK system.

6.14.7 Comparison of M-ary PSK and M-ary FSK :

Sr. No.	Parameter	M-ary PSK	M-ary FSK
1.	Expression of transmitted signal	$\sqrt{2P_s}$ $\cos(\omega_c t + \phi_m)$ $\phi_m = (2m+1)$ $\frac{\pi}{4}$, m = 0, 1, 2 ... M – 1	$\sqrt{2P_s} \cos(\omega_c t)$ with c = 1, 2, ... M – 1
2.	Number of bits per symbol	$N \times (M = 2^N)$	$N \times (M = 2^N)$
3.	Symbol duration	$T_s = N T_b$	$T_s = N T_b$
4.	Information is transmitted through	Change in phase of carrier	Change in frequency of carrier
5.	Demodulation method	Coherent (synchronous)	Noncoherent
6.	Bandwidth (BW)	$2 f_b / N$	$2^{N+1} f_b / N$
7.	Minimum distance between signal points	$d = 2\sqrt{E_s} \sin(\pi/M)$	$\sqrt{2NE_b}$
8.	Probability of error	More than that in M-ary FSK	Less than that in the M-ary PSK



(E-468) **Fig. 6.14.4 : Symbol error probability for coherently detected M- ary FSK**

6.15 Minimum Shift Keying (MSK) :

SPPU : Dec. 07, Dec. 11

University Questions

Q. 1 State basic principle of MSK with block schematic and suitable waveforms. Compare its performance with GMSK. **(Dec. 07, 10 Marks)**

Q. 2 Explain the necessity of continuous PSK. State and explain the basic principles of MSK with block schematic and suitable waveforms. **(Dec. 11, 10 Marks)**

To understand the MSK system we will compare MSK and QPSK systems on the basis of various points. The two major differences between QPSK and MSK are as follows :

1. In the MSK system, the baseband signal which is used to multiply the quadrature carriers is a "smooth" signal and not a rectangular signal as used for QPSK.
2. The spectrum of MSK has a much wider main lobe as compared to that of the QPSK system. Typically the main lobe of MSK is 1.5 times wider than the main lobe of QPSK. The side lobes of MSK are much smaller as compared to those of QPSK.

3. The waveforms of MSK has an important property called phase continuity. That means there are no abrupt changes in phase of MSK like QPSK. Due to this feature of MSK, the intersymbol interference caused by the nonlinear amplifiers is avoided completely.

6.15.1 Waveforms of MSK :

SPPU : Dec. 11

University Questions

Q. 1 Explain the necessity of continuous PSK. State and explain the basic principles of MSK with block schematic and suitable waveforms. **(Dec. 11, 10 Marks)**

The waveforms of MSK are as shown in Fig. 6.15.1. These waveforms are drawn with the following procedure.

Step 1 : The typical data bit stream b (t) is divided into an odd and even bit streams b_o (t) and b_e (t) as shown in Fig. 6.15.1(b) and (c). The even bit stream b_e (t) consists of the even numbered bits such as b_2 , b_4 ... etc. Whereas the odd bit stream b_o (t) consists of the odd numbered bits b_1 , b_3 ... etc. Each bit in both these streams has a time duration of two bit intervals i.e. for $T_s = 2 T_b$, where T_s = Symbol time.

Step 2 : The staggering process used in OQPSK is used in MSK as well. Therefore the signal b_o (t) and b_e (t) do not change simultaneously. They change one by one after every bit interval T_b.

Step 3 : Alongwith the even and odd bit streams, two more waveforms are generated at the MSK transmitter, which are :

$\sin 2\pi (t / 4 T_b)$ and $\cos 2 \pi (t / 4 T_b)$.

These two signals are produced in such a way that $\sin 2\pi (t / 4 T_b)$ passes through zero at the end of the symbol time in b_e (t) and $\cos 2\pi (t / 4 T_b)$ passes through zero at the end of the symbol time in b_o (t) as shown in Fig. 6.15.1(d).

Step 4 : Then product of b_e (t) with $\sin 2\pi (t / 4 T_b)$ and b_o (t) with $\cos 2\pi (t / 4 T_b)$ are generated as shown in Fig. 6.15.1(e) and (f).

6.15.2 Expression for the Transmitted MSK Signal :

In MSK the expression for the transmitted signal is given by,

$$V_{MSK} (t) = \sqrt{2 P_s} \, [b_e (t) \sin 2\pi (t / 4 T_b)] \cos \omega_c t$$

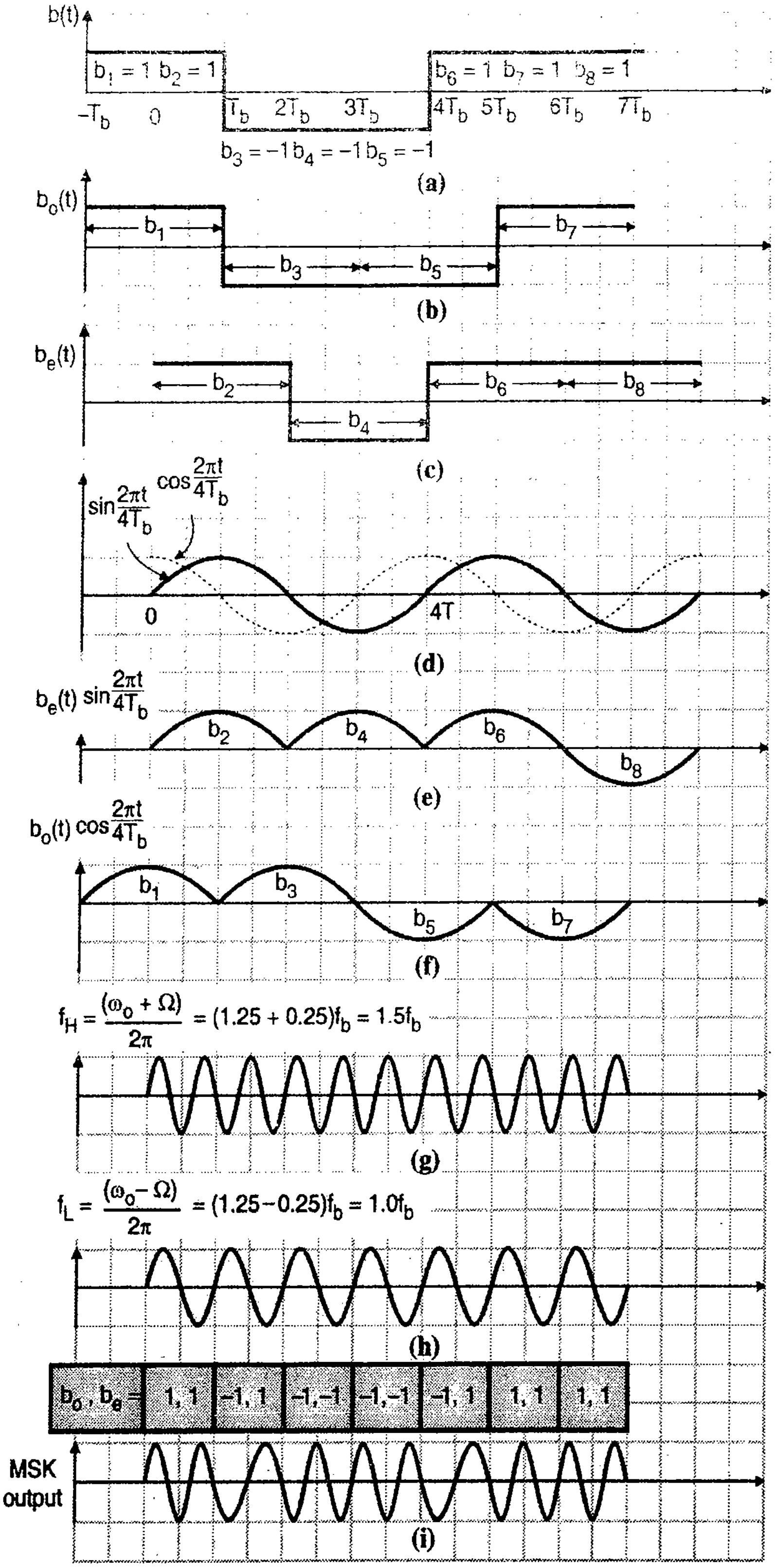
$$+ \sqrt{2 P_s} \, [b_o (t) \cos 2\pi (t / 4 T_b)] \sin \omega_c t \quad ...(6.15.1)$$

Conclusion :

In MSK the two quadrature carriers $\sin \omega_c t$ and $\cos \omega_c t$ are multiplied by "smooth" waveform of Figs. 6.15.1(e) and (f) and not by the abruptly changing waveforms like OQPSK. Due to this the side lobes generated in the MSK spectrum are much smaller and easy to suppress to avoid interchannel interference.

6.15.3 MSK is called as Shaped QPSK :

- Equation (6.15.1) and the waveforms shown in Fig. 6.15.1 indicate that MSK is very similar to an offset QPSK system.
- The only difference between the two is that in OQPSK the quadrature carriers are multiplied by the abruptly changing signals b_o (t) and b_e (t).
- But in MSK, the quadrature carriers are multiplied by smooth signals $b_o (t) \cos 2\pi (t / 4 T_b)$ and $b_e (t) \sin 2\pi (t / 4 T_b)$.
- We may imagine that the $\cos 2\pi (t / 4 T_b)$ and $\sin 2\pi (t / 4 T_b)$ signal are modifying or shaping the OQPSK signal to produce the MSK signal.
- Therefore MSK is called as shaped QPSK.
- After this discussion we may feel that MSK is a type of QPSK. But it is not true. Now we shall show that MSK is not QPSK but it is basically an FSK system.



(E-420) **Fig. 6.15.1 : MSK waveforms**

6.15.4 To Prove that MSK is FSK :

SPPU : Dec. 10

University Questions

Q. 1 Starting from signal expression of MSK find suitable values of f_H and f_L. **(Dec. 10, 8 Marks)**

- We know that,

$$\sin A \cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)]$$

and
$$\cos A \sin B = \frac{1}{2}[\sin(A+B)-\sin(A-B)]$$

- Using these standard identities, we can simplify Equation (6.15.1) as follows :

$$V_{MSK}(t) = \sqrt{2P_s}\, b_e(t)\,[\cos \omega_c t \times \sin 2\pi (t/4T_b)]$$

$$+ \sqrt{2\,P_s}\, b_o(t)\, [\sin \omega_c t \times \cos 2\pi\, (t/4\,T_b)]$$

$$= \frac{\sqrt{2\,P_s} \times b_e(t)}{2} \left[\sin\left(\omega_c + \frac{2\pi}{4\,T_b}\right) t - \sin\left(\omega_c - \frac{2\pi}{4\,T_b}\right) t \right]$$

$$+ \frac{\sqrt{2\,P_s}\, b_o(t)}{2} \left[\sin\left(\omega_c + \frac{2\pi}{4\,T_b}\right) t + \sin\left(\omega_c - \frac{2\pi}{4\,T_b}\right) t \right]$$

$$\therefore\ V_{MSK}(t) = \sqrt{2\,P_s} \left[\frac{b_e(t) + b_o(t)}{2} \right] \sin\left(\omega_c + \frac{2\pi}{4\,T_b}\right) t$$

$$+ \sqrt{2\,P_s} \left[\frac{b_o(t) - b_e(t)}{2} \right] \sin\left(\omega_c - \frac{2\pi}{4\,T_b}\right) t \qquad \text{...(6.15.2)}$$

- Substitute $\frac{2\pi}{4\,T_b} = \Omega = 2\pi \left(\frac{f_b}{4}\right)$ in the above expression to get,

$$V_{MSK}(t) = \sqrt{2\,P_s} \left[\frac{b_o(t) + b_e(t)}{2} \right] \sin(\omega_c + \Omega)\, t$$

$$+ \sqrt{2\,P_s} \left[\frac{b_o(t) - b_e(t)}{2} \right] \sin(\omega_c - \Omega)\, t \qquad \text{...(6.15.3)}$$

- Let $C_H(t) = \frac{b_o(t) + b_e(t)}{2}$ and

$$C_L(t) = \frac{b_o(t) - b_e(t)}{2}$$

- Substituting these in Equation (6.15.3) we get,

$$V_{MSK}(t) = \sqrt{2\,P_s}\, C_H(t) \sin(\omega_c + \Omega)\, t$$
$$+ \sqrt{2\,P_s} \times C_L(t) \sin(\omega_c - \Omega)\, t$$

$$\therefore\ V_{MSK}(t) = \sqrt{2\,P_s}\, C_H(t) \sin \omega_H t$$
$$+ \sqrt{2\,P_s}\, C_L(t)\, \sin \omega_L t \qquad \text{...(6.15.4)}$$

- This expression clearly shows that MSK is FSK. Now refer to Fig. 6.15.1(g), (h) and (i) which show you that the frequency of the transmitted MSK signal is dependent on the values of $b_o(t)$ and $b_e(t)$.

For example :

(a) If $b_o(t) = b_e(t)$, then $C_H = \pm 1$ and $C_L = 0$.

(b) If $b_o(t) = -\, b_e(t)$, then $C_H = 0$ and $C_L = \pm 1$.

Thus depending on the values of $b_o(t)$ and $b_e(t)$ in each bit interval, the transmitted signal will have a frequency of either f_H or f_L and the magnitude of MSK signal will always remain constant, at $\sqrt{2\,P_s}$ as shown in Fig. 6.15.1.

6.15.5 Values of f_H and f_L :

SPPU : Dec. 10

University Questions

Q. 1 Starting from signal expression of MSK find suitable values of f_H and f_L. **(Dec. 10, 8 Marks)**

- In MSK the frequencies f_H and f_L are selected carefully in such a way that the two possible output signals are orthogonal over one bit interval T_b. That means they should satisfy the following condition :

$$\int_0^{T_b} \sin \omega_H t \times \sin \omega_L t \, dt = 0 \qquad \text{...(6.15.5)}$$

- We know that,

$$\sin A \sin B = \frac{1}{2} [\cos(A + B) - \cos(A - B)]$$

- Applying this to Equation (6.15.5) we get,

$$\therefore\ \frac{1}{2} \int_0^{T_b} [\cos(\omega_H + \omega_L)\, t - \cos(\omega_H - \omega_L)\, t]\, dt = 0$$

$$\therefore\ \int_0^{T_b} \cos(\omega_H + \omega_L)\, t\, dt - \int_0^{T_b} \cos(\omega_H - \omega_L)\, t\, dt = 0$$

$$\therefore\ \sin(\omega_H + \omega_L)\, T_b = 0 \text{ and } \sin(\omega_H - \omega_L)\, T_b = 0$$

$$\therefore\ 2\pi\, (f_H + f_L)\, T_b = m\pi \qquad \text{...(6.15.6)}$$

$$\text{and } 2\pi\, (f_H - f_L)\, T_b = n\pi \qquad \text{...(6.15.7)}$$

where m and n are integers.

- But we know that,

$$\Omega = \frac{2\pi}{4\,T_b} = 2\pi \times \left(\frac{f_b}{4}\right) \qquad \text{...(6.15.8)}$$

$$\therefore\ f_H = f_C + \Omega = f_C + \frac{f_b}{4} \qquad \text{...(6.15.9)}$$

$$\text{and}\ f_L = f_C - \Omega = f_C - \frac{f_b}{4} \qquad \text{...(6.15.10)}$$

- Now refer to Equations (6.15.6) and (6.15.7). They can be written as,

$$2\,(f_H + f_L)\, T_b = m \qquad \text{and } 2\,(f_H - f_L)\, T_b = n$$

- But as $T_b = 1 / f_b$, the above equations will be modified to,

$$2\,(f_H + f_L) = m\, f_b \qquad \text{...(6.15.11)}$$

$$\text{and}\ 2\,(f_H - f_L) = n\, f_b \qquad \text{...(6.15.12)}$$

- Now refer to Equations (6.15.9) and (6.15.10). Add these equations to get,

$$(f_H + f_L) = 2\, f_C \qquad \text{...(6.15.13)}$$

- Subtract Equation (6.15.10) from Equation (6.15.9) we get,

$$(f_H - f_L) = 2\left(\frac{f_b}{4}\right) \qquad \text{...(6.15.14)}$$

- Substitute Equation (6.15.13) into Equation (6.15.11) to get,

$$2 \times 2\, f_C = m\, f_b$$

$$\therefore\ f_C = \frac{m}{4}\, f_b \qquad \text{...(6.15.15)}$$

- And substitute Equation (6.15.14) into Equation (6.15.12) to get,

$$2 \times \frac{f_b}{2} = n f_b$$

$$\therefore \quad n = 1 \qquad \text{...(6.15.16)}$$

- Substituting the value of f_C into Equations (6.15.9) and (6.15.10) we get,

$$f_H = (m+1)\frac{f_b}{4}$$

$$\text{and} \quad f_L = (m-1)\frac{f_b}{4} \qquad \text{...(6.15.17)}$$

Why is the name MSK ?

Equation (6.15.16) shows that n = 1 hence $(f_H - f_L) = f_b / 2$. Thus f_H and f_L are as close as possible so that the condition for their orthogonality is satisfied. This is the reason why this system is called as "minimum shift keying".

6.15.6 Signal Space Representation of MSK :

SPPU : Dec.-07

University Questions

Q. 1 For rectangular data pulses calculate the second null-to-null bandwidth of BPSK, QPSK, MSK, 16 PSK and 16 QAM. Discuss advantages and disadvantages of using each of these methods. **(Dec. 07, 6 Marks)**

The signal space representation of MSK is shown in Fig. 6.15.2.

- We know that the MSK signal is represented mathematically as,

$$V_{MSK}(t) = \sqrt{2P_s}\, C_H(t) \sin \omega_H t + \sqrt{2P_s}\, C_L(t) \sin \omega_L t \qquad \text{... from Equation (6.15.4)}$$

Let us rearrange this expression as follows :

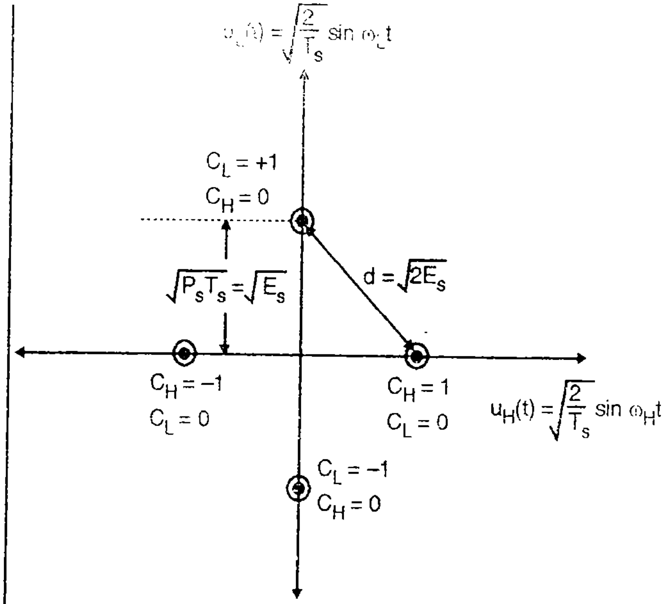
$$V_{MSK}(t) = C_H(t)\sqrt{P_s T_s} \times \sqrt{2/T_s} \sin \omega_H t + C_L(t)\sqrt{P_s T_s}\sqrt{2/T_s} \sin \omega_L t \qquad \text{...(6.15.18)}$$

- In Equation (6.15.18) let the orthonormal unit vectors of the co-ordinate system be given by,

$$u_H(t) = \sqrt{2/T_s} \sin \omega_H t \qquad \text{...(6.15.19)}$$

$$\text{and} \quad u_L(t) = \sqrt{2/T_s} \sin \omega_L t \qquad \text{...(6.15.20)}$$

- For different values of $C_H(t)$ and $C_L(t)$ we get different expressions for $V_{MSK}(t)$ as listed in the Table 6.15.1. These four possible combinations are plotted as shown in Fig. 6.15.2.



(E-421) **Fig. 6.15.2 : Signal space representation of MSK**

Table 6.15.1

C_H	C_L	$V_{MSK}(t)$
0	+ 1	$\sqrt{P_s T_s}\sqrt{2/T_s} \sin \omega_L t = \sqrt{P_s T_s}\, u_L(t)$
0	− 1	$-\sqrt{P_s T_s}\sqrt{2/T_s} \sin \omega_L t$ $= -\sqrt{P_s T_s}\, u_L(t)$
+ 1	0	$\sqrt{P_s T_s} \times \sqrt{2/T_s} \sin \omega_H t = \sqrt{P_s T_s}\, u_H(t)$
− 1	0	$-\sqrt{P_s T_s} \cdot \sqrt{2/T_s} \sin \omega_H t$

- The four possible signals are represented by the circled dots in Fig. 6.15.2. These four points have been plotted for different combinations of the values of $C_H(t)$ and $C_L(t)$.
- Looking at Fig. 6.15.2 we can calculate the minimum Euclidean distance "d" as,

$$d^2 = P_s T_s + P_s T_s = 2 P_s T_s$$

$$\therefore \quad d = \sqrt{2 P_s T_s} \qquad \text{...(6.15.21)}$$

- But $T_s = 2\, T_b$. Substituting this into Equation (6.15.21) we get,

$$d = \sqrt{4 P_s T_b} \qquad \text{...(6.15.22)}$$

- But $P_s T_b = E_b$ i.e. energy per bit. Hence Equation (6.15.22) get modified to,

$$d = \sqrt{4 E_b} \qquad \text{...(6.15.23)}$$

- This is the smallest distance between these signal points which is same as that for QPSK. Therefore the error probability of MSK will be same as that of QPSK.

6.15.7 Phase Continuity in MSK :

SPPU : May 07, May 08, Dec. 08, May 09, May 12

University Questions

Q. 1 Explain phase continuity in MSK. **(May 07, 6 Marks)**

Q. 2 Write a note on : Phase continuity in MSK. **(May 08, May 09, 5 Marks)**

Q. 3 Compare QPSK and MSK. Explain phase continuity of MSK. **(Dec. 08, 6 Marks, May 09, 8 Marks)**

Q. 4 Explain the performance of MSK with suitable block schematic and also explain how phase continuity is maintained in this system ? **(May 12, 8 Marks)**

- The most important feature of MSK is its phase continuity. This has been clearly illustrated in Fig. 6.15.1(g), (h) and (i).
- If we assume that $f_C = 5\frac{f_b}{4}$ then the values of f_H and f_L are given by,

$$f_H = 5\frac{f_b}{4} + \frac{f_b}{4} = 1.5\, f_b \qquad ...(6.15.24)$$

$$\text{and} \quad f_L = 5\frac{f_b}{4} - \frac{f_b}{4} = f_b \qquad ...(6.15.25)$$

- Now depending on the values of $b_e(t)$ and $b_o(t)$, we can get the values of $V_{MSK}(t) / \sqrt{2P_s}$ as shown in Table 6.15.2. Note that we have obtained these values from the following equation of $V_{MSK}(t)$ which states that,

$$V_{MSK}(t) = \sqrt{2P_s}\, C_H(t) \sin \omega_H t + \sqrt{2P_s}\, C_L(t) \sin \omega_L t \qquad ...(6.15.26)$$

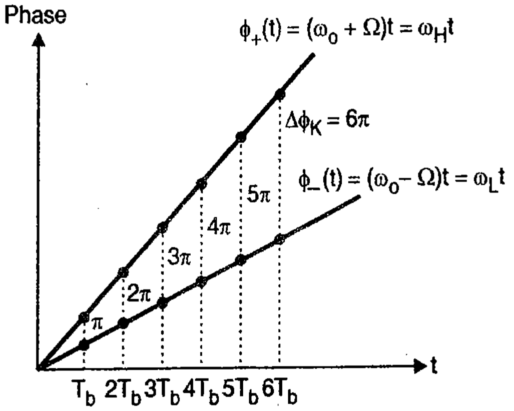
$$\text{Where} \quad C_H(t) = \frac{b_o(t) + b_e(t)}{2} \quad \text{and}$$

$$C_L(t) = \frac{b_o(t) - b_e(t)}{2}.$$

Table 6.15.2

$b_e(t)$	$b_o(t)$	$V_{MSK}(t)$
-1	-1	$-\sin \omega_H t$
-1	1	$\sin \omega_L t$
1	-1	$-\sin \omega_L t$
1	1	$\sin \omega_H t$

- Because of staggering, $b_o(t)$ and $b_e(t)$ do not change simultaneously. The waveform for $V_{MSK}(t)$ is generated in the following way :
- In each bit interval, we determine from the Table 6.15.2 whether to use the carrier of frequency f_H or of frequency f_L and whether the waveform is to be inverted or not.
- After following all the steps listed above, we observe that the waveform of $V_{MSK}(t)$ is a "smooth" waveform and it does not exhibit any abrupt changes in phase.
- Now let us prove that the phase continuity illustrated in Fig. 6.15.2 is a general characteristics of MSK. Therefore we refer to Table 6.15.2 to write down the expression for $V_{MSK}(t)$ as follows :



(E-422) **Fig. 6.15.3 : A figure illustrating the phase continuity in the MSK system**

$$V_{MSK}(t) = b_o(t) \times \sqrt{2P_s} \sin [\omega_c t + b_o(t)\, b_e(t)\, \Omega t] \qquad ...(6.15.27)$$

- This expression will yield the same results as those by Equation (6.15.26) which represents the MSK signal.

6.15.8 MSK Transmitter :

SPPU : Dec. 11, May 12

University Questions

Q. 1 Explain the necessity of continuous PSK. State and explain the basic principles of MSK with block schematic and suitable waveforms. **(Dec. 11, 10 Marks)**

Q. 2 Explain the performance of MSK with suitable block schematic and also explain how phase continuity is maintained in this system ? **(May 12, 8 Marks)**

The MSK transmitter is as shown in Fig. 6.15.4.

Operation :

- The carrier signal $\sin \omega_c t$ is multiplied with the carrier $(\cos \Omega t)$ in a balanced modulator (multiplier) to produce the following output,

  Output of the first multiplier = $\sin \omega_c t \cos \Omega t$

- Let us represent the two distinct values of this phase angle as,

$$\phi_+(t) = (\omega_c + \Omega)\, t \; ... \text{for } b_o(t) \times b_e(t) = +1 \qquad ...(6.15.28)$$

$$\text{and} \quad \phi_-(t) = (\omega_c - \Omega)\, t \; ... \text{for } b_o(t) \times b_e(t) = -1 \qquad ...(6.15.29)$$

The variation of $\phi_+ (t)$ and $\phi_- (t)$ has been plotted in Fig. 6.15.3.

- In MSK, both $b_o (t)$ and $b_e (t)$ will not change simultaneously. Thus $b_o (t)$ will change at times $t = k T_b$ with k odd and $b_e (t)$ will change at times $t = k T_b$ with k even.

Consider that $b_e (t)$ has changed its value.

- Therefore in Equation (6.15.27) there is an abrupt phase change in $\phi (t)$ of magnitude,

$$2 \times \Omega \times t = 2 \times \Omega \times k T_b \qquad ...(6.15.30)$$

$$= 2 \times 2\pi \left(\frac{f_b}{4}\right) \times k \times T_b$$

$$\therefore \quad 2 \Omega t = k\pi \qquad ...(6.15.31)$$

As $b_e (t)$ changes at times $t = k T_b$ with k even i.e. 2, 4, 6 .... the associated phase change will be 2π, 4π, 6π... i.e. the phase change is multiple of 2π, which is equivalent to no change in phase at all. **Thus with change in $b_e (t)$ there is absolutely no change in phase.**

- Now consider the effect of change in $b_o (t)$. In this case the phase change in $\phi (t)$ will be an odd multiple of π i.e. π , 3π, 5π ... etc. That means the phase change is equal to π. But in Equation (6.15.27) we have $b_o (t)$ as a multiplying factor. So it will also change its sign, which corresponds to an additional phase shift of π. Thus the net phase shift corresponding to change in b_o (t) also is zero. **Thus with change in $b_o (t)$ there is absolutely no change in phase.**

Conclusion : Thus we conclude that there are no phase discontinuities present in the output of an MSK system.

Generation and reception of MSK :

The MSK transmitter is as shown in Fig. 6.15.4.

Operation :

- The carrier signal sin ω_c is multiplied with the cos Ωt in a balanced modulator (multiplier) to produce the following output,

  Output of the first multiplier = sin ω $_c$t cos Ωt

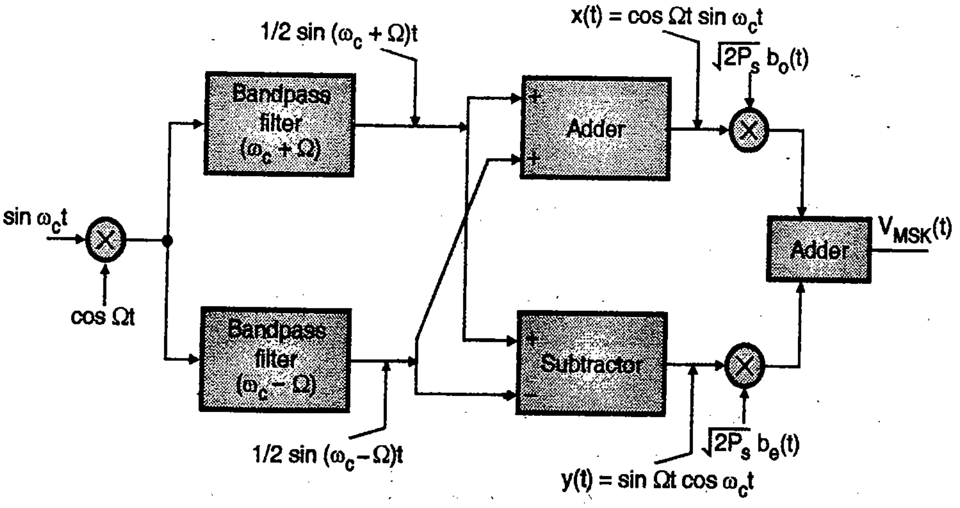
$$= \frac{1}{2} \sin (\omega_c + \Omega) t + \frac{1}{2} \sin (\omega_c - \Omega) t \qquad ...(6.15.32)$$

- Equation (6.15.32) shows that the multiplier output contains the sum and difference components of ω_c and Ω. This is applied to two bandpass filters with center frequencies at $(\omega_c + \Omega)$ and $(\omega_c - \Omega)$ respectively.
- The outputs of the two bandpass filters are given by,

  (a) Output of BPF_1 $= \frac{1}{2} \sin (\omega_c + \Omega)$

  (b) Output of BPF_2 $= \frac{1}{2} \sin (\omega_c - \Omega)$

Both these outputs are applied to an adder and a subtractor. At the output of these adder and subtractor we get the signals x (t) and y (t) respectively which are given by,



(E-423) **Fig. 6.15.4 : MSK transmitter**

$x(t) = \frac{1}{2} \sin(\omega_c + \Omega)t + \frac{1}{2} \sin(\omega_c - \Omega)$

$t = \sin \omega_c t \cos \Omega t$ ... as per Equation (6.15.32)

and $y(t) = \frac{1}{2} \sin(\omega_c + \Omega)t - \frac{1}{2} \sin(\omega_c - \Omega)t$

$= \frac{1}{2} [\sin \omega_c t \cos \Omega t + \cos \omega_c t$

$\sin \Omega t - \sin \omega_c t \cos \Omega t + \cos \omega_c t \sin \Omega t]$

$\therefore \quad y(t) = \cos \omega_c t \sin \Omega t$ ...(6.15.33)

- x (t) is then multiplied with $\sqrt{2 P_s}\, b_o(t)$ while y (t) is multiplied with $\sqrt{2 P_s}\, b_e(t)$. The outputs of these multipliers are then added together to produce the MSK signal given by,

$$V_{MSK}(t) = \sqrt{2 P_s}\, b_o(t) \sin \omega_c t \cos \Omega t$$

$$+ \sqrt{2 P_s}\, b_e(t) \cos \omega_c t \sin \Omega t \qquad ...(6.15.34)$$

$$= \sqrt{2 P_s}\, b_o(t) \times \frac{1}{2} [\sin(\omega_c + \Omega)t + \sin(\omega_c - \Omega)t]$$

$$+ \sqrt{2 P_s}\, b_e(t) \times \frac{1}{2} [\sin(\omega_c + \Omega)t - \sin(\omega_c - \Omega)t]$$

$$= \sqrt{2 P_s} \left[\frac{b_o(t) + b_e(t)}{2}\right] \sin(\omega_c + \Omega)t$$

$$+ \sqrt{2 P_s} \left[\frac{b_o(t) - b_e(t)}{2}\right] \sin(\omega_c - \Omega)t$$

$$V_{MSK}(t) = \sqrt{2 P_s}\, C_H(t) \sin \omega_H t + \sqrt{2 P_s}\, C_L(t) \times \sin \omega_L t \qquad ...(6.15.35)$$

- This expression is same as the one obtained in Equation (6.15.26). Thus the transmitter of Fig. 6.15.4 generates the MSK signal.

6.15.9 MSK Receiver :

SPPU : Dec. 11, May 12

University Questions

Q. 1 Explain the necessity of continuous PSK. State and explain the basic principles of MSK with block schematic and suitable waveforms. **(Dec. 11, 10 Marks)**

Q. 2 Explain the performance of MSK with suitable block schematic and also explain how phase continuity is maintained in this system ? **(May 12, 8 Marks)**

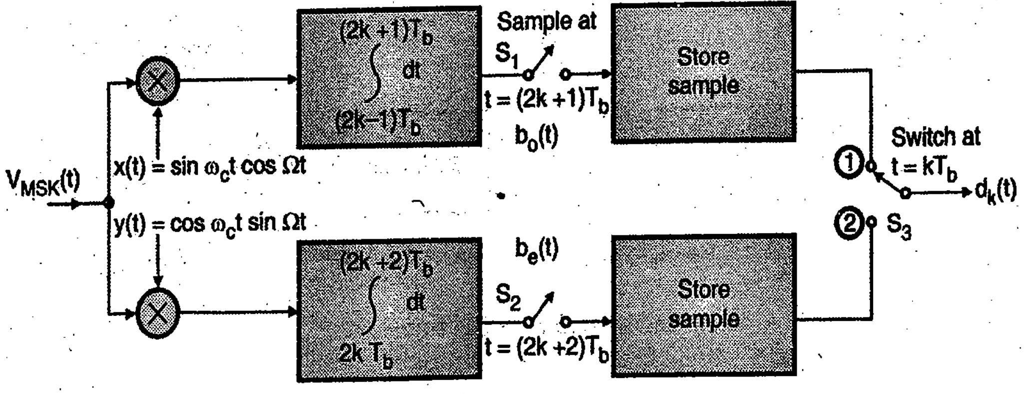
- The block diagram of an MSK receiver is as shown in Fig. 6.15.5. This is the synchronous type of detection. As discussed earlier, this type of detection is performed by multiplication and integration over the symbol interval.

Operation :

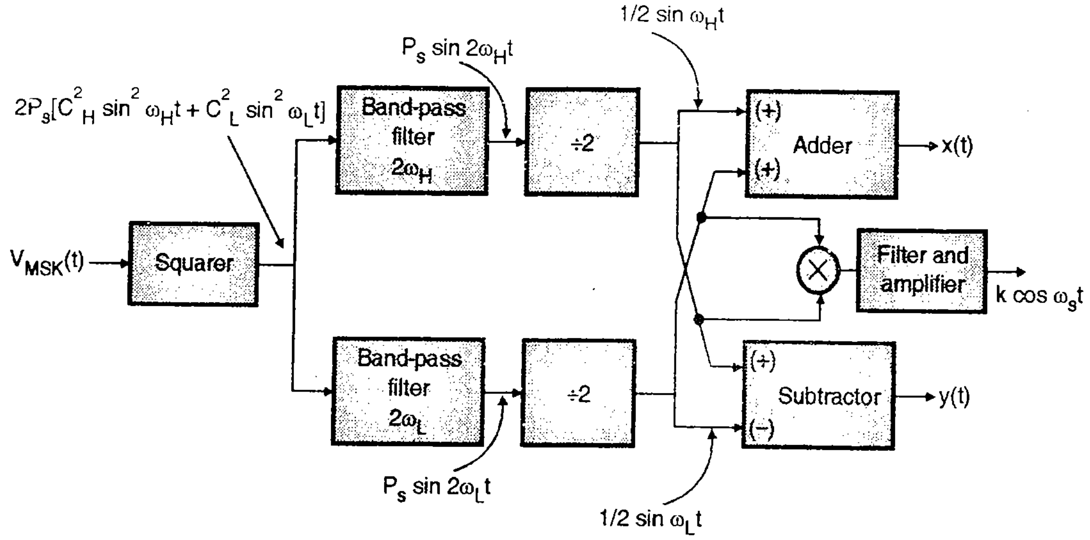
- The signals x (t) and y (t) are regenerated at the receiver. Then the incoming MSK signal is multiplied by the signals x (t) and y (t) in the two balanced modulators.
- The bit $b_o(t)$ is determined from the multiplier integrator chain which uses signal x (t) and the bit $b_e(t)$ is obtained from the multiplier integrator chain which uses signal y (t).
- Both the integrators will integrate over the symbol duration of 2 T_b seconds. At the end of each integration interval, the integrator outputs are sampled and stored.
- The switch S_3 at the output will then switch between the positions 1 and 2 at the rate equal to bit rate, so as to generate the original data bit stream $d_k(t)$.

6.15.10 Generation of the Signals x (t) and y (t) :

- The block diagram of Fig. 6.15.6 shows the scheme to generate the signals x (t) and y (t) at the MSK receiver.



(E-424) **Fig. 6.15.5 : MSK receiver**



(E-425) **Fig. 6.15.6 : Generation of the signal x (t) and y (t)**

Operation :

- The received MSK signal is first applied to a square law device (similar to the BPSK reception). The output of this square law device is given by,

Output of square law device

$$= 2 P_s \left[C_H^2 \sin^2 \omega_H t + C_L^2 \sin^2 \omega_L t\right] \quad ...(6.15.36)$$

- The output of the square law device consists of spectral components at frequencies 2 ω_H and 2 ω_L. This output is applied to two bandpass filters which have center frequencies at 2 ω_H and 2 ω_L respectively.
- These bandpass filters will allow only the signals of frequency 2 ω_H and 2 ω_L respectively, to pass through them.

$\therefore$ Output of bandpass filter 1 = $P_s \sin 2 \omega_H t$

and output of bandpass filter 2 = $P_s \sin 2 \omega_L t$

- The filter outputs are then passed to frequency dividers. A frequency division by 2 will produce the waveforms $\frac{1}{2} \sin \omega_H t$ and $\frac{1}{2} \sin \omega_L t$. From these waveforms, the signals x (t) and y (t) are generated by using adder and subtractor as shown in Fig. 6.15.6.
- The frequency divider outputs are then applied to a multiplier and a low pass filter to generate a signal at the symbol rate $f_s = f_b / 2$. This signal is used to operate the sampling switches S_1, S_2 and S_3 in Fig. 6.15.5 (MSK receiver).

6.15.11 Advantages of MSK as Compared to QPSK :

SPPU : Dec. 12

University Questions

Q. 1 State advantages of MSK over QPSK.

(Dec. 12, 10 Marks)

Some of the important advantages of MSK are :

1. The baseband waveforms used for multiplication with the quadrature carriers are smooth signals (sine and cosine waves.)
2. The MSK signal has a phase continuity. That means there are no phase changes in the MSK signal. In QPSK there are abrupt changes in the phase shifts ($\pi/2$ or π radians).
3. There are no amplitude variations, when a MSK signal is passed through a filter. Whereas there are sudden variations in the amplitude of the QPSK when it is passed through a filter.
4. If we compare the spectrums, then it is found that the main lobe of MSK is wider than that of QPSK. In addition to that, more energy is contained in the main lobe of MSK (about 99 % of the total energy) as compared to the QPSK (about 90 % of the total energy).
5. The size of MSK sidelobes is always small as compared to the sidelobes of QPSK. Hence the interchannel interference is reduced.
6. In QPSK, large amplitudes of sidelobes will result in interchannel interference. To reduce such an interference we have to use filters to suppress the sidelobes. No such filter is required to be used for the MSK system.

6.15.12 Disadvantages of MSK System :

1. The generation and detection of MSK signal is more complicated as compared to that of QPSK. The performance of the MSK system depends on the quality of synchronisation. Incorrect synchronisation can introduce phase errors.

2. The bandwidth of MSK is 1.5 f_b whereas the bandwidth of QPSK is only f_b. Eventhough on paper the bandwidth of MSK is more, actually this bandwidth is sufficient to pass about 99 % of the signal power. On the contrary QPSK will need a bandwidth of about 8 f_b in order to pass the same amount of power.

6.15.13 Power Spectral Density and Bandwidth of MSK :

SPPU : Dec. 07, May 13

University Questions

Q. 1 For rectangular data pulses calculate the second null-to-null bandwidth of BPSK, QPSK, MSK, 16 PSK and 16 QAM. Discuss advantages and disadvantages of using each of these methods. **(Dec. 07, 6 Marks)**

Q. 2 Draw signal space and spectral diagram of following digital CW modulation and state only the bandwidth requirement : 16 QAM, 16-ary PSK, QPSK and MSK. **(May 13, 6 Marks)**

- We will obtain the power spectral density (PSD) of the MSK signal by considering the expression for the baseband MSK signal (which is used to modulate the quadrature carrier sin 2 π f_c t) given by,

$$V_b(t) = \sqrt{2 P_s}\,[b_o(t) \cos(2\pi t / 4 T_b)]$$

$$= \sqrt{2 P_s}\, b_o(t) \cos(\pi f_b t / 2)$$

- The power spectral density of this signal is given by

$$S_b(f) = \frac{32 E_b}{\pi^2}\left[\frac{\cos(2\pi f T_b)}{1-(4 f T_b)^2}\right] \qquad ...(6.15.37)$$

- The normalized, power spectral density of the baseband signal is shown in Fig. 6.15.7. Due to normalization, the maximum amplitudes are scaled with respect to 1. The power spectral density of QPSK also is drawn alongwith it for comparison.

Bandwidth of MSK :

From Fig. 6.15.7, the bandwidth of MSK is given by,

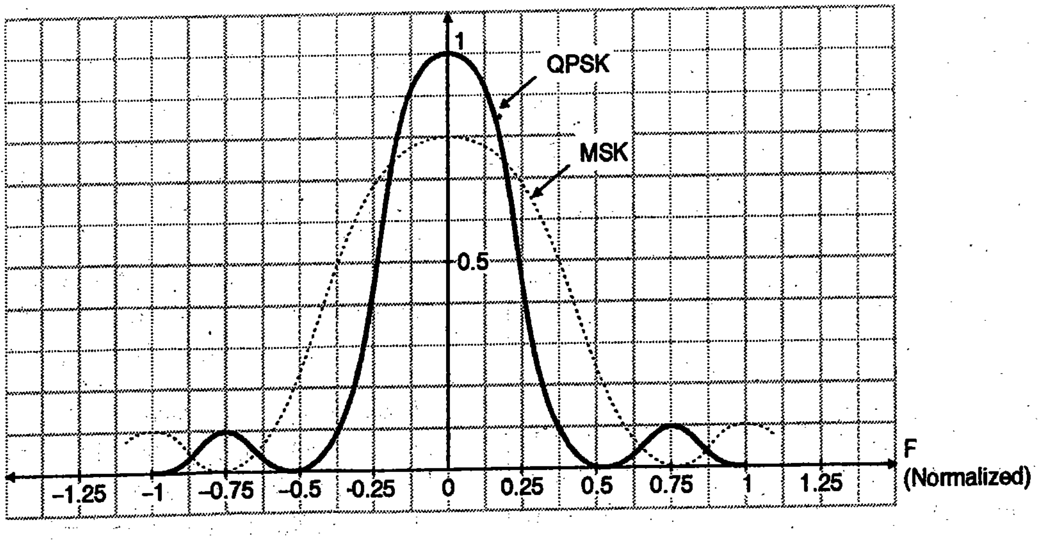
$$BW = \text{Width of the main lobe} = 0.75\ f_b - 0.75\ f_b$$

$$\therefore\ BW = 1.5\ f_b \qquad ...(6.15.38)$$

Thus bandwidth of MSK is higher than that of the QPSK.

6.15.14 Comparison of QPSK and MSK Spectra :

- The main lobe of MSK psd is wider than that of QPSK (see Fig. 6.15.7). Hence the bandwidth of MSK is more than that of QPSK.
- The main lobe of MSK contains about 99% of the signal energy whereas the main lobe of QPSK contains about 90% of the total signal energy.
- The sidelobe amplitude for MSK is much smaller as compared to that of QPSK.



(E-426) **Fig. 6.15.7 : Power spectral densities of MSK and QPSK**

6.15.15 Gaussian MSK :

SPPU : May 06, Dec. 06, May 07, Dec. 07, May 15

University Questions

Q. 1 What is GMSK ? Explain. **(May 06, May 07, 4 Marks)**

Q. 2 Explain with necessary waveforms, block diagram, constellation diagram and the bandwidth requirement, the operation of GMSK. Where it is used ? **(Dec. 06, 8 Marks)**

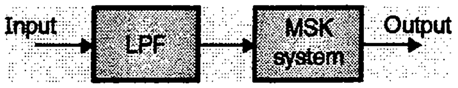
Q. 3 State basic principle of MSK with block schematic and suitable waveforms. Compare its performance with GMSK. **(Dec. 07, 10 Marks)**

Q. 4 Explain GMSK and its use in wireless data transmission. **(May 15, 8 Marks)**

In the preceding section we have discussed the performance of the minimal shift keying (MSK) system which discovered the following important points about MSK :

1. Phase continuity 2. Relatively narrow bandwidth
3. Performance equivalent to that of QPSK with coherent detection.

- However with all these plus points, the MSK system still cannot satisfy the stringent requirements of certain applications such as wireless communication.
- This happens because the baseband power spectral density of MSK signal does not reduce to low enough value as we go away from the center of its psd curve.
- This will result in the adjacent channel interference. Therefore we can not use the MSK system for the multiuser communication applications.
- The performance of the MSK system can be improved by making its baseband power spectrum more compact. This can be achieved by using a low pass filter as shown in Fig. 6.15.8. This filter is also called as baseband pulse shaping filter, and it is introduced before the MSK system.



(E-427) **Fig. 6.15.8 : Gaussian MSK system**

The pulse shaping filter must satisfy the following requirements :

1. It must have narrow bandwidth with sharp cut-off characteristics.
2. Its impulse response must have relatively low overshoots.

- These requirements can be satisfied by passing the non-return to zero (NRZ) binary data stream through a filter whose impulse response (and frequency response) is defined by a Gaussian function. Therefore this system is known as Gaussian MSK system.
- The transfer function of the pulse shaping filter is defined as :

$$H(f) = e^{\left[\frac{-\log 2}{2}(f/W)^2\right]} \qquad ...(6.15.39)$$

where W = 3 dB baseband bandwidth of the filter.

- From this expression, we can obtain the expression for the impulse response h (t) of the pulse shaping filter as :

$$h(t) = \sqrt{\frac{2\pi}{\log 2}} \times W \times e^{\left[-\frac{2\pi^2}{\log 2} W^2 t^2\right]} \qquad ...(6.15.40)$$

The disadvantage of Gaussian MSK system is that, the baseband pulse shaping filter bandlimits the digital signal applied at its input. But in doing so it "spreads outs" each bit beyond the interval T_b. The result of this pulse spreading is intersymbol interference. The Gaussian MSK is widely used in a standardized communication system known as GSM which is TDMA system.

Ex. 6.15.1 : For a bit sequence of 1 0 1 1 0 1 0 1 0 1 show that the MSK waveform has phase continuity.

Soln. :

The mathematical expression for MSK is as follows :

$$V_{MSK}(t) = \sqrt{2P_s}\, C_H(t) \sin \omega_H t + \sqrt{2P_s}\, C_L(t) \sin \omega_L t \qquad ...(1)$$

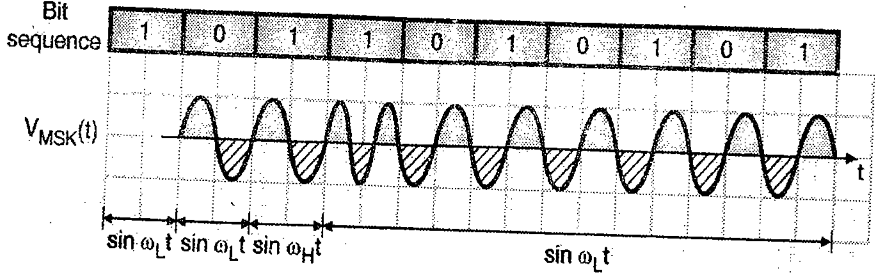
where $C_H(t) = \frac{b_o(t) + b_e(t)}{2}$ and

$C_L(t) = \frac{b_o(t) - b_e(t)}{2}$

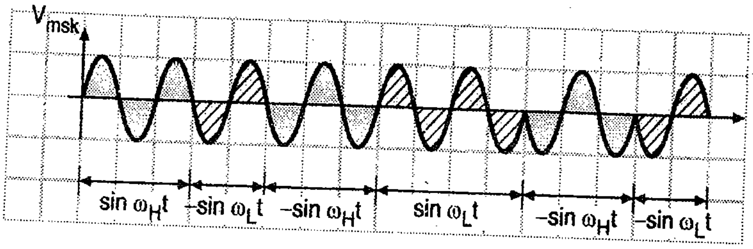
Let us prepare a table to obtain the transmitted MSK signal.

Table P. 6.15.1

Bit number	1	2	3	4	5	6	7	8	9	10
Bit sequence	1	0	1	1	0	1	0	1	0	1
$b_o(t)$	1	1	1	1	– 1	– 1	– 1	– 1	– 1	– 1
$b_e(t)$		– 1	– 1	1	1	1	1	1	1	1
$C_H(t)$		0	0	1	0	0	0	0	0	0
$C_L(t)$		1	1	0	– 1	– 1	– 1	– 1	– 1	– 1
V_{MSK}		$\sin \omega_L t$	$\sin \omega_L t$	$\sin \omega_H t$	$-\sin \omega_L t$	$-\sin \omega_L t$	$-\sin \omega_L t$	$-\sin \omega_L t$	$-\sin \omega_L t$	$-\sin \omega_L t$



(E-428) **Fig. P. 6.15.1**



(E-1609) **Fig. P. 6.15.2 : MSK Waveform**

Table P. 6.15.2

Bit number	1	2	3	4	5	6	7	8
Bit sequence	1	1	0	0	1	0	0	1
b_o (t)	1	1	– 1	– 1	1	1	– 1	– 1
b_e (t)	–	1	1	– 1	– 1	– 1	– 1	1
C_H (t)	–	1	0	– 1	0	0	– 1	0
C_L (t)	–	0	– 1	0	1	1	0	– 1
V_{MSK}	–	sin ω_H t	– sin ω_L t	– sin ω_H t	sin ω_L t	– sin ω_L t	– sin ω_H t	– sin ω_H t

The MSK waveform is as shown in Fig. P. 6.15.1.

The waveform in Fig. P. 6.15.1 shows the phase continuity for the MSK signal.

Ex. 6.15.2 : Sketch the waveforms of MSK for the given bit stream 11001001. **May 15, 8 Marks**

Soln. :

The given bit stream : 11001001

$$C_H(t) = \frac{b_o(t) + b_e(t)}{2}, \quad C_L(t) = \frac{b_o(t) - b_e(t)}{2}$$

$$V_{MSK} = \sqrt{2 P_S}\; C_H(t) \sin \omega_H(t) + \sqrt{2 P_S}\; C_L(t) \sin \omega_L t.$$

Prepare a table as follows to obtain V_{MSK} signal.

The MSK waveform is as shown in Fig. P. 6.15.2.

6.16 Comparison of Digital CW Systems :

SPPU : May 06, May 07, Dec. 07, Dec. 08, May 09, Dec. 12, Dec. 13, Dec. 14

University Questions

Q. 1 Compare MSK and QPSK systems. **(May 06, 6 Marks)**

Q. 2 Compare QPSK and ASK. **(May 07, 4 Marks)**

Q. 3 Diagram the geometric representation of orthogonal QPSK, non- orthogonal QPSK, M-ary PSK, M-ary FSK and QASK. What is the importance of Euclidean distance ? Write its expression for above representation and compare them. **(Dec. 07, 10 Marks)**

Q. 4 Compare QPSK and MSK. Explain phase continuity of MSK. **(Dec. 08, 6 Marks, May 09, 8 Marks)**

Q. 5 Compare the performance of BPSK, FSK, M-ary PSK, M-ary FSK with respect to bandwidth, Euclidian distance and probability of error. **(Dec. 12, 10 Marks)**

Q. 6 Compare the Euclidian distance 'd' and Bandwidth of M-Ary PSK, M-Ary FSK and QAM with $M = 2^n$ for n = 3,4. **(Dec. 13, 10 Marks)**

Q. 7 Compare BPSK, QPSK and M'ary PSK with the help of equations, signal space representation, symbol rate and bandwidth. **(Dec. 14, 9 Marks)**

The comparison of some of the important digital CW systems is given in Table 6.16.1.

Table 6.16.1 : Comparison of digital CW systems

Sr. No.	Parameter	ASK	BPSK	QPSK	QAM	M-ary PSK	BFSK	M-ary FSK	MSK
1.	Information is transmitted by change in	Amplitude	Phase	Phase	Amplitude and phase	Phase	Frequency	Frequency	Frequency
2.	Expression for transmitted signal	$\sqrt{2 P_s}$ $\cos \omega_c t$ for symbol 1 0 for symbol 0	$\sqrt{2 P_s}$ b (t) $\cos \omega_c t$ b (t) = 1 for 1 = –1 for 0	$\sqrt{2 P_s} \cos \left[\omega_c t + (2m+1)\frac{\pi}{4}\right]$ m = 0, 1, 2, 3	$k_1 \sqrt{0.2 P_s} \cos \omega_c t + k_2$				
3.	Number of bits per symbol	N = 1	N = 1	N = 2	N	N	N = 1	N	N = 2
4.	Number of possible symbols $M = 2^N$	Two	Two	Four	$M = 2^N$	$M = 2^N$	Two	$M = 2^N$	Four
5.	Detection method	Coherent	Coherent	Coherent	Coherent	Coherent	Non-Coherent	Non-Coherent	Coherent
6.	Minimum Euclidean distance	$\sqrt{E_b}$	$2\sqrt{E_b}$	$2\sqrt{E_b}$	$\sqrt{0.4 E_s}$ for M = 16	$2\sqrt{E_s} \sin \frac{\pi}{M}$	$\sqrt{2 E_b}$	$\sqrt{2 N E_b}$	$2\sqrt{E_b}$
7.	Minimum bandwidth	$2 f_b$	$2 f_b$	f_b	$\frac{2 f_b}{N}$	$\frac{2 f_b}{N}$	$4 f_b$	$\frac{2^{N+1} f_b}{N}$	$1.5 f_b$
8.	Symbol duration T_s	T_b	T_b	$2 T_b$	$N T_b$	$N T_b$	T_b	$N T_b$	$2 T_b$

6.17 Comparison of Digital Modulation Schemes Using a Single Carrier :

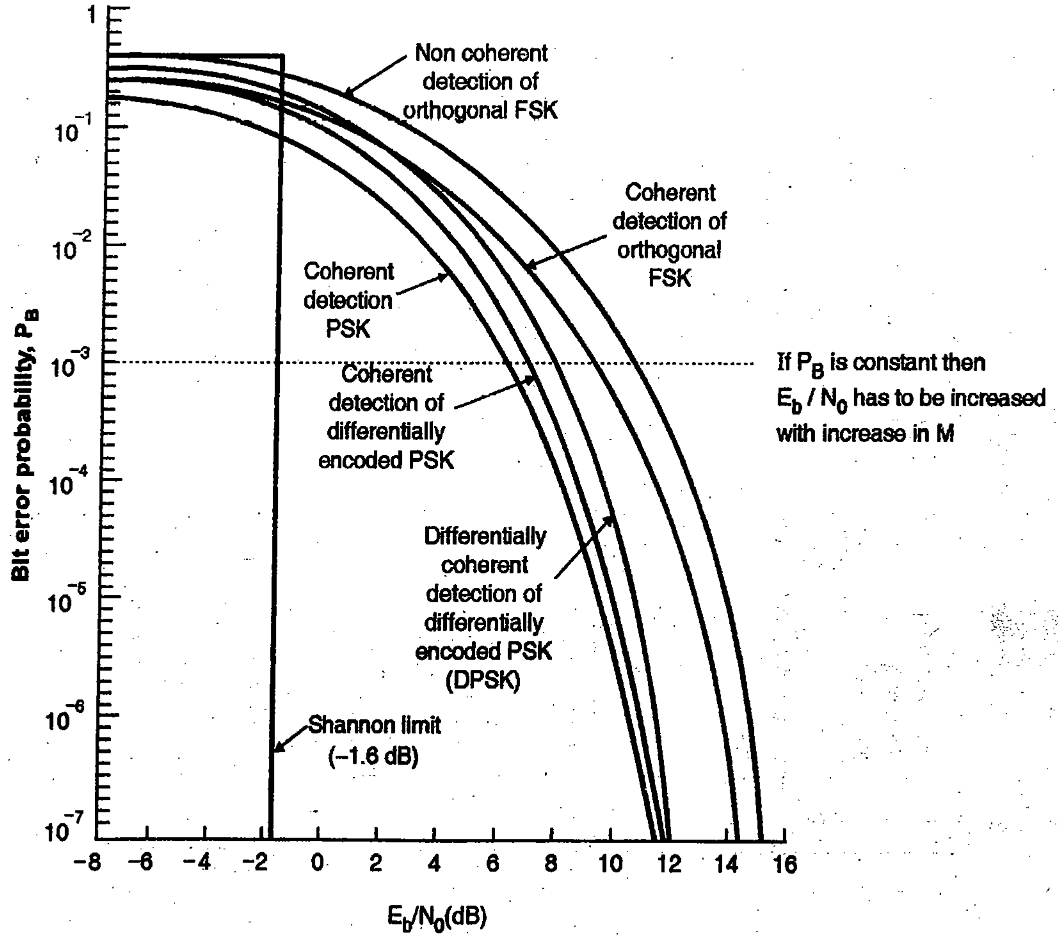
- The comparison of different binary and quaternary modulation systems is based on the average probability of symbol error expressed as a function of the bit energy - to - noise density ratio .
- Table 6.17.1 summarizes the expressions of average error probability for various systems.

Table 6.17.1 : Symbol error probability of various systems

Sr. No.	Name of the system	Error probability
1.	Coherent BPSK	$(1/2)\ \mathrm{erfc}\sqrt{E_b/N_0}$
2.	Coherent BFSK	$(1/2)\ \mathrm{efrc}\ (\sqrt{E_b/2N_0}\ )$
3.	DPSK	$(1/2)\ e^{-(E_b/2N_0)}$
4.	Noncoherent FSK	$\left(\frac{1}{2}\right) e^{-(E_b/N_0)}$
5.	QPSK }	$\mathrm{efrc}\ (\sqrt{E_b/N_0}\ )$ $-\frac{1}{4}\mathrm{efrc}^2\ (\sqrt{E_b/N_0}\ )$
6.	MSK }	

- Now consider the curves shown in Fig. 6.17.1 for various binary and quaternary systems.
- Based on Fig. 6.17.1 and Table 6.17.1 we will compare various systems.

1. The error rate (or error probability) decreases with increase in the values of E_b / N_0 for all the systems.
2. For any value of E_b / N_0, the coherent BPSK and QPSK have the smallest error rates. So they are the optimum systems.
3. The value of E_b / N_0 required by the coherent PSK and DPSK is 3 dB less than that required for coherent FSK. So the noise performance of PSK and DPSK is better than FSK.
4. The error rate of coherent BPSK and QPSK is almost the same at higher values of (E_b / N_0).
5. The error rate of MSK is same as that of QPSK.
6. The receiver of MSK has a memory and it makes a decision based on the observastions over two successive bit intervals.



(E-866) **Fig. 6.17.1 : Bit error probability of different binary and quaternary systems**

6.17.1 Bandwidth Efficiency of M-ary Modulation Techniques :

- Table 6.17.2 summarizes the values of power-bandwidth requirement for coherent binary and M-ary PSK schemes.
- It is assumed that an average error probability and the noise environment are exactly the same for all systems.

Table 6.17.2

M	$\frac{(\mathrm{BW})_{\mathrm{M\,ary}}}{(\mathrm{BW})_{\mathrm{Binary}}}$	$\frac{(\text{Average Power})_{\text{M-ary}}}{(\text{Average Power})_{\text{Binary}}}$
4	0.5	0.34 dB
8	0.333	3.91 dB
16	0.25	8.52 dB
32	0.2	13.52 dB

- From this table we conclude that QPSK (M = 4) has got the best trade off between power and bandwidth requirements. That is why QPSK is the most widely used system in practice.
- For the M-ary systems with M > 8, the power requirement is too large. So these schemes are not practically used.
- M-ary PSK and M-ary QAM have similar spectral characteristics and bandwidth requirement. For M > 4 they have different constellation diagrams.
- Due to this reason, for M > 4, the noise performance of M-ary QAM is better than that of M-ary PSK.
- Now consider M-ary FSK system. If the error probability is constant, then with increase in M, there is a reduction in power requirement. But there is an increase in the channel bandwidth. So M-ary FSK behaves exactly in the opposite manner as that of M-ary PSK.

6.18 Effect of Intersymbol Interference :

- Till now we have understood the effect of white noise on the performance of digital systems.
- But there is another factor which needs to be considered which is ISI i.e. the intersymbol interference due to the limited bandwidth occupied by the systems.
- We must consider the effect of ISI on the error rate of a system.
- When ISI is present, the correlator receiver or matched filter receiver will not behave as optimum filter. This degrades the actual error rate.
- In order to obtain the effect of ISI we have to take help of computer simulation. We have to use the discrete Fourier transform (DFT) and fast Fourier transform (FFT) alongwith the computer simulation.

6.19 Solved University Examples :

Ex. 6.19.1 : In a BFSK system a bit rate of 2 kbps is used. If the lower frequency signal is 10 kHz find higher frequency signal if minimum separation is used between the two signals.

May 06, 6 Marks

Soln. :

Given : Bit rate $f_b = 2$ kbps, $f_L = 10$ kHz

To find : f_H

With minimum separation, the difference between f_H and f_L is $2f_b$.

$$\therefore \; f_H - f_L = 2 f_b$$

$$\therefore \; f_H = f_L + 2 f_b = 10 \text{ kHz} + (2 \times 2\text{k}) = 14 \text{ kHz}.$$

Ex. 6.19.2 : If the digital message input data rate is 10 kbps and average energy per bit is 0.02 unit find bandwidth and Euclidean distance for the following schemes :

1. BPSK 2. 16-MPSK
3. MSK 4. 16-QAM

May 11, 6 Marks

Soln. :

Given : $f_b = 10$ kHz, $E_b = 0.02$.

1. **BPSK :**

For BPSK,B. W. $= 2 f_b = 2 \times 10 \text{ kHz} = 20 \text{ kHz}$ **...Ans.**

And $d = 2\sqrt{E_b} = 2\sqrt{0.02} = 0.2828$ **...Ans.**

2. **16 MPSK :**

Here M = 16, and $M = 2^N \; \therefore \; N = 4$

$$\text{B.W.} = \frac{2f_b}{N} = \frac{2 \times 10 \text{ kHz}}{4} = 5 \text{ kHz}$$ **...Ans.**

$$d = 2\sqrt{E_s} \sin(\pi/M)$$

But $E_s = 4 E_b$

$$\therefore \; d = 2\sqrt{4 E_b} \sin(180/16) = 0.06128$$ **...Ans.**

3. **MSK :**

$$\text{B.W.} = 1.5 f_b = 1.5 \times 10 = 15 \text{ kHz}$$ **...Ans.**

$$d = \sqrt{4 E_b} = \sqrt{4 \times 0.02} = 0.2828$$ **...Ans.**

4. **16 QAM :**

$$\text{B.W.} = \frac{2f_b}{N} = 5 \text{ kHz}$$ **...Ans.**

$$d = 2 [0.1 E_s]^{1/2} = 2 [0.1 \times 4 E_b]^{1/2}$$

$$= 2 [0.1 \times 4 \times 0.02]^{1/2} = 0.1789$$ **...Ans.**

Table P. 6.19.2

System	BW	D	Comment
1. BPSK	20 kHz	0.2828	Good error performance, less BW efficiency.
2. 16 MPSK	5 kHz	0.06128	Bad error performance, very good BW efficiency.
3. MSK	15 kHz	0.2828	Good error performance, moderate BW efficiency.
4. 16 QAM	5 kHz	0.1789	Moderate error performance, very good BW efficiency.

Ex. 6.19.3 : If the digital message input data rate is 8 kbps and average energy per bit is 0.01 unit, find :

1. Bandwidth required for transmission of the message through BPSK, QPSK, 16-MPSK, orthogonal BPSK, MSK and 16-MFSK.
2. Put these schemes in order of their susceptibility to noise after calculating minimum separation in signal space.

Dec. 08, 8 Marks

Ans. : Similar to Ex. 6.12.2.

Ex. 6.19.4 : An FSK system transmits binary data at the rate of 2.5×10^6 bits per second. During the course of transmission, Gaussian noise of zero mean and power spectral density 10^{-20} Watts/Hz is added to the signal. In the absence of noise, the amplitude of received sinusoidal wave for digit 1 or 0 is 1 microvolt. Determine the average probability of symbol error, assuming coherent detection.

Soln. : Given :

1. Bit rate $= 25 \times 10^6$ bits/sec.

$$\therefore \quad T_b = \frac{1}{\text{Bit rate}} = \frac{1}{2.5 \times 10^6}$$

$$= 0.4 \times 10^{-6} \text{ sec or } 0.4\ \mu S$$

2. Power spectral density of noise

$$= \frac{N_0}{2} = 10^{-20} \text{ Watts/Hz.}$$

$$\therefore \quad N_0 = 2 \times 10^{-20}$$

3. Amplitude of the signal $= A = 1\ \mu V = 1 \times 10^{-6}$

$$\therefore \text{ Normalized power } P_s = \frac{A^2}{2} = \frac{1 \times 10^{-12}}{2}$$

The error probability of FSK with coherent detection is given by,

$$P_e = \frac{1}{2} \operatorname{erfc}\left[\frac{0.6\,E_b}{N_0}\right]^{1/2}$$

$$\text{But} \quad E_b = P_s T_b \quad \therefore \quad P_e = \frac{1}{2} \operatorname{erfc}\left[\frac{0.6\,P_s\,T_b}{N_0}\right]^{1/2}$$

Substituting the values we get,

$$P_e = \frac{1}{2} \operatorname{erfc}\left[\frac{0.6 \times 1 \times 10^{-12} \times 1}{2 \times 2 \times 10^{-20} \times 2.5 \times 10^6}\right]^{1/2}$$

$$= \frac{1}{2} \operatorname{erfc}\,[7.745]^{1/2}$$

$$\therefore \quad P_e = \frac{1}{2} \operatorname{erfc}\,[2.78]$$

$$\therefore \quad P_e \approx 1.5 \times 10^{-4}$$ ...Ans.

Ex. 6.19.5 : Error probability of 10^{-5} is desired and a channel bandwidth of 20 kHz is available for 16 QAM or QPSK system. Calculate the value of E_b / η required for each of these systems.

May 04, 8 Marks

Soln. :

Given : $P_e = 10^{-5}$, $B = 20$ kHz, Bit Rate = 80 kbps

Part I : E_b / N_0 for QPSK :

The error probability of a QPSK sys

Dec. 06, 8 Marks

$$P_e = \operatorname{erfc}\sqrt{E_b / N_0}$$

$$\therefore \quad 10^{-5} = \operatorname{erfc}\sqrt{E_b / N_0}$$

$$\text{But} \quad \operatorname{erf}(u) = 1 - \operatorname{erfc}(u)$$

$$\therefore \quad 1 - 10^{-5} = 1 - \operatorname{erfc}\sqrt{E_b / N_0} = \operatorname{erf}\sqrt{E_b / N_0}$$

$$\therefore \quad 0.99999 = \operatorname{erf}\sqrt{E_b / N_0}$$

Referring to Appendix E we get,

$$\sqrt{E_b / N_0} = 3$$

$$\therefore \quad E_b / N_0 = 9$$ ...Ans.

Ex. 6.19.6 : Binary data is transmitted at rate of 10 Mbps over a channel whose bandwidth is 8MHz. Find signal energy per bit at the receiver input for coherent BPSK and DPSK to achieve probability error $P_e \le 10^{-4}$.

Assume $\frac{N_0}{2} = 10^{-10}$ Watts/Hz.

May 06. Dec. 07. May 09. 8 Marks

Soln. :

Given : Data rate $= 10 \times 10^6$ bit/sec

$$N_0 = 2 \times 10^{-10} \text{ W/Hz}, \; P_e \le 10^{-4}$$

1. **BPSK system :**

$$P_e = \frac{1}{2} \text{erfc} \sqrt{E / N_0} = 2 Q [\sqrt{2 E / N_0}]$$

$$\therefore \quad 10^{-4} = 2 Q [\sqrt{2 E / 2 \times 10^{-10}}]$$

$$5 \times 10^{-4} = Q [\sqrt{E \times 10^{-10}}]$$

$$\therefore \quad \sqrt{E \times 10^{-10}} \approx 4.5$$

$$\therefore \quad E = 2.12 \times 10^{-10} \text{ Joules.}$$

2. **DPSK system :**

$$P_e = \frac{1}{2} e^{-E_b / N_0}$$

$$\therefore \quad 2 \times 10^{-4} = e^{-E_b / 2 \times 10^{-10}}$$

$$\therefore \quad ln(2 \times 10^{-4}) = \frac{-E_b}{2 \times 10^{-10}}$$

$$\therefore \quad -8.517 \times 2 \times 10^{-10} = E_b$$

$$\therefore \quad E_b = 1.7 \times 10^{-9} \text{ Joules.}$$

Ex. 6.19.7 : It is required to transmit 2.08×10^6 binary digits per second with $P_b \le 10^{-6}$. Three possible schemes are considered.

1. binary 2. 16-Ary ASK
3. 16-Ary PSK

The channel noise PSD is $S_n(\omega) = 10^{-8}$. Determine the transmission bandwidth and the signal power required at receiver input in each case. **Dec. 07, 10 Marks**

Soln. :

Given : Bit rate $= 2.08 \times 10^6$ bit/sec

$$N_0 / 2 = 10^{-8} \text{ W/Hz}$$

$$\therefore \quad N_0 = 2 \times 10^{-8} \text{ W/Hz}$$

$$P_e \le 10^{-6}$$

To find : Signal power and transmission B.W. for BPSK, 16-ary ASK and 16 ary PSK.

1. **For BPSK :**

Signal power :

The error probability of BPSK is

$$P_e = \frac{1}{2} \text{erf}_c \left[\frac{E_b}{N_0} \right]^{1/2}$$

$$\therefore \quad 10^{-6} = \frac{1}{2} \text{erf}_c \left[\frac{E_b}{2} \times 10^{-8} \right]^{1/2}$$

$$\therefore \quad 2 \times 10^{-6} = \text{erf}_c [5 \times 10^7 E_b]^{1/2}$$

But $1 - \text{erf}_c(u) = \text{erf}(u)$

$$\therefore \quad 1 - 2 \times 10^{-6} = 1 - \text{erf}_c [5 \times 10^7 E_b]^{1/2}$$

$$\therefore \quad 0.999998 = \text{erf} [5 \times 10^7 E_b]^{1/2}$$

From Appendix E we get,

$$0.999998 = \text{erf} [3.3]^{1/2}$$

$$\therefore \quad 5 \times 10^7 E_b = \sqrt{3.3}$$

$$\therefore \quad E_b = \frac{\sqrt{3.3}}{5 \times 10^7} = 3.633 \times 10^{-8}$$

$$E_b = P_s T_b$$

$$\therefore \quad \text{Signal power } P_s = \frac{E_b}{T_b}$$

$$\text{But } T_b = \frac{1}{\text{Bit rate}}$$

$$\therefore \quad P_s = E_b \times \text{Bit rate}$$

$$= 3.633 \times 10^{-8} \times 2.08 \times 10^{-6}$$

$$= 0.07557 \text{ W} = 75.57 \text{ mW} \quad \text{...Ans.}$$

Bandwidth :

$$BW = 2 f_b = \frac{2}{T_b} = 2 \times \text{Bit rate}$$

$$= 2 \times 2.08 \times 10^6 = 4.16 \text{ MHz} \quad \text{...Ans.}$$

2. **For 16-ary PSK :**

$$P_e = \text{erf}_c \left[\sqrt{E_s / N_0} \sin \frac{\pi}{M} \right] \quad M = 16$$

$$\therefore \quad P_e = \text{erf}_c [\sqrt{E_s / N_0} \sin 11.25°]$$

$$\therefore \quad 10^{-6} = \text{erf}_c [\sqrt{E_s / N_0} \times 0.195]$$

$$1 - 10^{-6} = 1 - \text{erf}_c [\sqrt{E_s / N_0} \times 0.195]$$

$$\therefore \quad 0.999999 = \text{erf} [\sqrt{E_s / N_0}] \times 0.195$$

Referring Appendix E we get,

$$0.999999 = \text{erf} [3.3]$$

$$\therefore \quad 0.195 \sqrt{E_s / N_0} = 3.3$$

$$\therefore \quad \sqrt{E_s / N_0} = 16.923$$

$$\therefore \quad E_s = 286.39 \times N_0 = 286.39 \times 2 \times 10^{-8}$$

$$\therefore \quad E_s = 5.7278 \times 10^{-6}$$

$$\text{But } E_s = 2\,P_s\,T_b = \frac{2\,P_s}{\text{Bit rate}}$$

$$\therefore\ P_s = \frac{E_s \times \text{Bit rate}}{2}$$

$$= \frac{5.7278 \times 10^{-6} \times 2.08 \times 10^{6}}{2} = 5.9569\ \text{W}$$

$$\text{Bandwidth} = \frac{2\,f_b}{N} = \frac{2 \times \text{Bit rate}}{4}$$

$$\therefore\ \text{BW} = \frac{2.08 \times 10^{6}}{2} = 1.04\ \text{MHz}$$

Note that required signal power for the same value of P_e increases for M-ary system as compared to binary. But the B.W of M-ary system is less than that of binary.

Ex. 6.19.8 : Binary data is transmitted using PSK at a rate 2 Mbps over RF link having bandwidth 2 MHz. Find signal power required at receiver input so that error probability is less than or equal to 10^{-4}. Assume noise PSD to be 10^{-10} watt/Hz. (Q (3.71) = 10^{-4}).

Dec. 10, 8 Marks

Soln. :

Given : 1. Data rate = 2 Mbps, 2. $P_e \le 10^{-4}$, 3. $\frac{N_0}{2} = 10^{-10}$ Watts/Hz

$$\therefore\ N_0 = 2 \times 10^{-10}$$

The average error probability of PSK system is

$$P_B = 2Q\sqrt{\frac{2E}{N_0}}$$

$$10^{-4} = 2Q\left[\sqrt{2E/2 \times 10^{-10}}\right]$$

$$\frac{1}{2} \times (10^{-4}) = Q\left[\sqrt{E/10^{-10}}\right]$$

$$\therefore\ \sqrt{E \times 10^{+10}} = \frac{1}{2}(3.71)$$

$$\sqrt{E \times 10^{+10}} = 1.855$$

$$E \times 10^{+10} = 3.44$$

$$E = 3.44 \times 10^{-10}\ \text{Joules}$$

$$\text{But } E = P \times T \quad \text{and } T = \frac{1}{\text{Bit rate}} = \frac{1}{2 \times 10^{6}}$$

$$\therefore\ P = \frac{E}{T} = 3.44 \times 10^{-10} \times 2 \times 10^{6}$$

$$\therefore\ P = 6.882 \times 10^{-4}$$

$\therefore$ Signal power P = 0.6882 mW ...**Ans.**

Ex. 6.19.9 : Binary data is transmitted using M-ary PSK at a rate 2 Mbps over RF link having bandwidth 2 MHz. Find signal power required at receiver input so that bit error probability is less than or equal to 10^{-5}. The channel noise PSD is 10^{-8} Watt/Hz.

Calculate for M = 16 and M = 32.

Given erf (0.99996) = 3.1

erf (0.99995) = 3.2

Dec. 10, 8 Marks

Soln. :

Given : Bit rate = 2 Mbps, B.W. = 2 MHz, $P_e \le 10^{-5}$, $N_0/2 = 10^{-8}$ Watts/Hz

$$\therefore\ N_0 = 2 \times 10^{-8}\ \text{Watts / Hz}$$

1. Signal power for M = 16 :

$$P_e = \text{erfc}\left[\sqrt{E_s/N_0}\,\sin\frac{\pi}{M}\right]$$

$$P_e = \text{erfc}\left[\sqrt{E_s/N_0}\,\sin 11.25^\circ\right]$$

$$10^{-5} = \text{erfc}\left[\sqrt{E_s/N_0} \times 0.195\right]$$

$$1 - 10^{-5} = 1 - \text{erfc}\left[\sqrt{E_s/N_0} \times 0.195\right]$$

$$0.9999 = \text{erf}\,[3.1]$$

$$0.195\sqrt{E_s/N_0} = 3.1$$

$$\sqrt{E_s/N_0} = 15.897$$

$$\therefore\ E_s = 252.72 \times N_0 = 252.75 \times 2 \times 10^{-8}$$

$$\therefore\ E_s = 5.05 \times 10^{-6}$$

$$E_s = 2\,P_s\,T_b = \frac{2P_s}{\text{Bit rate}}$$

$$P_s = \frac{E_s \times \text{Bit rate}}{2}$$

$$= \frac{5.05 \times 10^{-6} \times 2 \times 10^{6}}{2}$$

$$P_s = 5.05\ \text{W}$$ ...**Ans.**

2. Signal power for M = 32 :

$$P_e = \text{erfc}\left[\sqrt{E_s/N_0}\,\sin\frac{\pi}{M}\right]$$

where M = 32

$$P_e = \text{erfc}\left[\sqrt{E_s/N_0}\,\sin(5.625)^\circ\right]$$

$$10^{-6} = \text{erfc}\left[\sqrt{E_s/N_0} \times 0.098\right]$$

$$1 - 10^{-6} = 1 - \text{erfc}\left[\sqrt{E_s / N_0} \times 0.098\right]$$

$$0.9999999 = \text{erfc}\left[\sqrt{E_s / N_0}\right] \times 0.098$$

$$0.999999 = \text{erf}\,[3.2]$$

$$0.098\sqrt{E_s / N_0} = 3.2$$

$$\sqrt{E_s / N_0} = 32.653$$

$$\therefore\ E_s = 1066.22 \times N_o = 1066.22 \times 2 \times 10^{-8}$$

$$= 2132.4448 \times 10^{-8} = 21.32 \times 10^{-6}$$

$$\therefore\ E_s = 2P_s\,T_b = \frac{2P_s}{\text{Bit rate}}$$

$$P_s = \frac{E_s \times \text{Bit rate}}{2}$$

$$= \frac{21.32 \times 10^{-6} \times 2 \times 10^{6}}{2}$$

$$P_s = 21.32\ \text{W}$$ **...Ans.**

Ex. 6.19.10 : A QPSK signal is received at the input of a coherent optimal receiver with amplitude 10 mV and frequency 2 MHz. The signal is corrupted with white noise of PSD 10^{-11} W/Hz. If data rate is 10^4 bits/sec find the probability of error, also find the probability of error for BPSK system if the local oscillator has a phase shift of $\pi/6$ rad with the input signal. Ref. Table P. 6.19.10.

May 11, 8 Marks

Table P. 6.19.10

Z	Q (Z)
2.5	0.0062100
2.8	0.0025600
3.0	0.0013500
3.2	0.0006900
3.4	0.0003400
3.6	0.0001690
3.68	0.0001660
3.8	0.0000700
4.0	0.0000300
4.3	0.0000100
4.7	0.0000010
5.2	0.0000001

Soln. :

Given : QPSK system, Assume A = 1 mV,

$f = 2$ MHz, $N_0 = 10^{-11}$ W/Hz

$f_b = 10^4$ bits/sec.,

$$\therefore\ T_b = \frac{1}{10^4} = 1 \times 10^{-4} = 100\ \mu\text{sec.}$$

To find : P_e

$$\text{For QPSK}\quad P_e = \text{erfc}\sqrt{E/2N_0}$$

$$= \text{erfc}\,\frac{\left(\sqrt{E/N_0}\right)}{\sqrt{2}} \qquad ...(1)$$

$$\text{But } Q\,(Z) = \frac{1}{2}\,\text{erfc}\left(\frac{Z}{\sqrt{2}}\right) \qquad ...(2)$$

$$\therefore\ P_e = \text{erfc}\left[\frac{\sqrt{E/N_0}}{\sqrt{2}}\right] = 2\,QZ \qquad ...(3)$$

With Z =

$$\text{But } E = 2 \times E_b = 2 \times P_s \times T_b$$

$$\text{But } P_s = \frac{A^2}{2}$$

$$\therefore\ E = 2 \times \frac{A^2}{2} \times T_b = A^2\,T_b \qquad ...(4)$$

$$\therefore\ E = (1 \times 10^{-3})^2 \times 100 \times 10^{-6}$$

$$= 1 \times 10^{-10}\ \text{Joules}$$

$$N_0 = 10^{-11}\ \text{W/Hz,}$$

$$\therefore\ N_0 = 2 \times 10^{-11}\ \text{W/Hz}$$

$\therefore$ Substituting these values into Equation (3) we get,

$$P_e = 2Q\left[\sqrt{\frac{1 \times 10^{-10}}{10^{-11}}}\right] = 2Q\,[3.16]$$

From the given table we get, Q [3.2] = 0.00069

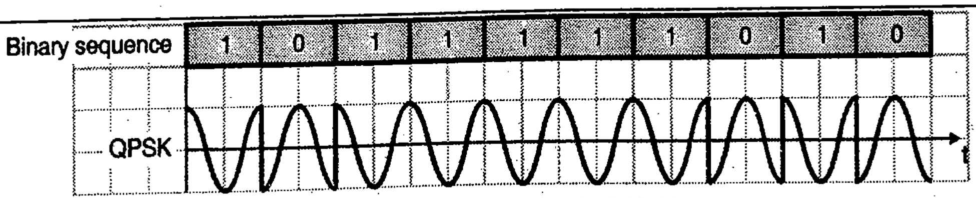
$$\therefore\ P_e = 2 \times 0.00069 = 1.38 \times 10^{-3}$$ **...Ans.**

Ex. 6.19.11 : Explain the working of QPSK coherent receiver. Sketch the waveform of the inphase and quadrature components of a QPSK signal for binary sequence 1011111010.

May 13, 10 Marks

Soln. :

For QPSK receiver Refer Section 6.9.3.



(E-1359) **Fig. P. 6.19.11**

Ex. 6.19.12 : Binary data is transmitted using PSK at a rate 3 Mbps over RF link having bandwidth 10 MHz. Find signal power required at receiver input so that error probability is less than or equal to 10^{-4}. Assume noise PSD to be 10^{-10} Watts/Hz. ($Q(3.71) = 10^{-4}$).

May 13, Dec. 15, 8 Marks

Soln. :

Given : 1. Data rate = 3 Mbps, 2. $P_e \leq 10^{-4}$,

3. $\frac{N_o}{2} = 10^{-10}$ Watts/Hz

$$\therefore \quad N_o = 2 \times 10^{-10}$$

The average error probability of PSK system is,

$$P_B = 2Q\sqrt{\frac{2E}{N_o}}$$

$$10^{-4} = 2Q\left[\sqrt{2E/2\times 10^{-10}}\right]$$

$$\frac{1}{2}\times(10^{-4}) = Q\left[\sqrt{E/10^{-10}}\right]$$

$$\therefore \quad \sqrt{E\times 10^{+10}} = \frac{1}{2}(3.71)$$

$$\sqrt{E\times 10^{+10}} = 1.855$$

$$E\times 10^{+10} = 3.44$$

$$E = 3.44\times 10^{-10} \text{ Joules}$$

$$\text{But } E = P\times T \quad \text{and } T = \frac{1}{\text{Bit rate}} = \frac{1}{3\times 10^{6}}$$

$$\therefore \quad P = \frac{E}{T} = 3.44\times 10^{-10}\times 3\times 10^{6}$$

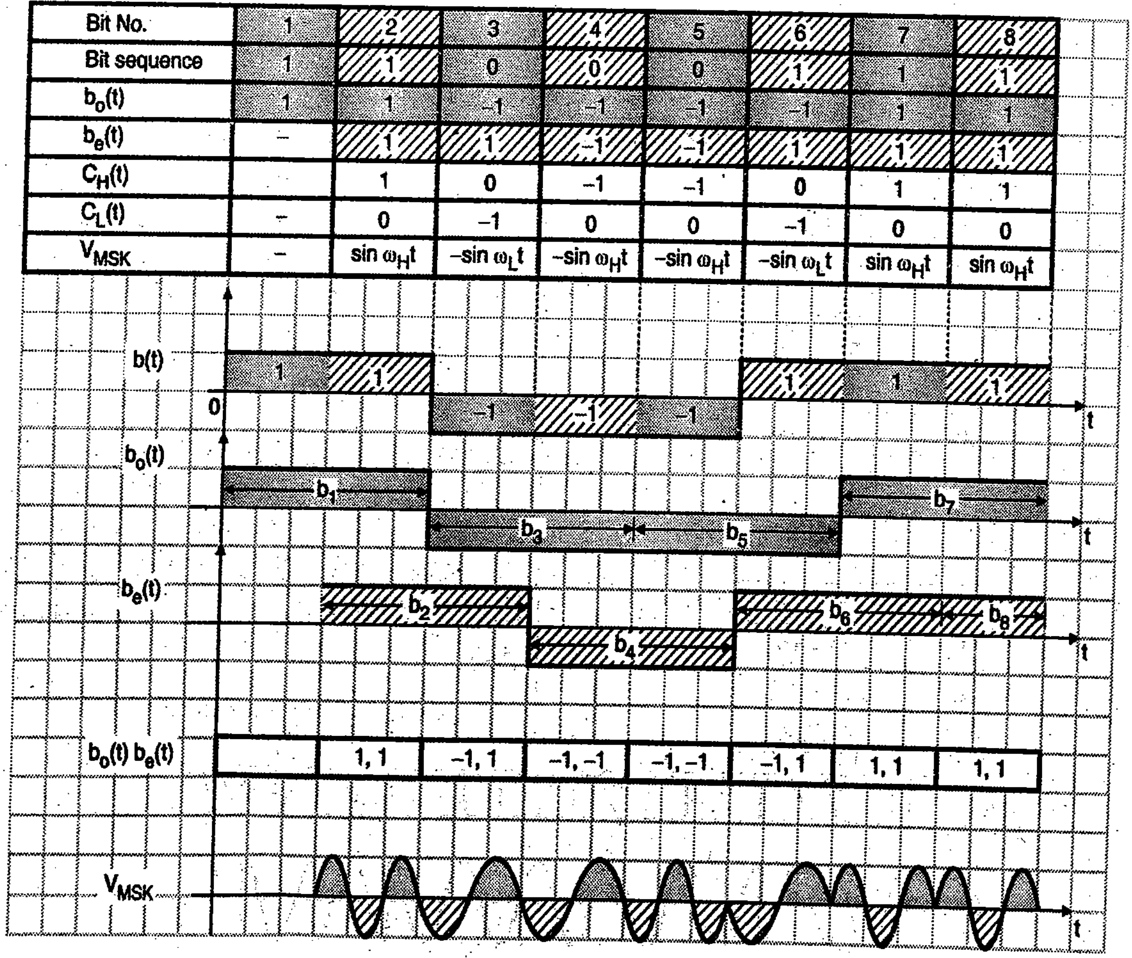
$$\therefore \quad P = 1.032\times 10^{-3}$$

$\therefore$ Signal power P = 1.032 mW **...Ans.**

Ex. 6.19.13 : Explain MSK with the help of waveforms for input sequence 11000111 along with respective mathematical representation. Compare it with QPSK. **Dec. 13, 10 Marks**

Soln. :

Refer to the waveforms of Fig. P. 6.19.13. Refer section 6.15.1 for the explanation. Refer sections 6.15.3 and 6.15.14 for comparison with QPSK.



Bit No.	1	2	3	4	5	6	7	8
Bit sequence	1	1	0	0	0	1	1	1
$b_o(t)$	1	1	–1	–1	–1	–1	1	1
$b_e(t)$	–	1	1	–1	–1	1	1	1
$C_H(t)$		1	0	–1	–1	0	1	1
$C_L(t)$	–	0	–1	0	0	–1	0	0
V_{MSK}	–	$\sin \omega_H t$	$-\sin \omega_L t$	$-\sin \omega_H t$	$-\sin \omega_H t$	$-\sin \omega_L t$	$\sin \omega_H t$	$\sin \omega_H t$

(E-1373) **Fig. P. 6.19.13 : MSK waveforms**

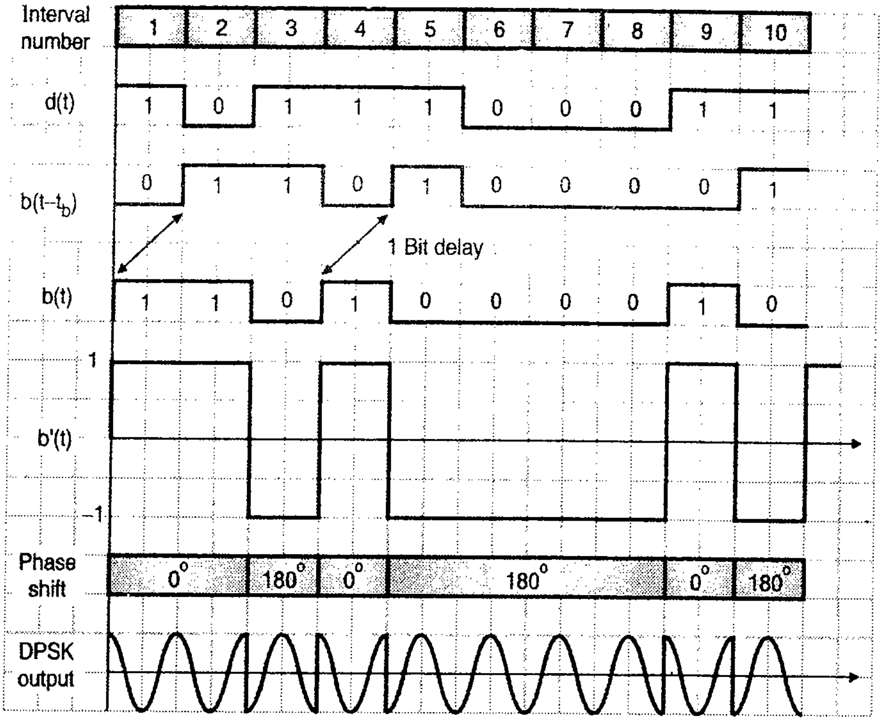
Ex. 6.19.14 : The following bit streams are be transmitted using DPSK scheme :

1. 1011100011 2. 0101000111

Determine and sketch the encoded sequence and transmitted phase sequence. **Dec. 13, 8 Marks**

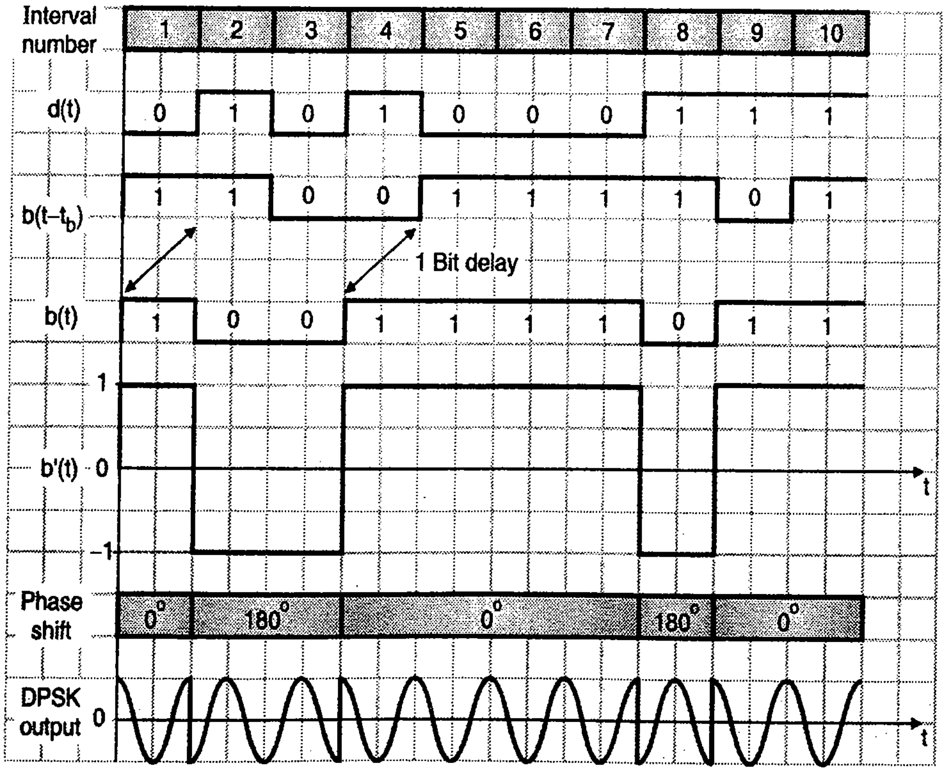
Soln. :

1. Given sequence : 1011100011 :

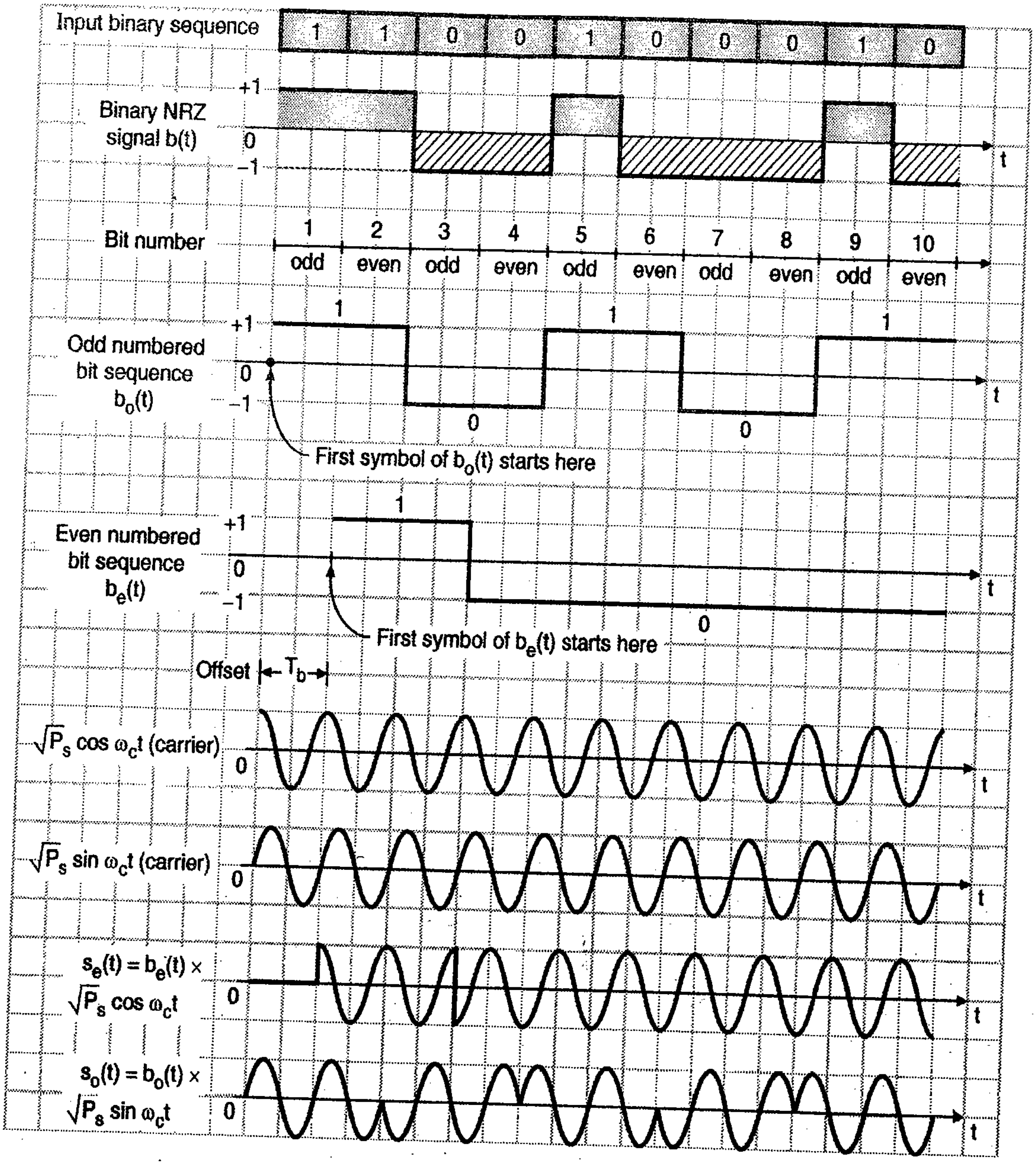


(E-1369) **Fig. P. 6.19.14(a)**

2. Given sequence : 0101000111 :



(E-1370) **Fig. P. 6.19.14(b)**



(E-1477) **Fig. P. 6.19.15 : QPSK waveforms**

Ex. 6.19.15 : Given the input binary sequence 1100100010, sketch the waveforms of the in-phase and quadrature components of a modulated wave obtained by using the QPSK scheme. Dec. 14, 9 Marks

Soln. : The required waveforms are as shown in Fig. P. 6.19.15.

Ex. 6.19.16 : Find error probability of coherent FSK when amplitude of I/P at coherent optimal receiver is 10 mV and frequency 1 MHz, the signal corrupted with white noise of PSD 10^{-9} W/Hz. The data rate is 100 kbps. [erfc (1.01) = 0.1531, erfc (1.11) = 0.1164, erfc (1.22) = 0.0844 and erfc (1.33 = 0.0599)] Dec. 15, 10 Marks

Soln. :

Given :

$$\text{Coherent FSK, } A = 10 \text{ mV}, \frac{N_o}{2} = 10^{-9} \text{ W/Hz},$$

$$\therefore N_o = 2 \times 10^{-9} \text{ W/Hz.}$$

$$\text{Bit rate} = 10 \times 10^3 \text{ bps}, \; f = 1 \text{ MHz.}$$

1. Bit duration $T_b = \dfrac{1}{\text{Bit rate}} = \dfrac{1}{10 \times 10^3} = 1 \times 10^{-4}$ sec.

2. PSD of white noise $\dfrac{N_o}{2} = 10^{-9}$ W/Hz

$$\therefore N_o = 2 \times 10^{-9} \text{ W/Hz.}$$

3. Amplitude of signal $A = 10 \text{ mV} = 1 \times 10^{-2}$

$\therefore$ Normalized power $P_S = \frac{A^2}{2} = \frac{1 \times 10^{-4}}{2}$

4. Error probability of coherent FSK is,

$$P_e = \frac{1}{2} \text{erfc} \left[\frac{0.6\, E_b}{N_o}\right]^{1/2}$$

$$\text{But } E_b = P_S\, T_b = 0.5 \times 10^{-4} \times 10^{-4}$$

$$= 0.5 \times 10^{-8}$$

$$\therefore P_e = \frac{1}{2} \text{erfc} \left[\frac{0.6 \times P_s\, T_b}{N_o}\right]^{1/2}$$

$$\therefore P_e = \frac{1}{2} \text{erfc} \left[\frac{0.6 \times 0.5 \times 10^{-8}}{2 \times 10^{-9}}\right]^{1/2}$$

$$= \frac{1}{2} \text{erfc}\, [1.22]$$

$$\therefore P_e = \frac{1}{2} \times 0.0844 = 0.0422 \quad \text{...Ans.}$$

Ex. 6.19.17 : If the digital message input data rate is 24 kbps and average energy / bit is 0.05 unit. Find bandwidth and Euclidean distance for the following modulation schemes :

1. BPSK 2. 8-PSK
3. MSK 4. 16 QAM

May 16, 8 Marks

Soln. :

Given : $f_b = 24$ kHz, $E_b = 0.05$.

1. BPSK :

For BPSK, $\text{B. W.} = 2\, f_b = 2 \times 24 \text{ kHz} = 48 \text{ kHz}$ **...Ans.**

And $d = 2\sqrt{E_b} = 2\sqrt{0.05} = 0.4472$ **...Ans.**

2. 8-PSK :

Here $M = 8$, and $M = 2^N$ $\therefore N = 3$

$$\text{B.W.} = \frac{2f_b}{N} = \frac{2 \times 24 \text{ kHz}}{3} = 16 \text{ kHz} \quad \text{...Ans.}$$

$$d = 2\sqrt{E_s} \sin(\pi/M)$$

$$\text{But } E_s = 4\, E_b$$

$$\therefore d = 2\sqrt{4\, E_b} \sin(180/8)$$

$$= 0.3422 \quad \text{...Ans.}$$

3. MSK :

$$\text{B.W.} = 1.5\, f_b = 1.5 \times 24 = 36 \text{ kHz} \quad \text{...Ans.}$$

$$d = \sqrt{4\, E_b} = \sqrt{4 \times 0.05}$$

$$= 0.4472 \quad \text{...Ans.}$$

4. 16 QAM :

$$\text{B.W.} = \frac{2f_b}{N} = \frac{2 \times 24}{3} = 16 \text{ kHz} \quad \text{...Ans.}$$

$$d = 2\, [0.1\, E_s]^{1/2} = 2\, [0.1 \times 4\, E_b]^{1/2}$$

$$= 2\, [0.1 \times 4 \times 0.05\,]^{1/2}$$

$$= 0.2828 \quad \text{...Ans.}$$

Ex. 6.19.18 : Compare the performance of BPSK, FSK, M-ary PSK M-ary FSK with respect to following parameters :

1. Bandwidth
2. PSD
3. Probability of error.

May 16, 8 Marks

Soln. :

Comparison of BPSK, BFSK, MPSK and MFSK :

1. Bandwidth :

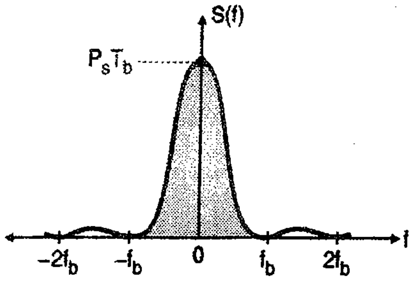
Table P. 6.19.18(a)

Sr. No.	System	BW
1.	BPSK	$2\, f_b$
2.	BFSK	$4\, f_b$
3.	MPSK	$2\, f_b/N$
4.	MFSK	$2^{N+1}\, (f_b/N)$

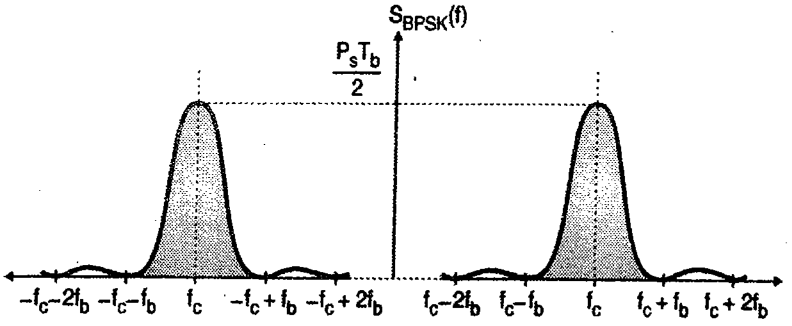
This shows that M-PSK system has the minimum bandwidth. It is the most BW efficient system.

2. PSD :

The PSDs of these four systems are as shown in Fig. P. 6.19.18. It shows that the FSK systems have multiple lobes. So there is a possibility of ISI.

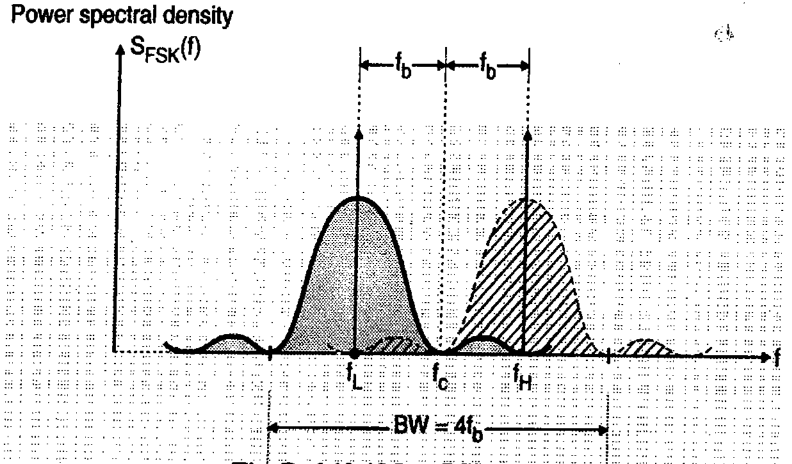


(a) Power spectral density of the NRZ data b(t)

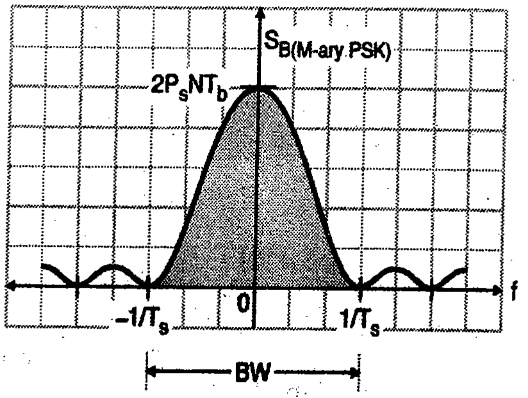


(b) Spectrum of BPSK

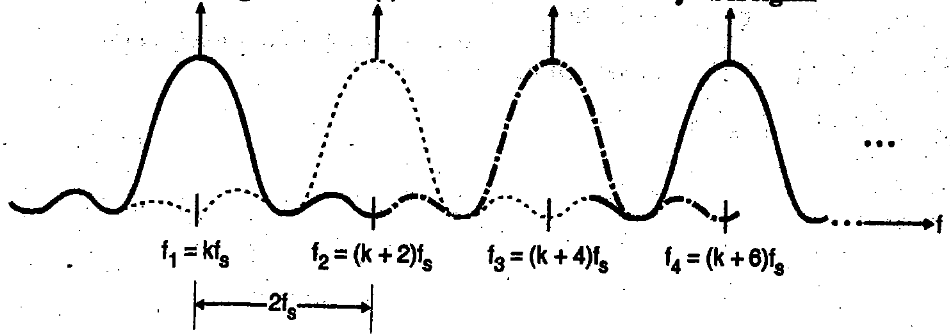
(E-374) **Fig. P. 6.19.18(a) : PSD of BPSK**



(E-410) **Fig. P. 6.19.18(b) : PSD of BFSK**



(E-396) **Fig. P. 6.19.18(c) : PSD of baseband M-ary PSK signal**



(E-418) **Fig. P. 6.19.18(d) : PSD of M-ary FSK**

3. **Probability of error :**

Table P. 6.19.18(b)

Sr. No.	System	Error Probability
1.	BPSK	$P_e = \frac{1}{2} \text{ erfc} \sqrt{E_b/N_o}$
2.	BFSK	$P_e = \frac{1}{2} \text{ erfc} \sqrt{0.6 \; E/N_o}$
3.	MPSK	$P_e = \text{erfc}\left[\sqrt{E_s/N_o} \sin \frac{\pi}{M}\right]$
4.	MFSK	$P_e = (M-1) \times \frac{1}{2} \text{ erfc} \left[\sqrt{E_s/2\,N_o}\,\right]$

This shows that the error performance of binary systems is better than that of the M-ary systems. BPSK has the best error performance while MFSK is the worst. The error performance becomes poor with increase in the value of M.

Review Questions

Q. 1 Derive a general expression for the Euclidean distance for M-ary PSK.

Q. 2 Compare the binary modulation systems in terms of power efficiency and spectrum efficiency.

Q. 3 Assume you are required to transmit f_b = 90 Mb / s data in an authorised bandwidth of 20MHz. Which modulation techniques would you consider ? Explain why ?

Q. 4 For the input binary sequence { b (k) } = { 1, – 1, 1, – 1, – 1, – 1, 1, 1 } find the transmitted phase sequence and sketch the transmitted waveform for QPSK.

Q. 5 Show that the detection of DPSK does not require a differential decoder.

Q. 6 Show that MSK signal has a constant envelope.

Q. 7 Show that output of the BPSK receiver over n^{th} bit slot is proportional to the status of n^{th} in the data bit sequence.

Q. 8 If a data bit sequence consists of a following string of bits, what will be the nature of waveform transmitted by QPSK transmitter ? The data bit sequence is 1 0 1 1 1 0 1 0 1 1 1.

Q. 9 Derive an expression for MSK signal.

Q. 10 For a bit sequence of 1 0 1 1 0 1 0 1 0 1 show that the MSK waveform has phase continuity.

Q. 11 Explain how QPSK System requires minimum bandwidth.

Q. 12 Write note on : Imperfect phase synchronization in BPSK.

Q. 13 Explain the advantage of differential encoding of the input in PSK system.

Q. 14 Explain the advantage of gray coding of the input to QPSK system.

Q. 15 Compare the constellation diagram showing allowed state transitions for QPSK and OQPSK.

Q. 16 Show that, in an AWGN channel, the detection performance of equal energy signals depends only on the 'distance' between the two pertinent message points in the signal space.

Q. 17 For a data bit sequence of 10110101, sketch the MSK waveform. Give relevant expressions.

Q. 18 State and explain the condition for orthogonality of BFSK signals. Determine their spectrum and hence the bandwidth required for transmission of signals.

Q. 19 MSK is called 'Shaped QPSK'. Justify with relevant expression or waveforms. Discuss the merits and demerits of MSK as compared to QPSK.

Q. 20 Give expressions for QPSK and MSK. Hence justify why MSK can be called `Shaped QPSK'.

Q. 21 Show that MSK has a constant envelope, starting from the expression for MSK.

Q. 22 Compare spectra of QPSK and MSK.

Q. 23 Why is differential encoding used in PSK systems ?

Q. 24 Explain the need of continuous wave (CW) modulation.

Q. 25 What is the application of CW modulation systems ?

Q. 26 What is the communication medium used in such systems ?

Q. 27 State the bandwidth requirement of ASK system.

Q. 28 State merits and demerits of BASK.

Q. 29 What type of receiver is used for the BPSK detection ?

Q. 30 What is the advantage of synchronous detection ?

Q. 31 What is the role of bit synchronizer in the BPSK receiver ?

Q. 32 What is the maximum B.W. of BPSK system ?

Q. 33 State the value of Euclidean distance for BPSK.

Q. 34 State advantages and disadvantages of BPSK.

Q. 35 Draw the BPSK signal for the following binary signal.

1 0 1 1 1 0 1 0

Q. 36 What is the merit of DPSK on BPSK ?

Q. 37 How many phases are transmitted in DPSK ?

Q. 38 What is the symbol duration in DPSK ?

Q. 39 What is the bandwidth of DPSK ?

Q. 40 State important disadvantages of DPSK.

Q. 41 Express QPSK mathematically.

Q. 42 How many phases are transmitted in QPSK ?

Q. 43 What is the phase difference between the adjacent messages in QPSK ?

Q. 44 What are advantages of QPSK system ?

Q. 45 What is the type of demodulation used for QPSK ?

Q. 46 What is the main advantage of M-ary PSK ?

Q. 47 Compare error probability of M-ary PSK and QPSK.

Q. 48 State the expression for Euclidean distance for M-ary PSK.

Q. 49 What is the value of "d" for an 8-ary PSK ?

Q. 50 What is the BW of M-ary PSK ?

Q. 51 State advantages and disadvantages of M-ary PSK.

Q. 52 What is QASK ?

Q. 53 Why is amplitude changed alongwith phase ?

Q. 54 State the expression for Eucledian distance for 16-QAM system.

Q. 55 Compare 16-QAM with QPSK on the basis of error rate.

Q. 56 What is the BW of QASK ?

Q. 57 State the expression for BFSK.

Q. 58 How is a message transmitted in BFSK ?

Q. 59 What is the BW of BFSK ?

Q. 60 What type of receiver is used for BFSK reception ?

Q. 61 What is meant by orthogonal BFSK ?

Q. 62 What is the advantage of M-ary FSK over BFSK ?

Q. 63 State the expression for BW of an M-ary FSK.

Q. 64 What are disadvantages of M-ary FSK ?

Q. 65 What type of reception technique is used for M-ary FSK ?

Q. 66 What is the effect of abrupt phase change on amplitude in QPSK ?

Q. 67 What is the operating principle of MSK ?

Q. 68 Compare error probability of MSK with QPSK.

Q. 69 State advantages and disadvantages of MSK.

Q. 70 What is the BW of MSK ?

6.20 University Questions and Answers :

Q. 1 Compare following digital modulation schemes :

1. QPSK 2. DPSK 3. FSK **(May 2015, 10 Marks)**

Ans. :

Sr. No.	Parameter	DPSK	QPSK	FSK	
				BFSK	M-ary FSK
1.	Bits per symbol.	One	Two	One	N
2.	Number of possible symbols $M = 2^N$.	Two	Four	Two	$M \cdot 2^N$
3.	Detection method	Non-coherent	Coherent	Non-Coherent	Non-Coherent
4.	Minimum Euclidean distance	–	$2\sqrt{E_b}$	$\sqrt{2E_b}$	$\sqrt{2NE_b}$
5.	Minimum bandwidth (BW)	f_b	f_b	$4f_b$	$\frac{2^{N+1}}{N} f_b$
6.	Symbol duration (T_S)	$2T_b$	$2T_b$	T_b	NT_b
7.	Modulation	Phase	Phase	Frequency	Frequency

□□□

CHAPTER 7

Unit VI

Spread Spectrum Modulation

Syllabus :

Introduction, Pseudo noise sequences, A notion of spread spectrum, Direct sequence spread spectrum with coherent BPSK, Signal space dimensionality and processing gain, Probability of error, Concept of jamming, Frequency hop spread spectrum.

7.1 Introduction :

SPPU : May 08, May 16

University Questions

Q. 1 What is spread spectrum technique ? How are they classified ? **(May 08, 8 Marks)**

Q. 2 What is need of spread spectrum modulation technique ? **(May 16, 9 Marks)**

Till now we have discussed a number of digital communication systems. The focus of our attention while discussing those systems was on two important factors, viz :

1. How to utilize the channel bandwidth efficiently ?
2. How to minimize the amount of transmitted power ? However the efficient utilization of bandwidth and minimizing the transmitted power are not the "only" problems faced by a communication system. Some other problems encountered by it are as follows :

Problems encountered by a communication system :

1. In the areas such as "military communication", the information has to be "secured". That means an unauthorized user is not expected to access the information. Also he should not be allowed to interfere the communication by any means.
2. Sometimes a hostile transmitter (say used by terrorists) can "jam" the desired or legitimate transmission. To avoid this the channel should be "immune" to any external interference.
3. Even in the non-military communications an unintentional interference is caused by a user who is transmitting its information through a channel which is already being used.

Remedy :

These problems can be successfully solved by using a technique called **"Spread Spectrum Modulation"**. We are going to discuss this modulation technique in detail in this chapter.

7.2 How is the SS Signal Different from the Normal Signal ?

The spread Spectrum Signal (SS) is different from a normal signal, in the following aspects :

- This signal occupies a larger bandwidth than that of a normal signal. (Therefore the name spread spectrum).
- The spread spectrum signal invariably uses some kind of coding. The spectrum spreading at the transmitter and despreading (opposite to spreading) at the receiver is obtained with the help of this code word.
- The code word associated with an SS signal is independent of the information carried by the signal.
- The most important point is that the SS signal is **"pseudorandom"** in nature.
- This makes it appear like **"random noise"**. Therefore the normal receiver cannot demodulate the SS signal.
- Only a specially designed receiver can demodulate it to recover the information. Due to this characteristics the SS signal appears as noise to any unintended receiver.

7.3 Applications of Spread Spectrum Modulation :

The spread spectrum signals are used in the following applications :

1. To avoid the intentional interference called as jamming.
2. To reject the unintentional interference from some other user : This is possible to achieve by assigning a different code for the signals from various users. This type of communication which allows multiple users to share a common channel for transmission of information is called as Code Division Multiple Access (CDMA).

3. **To avoid the self interference due to multipath propagation :** A signal can take multiple paths while travelling over a communication channel from transmitter to receiver. The signal components following different path lengths will result in a dispersed signal at the receiver. This is known as the self-interference. This type of interference also can be suppressed by using the SS modulation.
4. **In Low Probability of Intercept (LPI) signals :** A message can be hidden in the background noise by spreading its bandwidth using the code word and then transmitting the coded signal at a low power level. Due to these modifications, the probability that such a signal be intercepted (detected) is reduced to a great extent. Hence such a spread and coded signal is called as the low probability-of-intercept (LPI) signal.
5. **In obtaining the message privacy :** The message privacy can be obtained by superimposing a pseudorandom pattern on the transmitted message.

7.4 Classification of the Spread Spectrum Modulation Techniques :

SPPU : May 08, May 11

University Questions

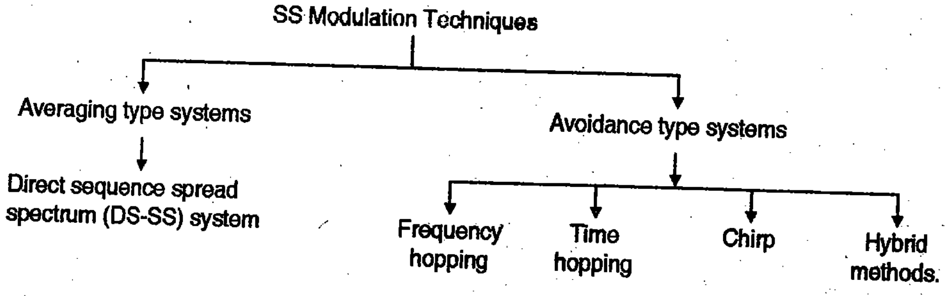
Q. 1 What is spread spectrum technique ? How are they classified ? **(May 08, 8 Marks)**

Q. 2 State classification of spread spectrum and explain FHSS in detail. **(May 11, 7 Marks)**

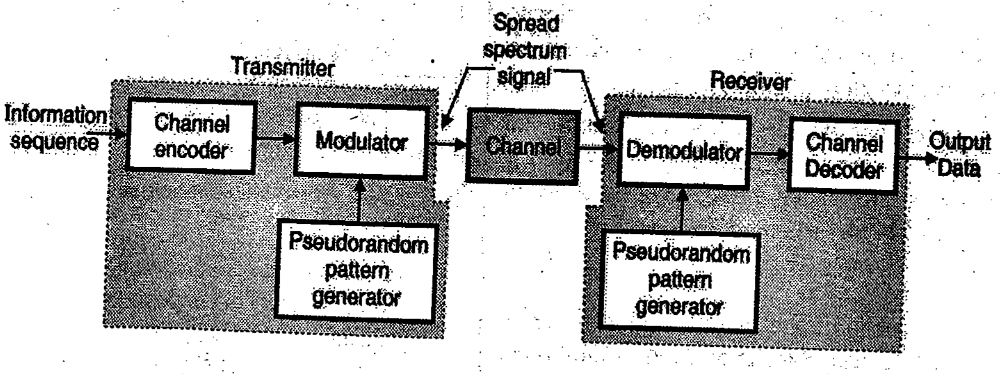
- The SS modulation techniques are broadly categorized into two categories namely the averaging type systems and the avoidance type systems as shown in Fig. 7.4.1.
- The averaging systems reduce the interference by averaging it over a long period. The Direct Sequence Spread Spectrum (DS-SS) system is an averaging system.
- The avoidance systems reduce the interference by making the signal avoid the interference over a large fraction of time. The avoidance systems are further classified depending on the type of modulation used.
- Some of the avoidance type systems using different modulation techniques are :
  1. Frequency hopping system
  2. Time hopping system
  3. Chirp
  4. Hybrid modulation system
- We will discuss all these systems in detail later on in this chapter.

7.5 Model of Spread Spectrum Digital Communication System :

The block diagram shown in Fig. 7.5.1 illustrates the basic elements of a spread spectrum digital communication system.



(E-470) Fig. 7.4.1 : Classification of spread spectrum technique



(E-471) Fig. 7.5.1 : Model of spread spectrum digital communication system

- The inputs to the parity generator are the outputs from the flip-flops i.e. Q_0, Q_1 ... Q_m. However it is not necessary to connect all the Q outputs to the input of parity generator.

Output of Parity Generator	Status of Inputs
Logic "0"	Even number of inputs are at logic "0"
Logic "1"	Odd number of inputs are at logic "1"

- The character generated by a PN sequence generator (Q_0, Q_1 ... Q_m) depends on the number of flip-flops used (m) and on the selection of which flip-flop outputs are connected to the inputs of parity generator.
- The state of each flip-flop changes and gets shifted to the next flip-flop corresponding to each pulse of the clock.
- A shift register of "m" flip-flops will have 2^m number of states; i.e.

  $Q_0 Q_1 \ldots Q_{m-1} = 000 \ldots 0$ to $Q_0 Q_1 \ldots Q_{m-1} = 111 \ldots 1$. Thus the output sequence will repeat itself after every 2^m bits.
- The PN generator of Fig. 7.6.1 cannot generate a truly random sequence because this structure is a deterministic structure. This is the reason why, the sequence repeats itself.
- In order to make the random sequence "look like" truly random, its length should be sufficiently large i.e. a large number of flip flops should be included. Typically upto 2000 flip flops are used.
- The maximum length of the sequence will be $2^m - 1$. This is because the state 000...0 is not to be considered.
- If all zero state is allowed to exist then the EX-OR gates used in the parity generator will produce a zero output all the time. To avoid this the all zero state should be excluded.

7.6.1 Definition of a PN Sequence :

SPPU : May 15, Dec. 16

University Questions

Q. 1 What is PN sequence ? State the properties of PN sequence with the help of 4 stage shift register. **(May 15, 8 Marks)**

Q. 2 What is PN sequence ? Verify the three properties of PN sequence with the help of shift register. **(Dec. 16, 6 Marks)**

A Pseudo-Noise (PN) sequence is defined as a coded sequence of 1s and 0s with certain auto-correlation properties.

7.6.2 The Maximum Length Sequence :

- The maximum length sequence is a type of cyclic code which represents a commonly used periodic PN sequence.
- Such a sequence has long periods and requires a linear feedback shift register for its generation. A PN sequence generator with m-register linear feedback shift register produces a sequence of length $2^m - 1$ as discussed earlier.
- A shift register of length m consists of m flip-flops and all of them operate on the same clock.
- At each clock pulse, the state of each flip-flop is shifted to the next one.
- So as to prevent the shift register from getting emptied, we have to apply an input continuously to the first flip-flop.
- This input (called feedback) is calculated by taking into account the states of all the flip-flops.
- For a linear type feedback register, a feedback signal is obtained by using modulo-2 addition (EX-OR gate) of the outputs of various flip-flops as shown in Fig. 7.6.2.
- For m = 3 as shown in Fig. 7.6.2 (3 flip-flops), the maximum length sequence at the generator output will always be periodic with a period of

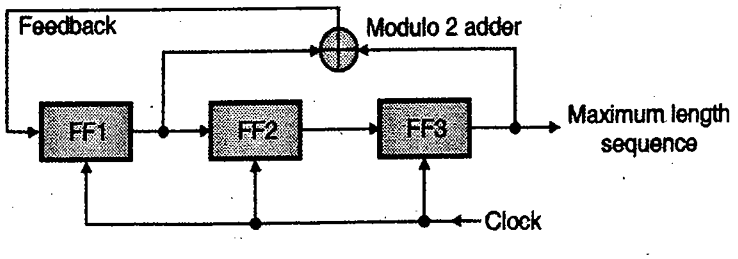
$$N = 2^m - 1 = 7 \qquad \ldots(7.6.1)$$

where m = length of shift register

- Table 7.6.1 highlights the relation between the register length m and PN sequence length N.
- Note that with increase in the value of m, the sequence length N increase and the PN sequence starts looking like random in true sense.

Table 7.6.1

Length of register m	7	8	9	10	11	12	13
PN sequence length N	127	255	511	1023	2047	4095	8191



(E-473) **Fig. 7.6.2 : Maximum length sequence generator**

7.6.3 Properties of Maximum-Length Sequences :

SPPU : May 06, Dec. 08, Dec. 10
Dec. 12, Dec. 14, May 15, Dec. 16

University Questions

Q. 1 State and explain properties of PN sequence. **(May 06, Dec. 10, 6 Marks)**

Q. 2 What is PN sequence ? Draw a suitable PN sequence generator and prove the properties of PN sequence and sketch autocorrelation function of same. **(Dec. 08, 8 Marks)**

Q. 3 What is PN sequence ? Verify the three properties of PN sequence with the help of 4 stage shift register. **(Dec. 12, 8 Marks)**

Q. 4 What are the properties of maximum length sequences ? Give the graphical representation of auto correlation property of random data and a PN sequence and explain. **(Dec. 14, 8 Marks)**

Q. 5 What is PN sequence ? State the properties of PN sequence with the help of 4 stage shift register. **(May 15, 8 Marks)**

Q. 6 What is PN sequence ? Verify the three properties of PN sequence with the help of shift register. **(Dec. 16, 6 Marks)**

Maximum length sequences have many properties possessed by a truly random sequence. Some of the important properties are :

1. Balance property
2. Run property
3. Correlation property

Balance property :

In each period of a maximum-length sequence, i.e. N the number of 1s is always one more than the number of 0s. This is called as the balance property. So if there are four 0's then there will be five 1's.

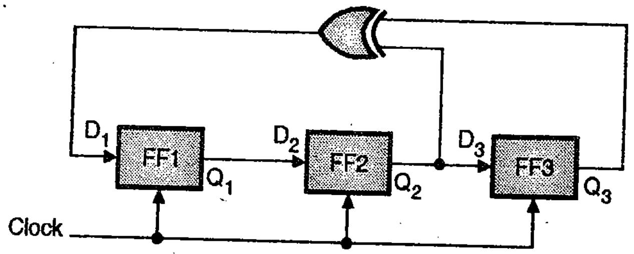
Run property :

- This property states that among the "runs" of 1s and 0s in each period of a maximum length sequence, one half the runs of a each kind are of length one (only one 0 or only one 1), one-fourth are of length two, (i.e.00 or 11) one eighth are of length three, and so on.
- In this statement, the word "run" means a subsequence of identical symbols (1s or 0's) within one period of the sequence. That means 000, 111 or 00, 11 or only 0, 1.... etc.
- And the length subsequence is the length of the run. The total number of runs will be (m + 1) / 2 if an m stage feedback shift register is being used.

Correlation property :

This property states that the auto-correlation of a maximum length sequence is periodic and has two possible values (i.e. it is binary valued).

Ex. 7.6.1 : For the PN sequence generator of Fig. P. 7.6.1(a) obtain and draw the PN sequence.



(E-474) **Fig. P. 7.6.1(a) : PN sequence generator**

Soln. :

- Assume that the initial state of the shift register is

$$Q_3 Q_2 Q_1 = 001$$

- The outputs Q_2 and Q_3 are connected to a modulo-2 adder i.e. an EX-OR gate. Table P. 7.6.1 summarizes the operation of the PN generator.

Important observations :

Some of the important observations from Fig. P. 7.6.1(b) are as follows :

1. The length of the PN sequence obtained at the Q_3 output is equal to $2^m - 1$. In this case m = 3 therefore the maximum length of the sequence is $N = 2^3 - 1 = 7$.
2. The PN sequence repeats itself after every 7 clock cycles.
3. The PN sequence is an NRZ type signal with logic 1 represented by + 1 and binary 0 is represented by – 1.
4. The duration of every bit is known as the **chip duration T_c and the chip rate R_c** is defined as the number of bits (chips) per second.

$$\therefore \quad T_c = \frac{1}{R_c} \quad ...(7.6.1(a))$$

$$\text{OR} \quad R_c = \frac{1}{T_c} \quad ...(7.6.2)$$

5. The period of the PN sequence is given by,

$$T_b = N T_c$$

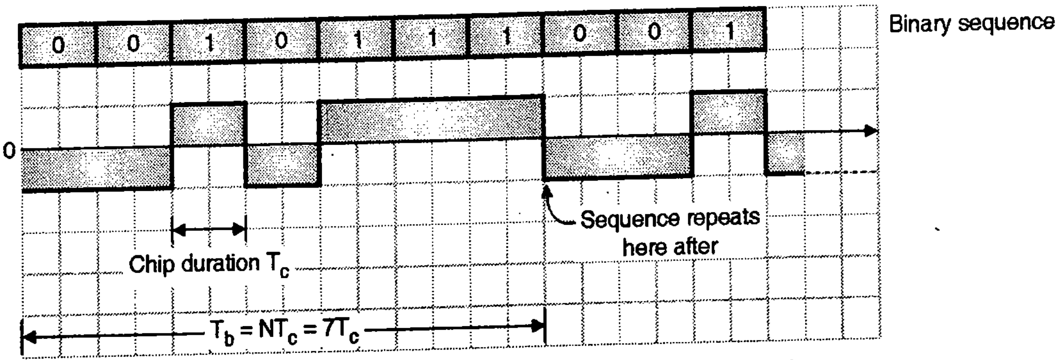
Note that the choice of 100 as the initial state is an arbitrary choice. Any other state of the remaining six can be treated as the initial state.

(E-574) Table P. 7.6.1 : Operation of the PN sequence generator

Clock Pulse Number	Shift register outputs			EX-OR gate output	PN sequence
	Q_3	Q_2	Q_1	$Q_3 \oplus Q_2$	Q_3
0	0	0	1	$0 \oplus 0 = 0$	0
1	0	1	0	$0 \oplus 1 = 1$	0
2	1	0	1	$1 \oplus 0 = 1$	1
3	0	1	1	$0 \oplus 1 = 1$	0
4	1	1	1	$1 \oplus 1 = 0$	1
5	1	1	0	$1 \oplus 1 = 0$	1
6	1	0	0	$1 \oplus 0 = 1$	1
7	0	0	1	$0 \oplus 0 = 0$	0
8	0	1	0	$0 \oplus 1 = 1$	0
9	1	0	1	$1 \oplus 0 = 1$	1
10	0	1	1	$0 \oplus 1 = 1$	0

The sequence repeats after this.

The PN sequence obtained at the Q_3 output is as shown in Fig. P. 7.6.1(b).



(E-475) Fig. P. 7.6.1(b) : PN sequence obtained at Q_3

7.6.4 Auto-correlation Function of PN Sequences :

SPPU : Dec. 08. Dec. 14

University Questions

Q. 1 What is PN sequence ? Draw a suitable PN sequence generator and prove the properties of PN sequence and sketch autocorrelation function of same. **(Dec. 08, 8 Marks)**

Q. 2 What are the properties of maximum length sequences ? Give the graphical representation of auto correlation property of random data and a PN sequence and explain. **(Dec. 14, 8 Marks)**

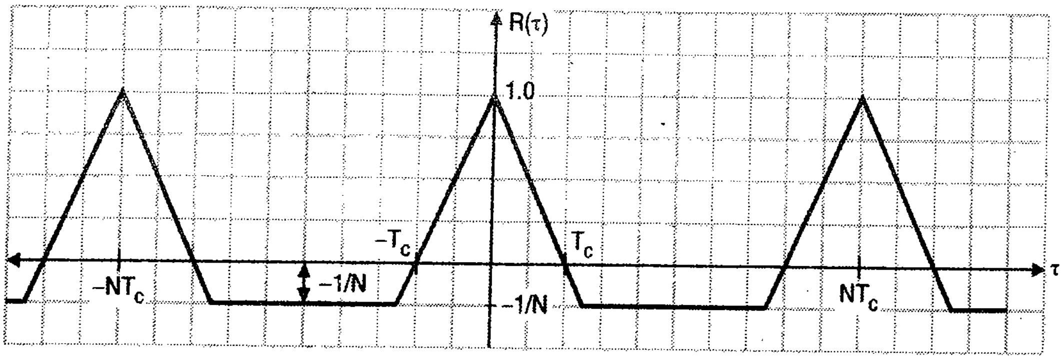
- The auto-correlation function R (τ) of the PN sequence c (t) is given by :

$$R(\tau). = \frac{1}{T_b} \int_{-T_b/2}^{T_b/2} c(t) \cdot c(t-\tau)\, dt \quad ...(7.6.3)$$

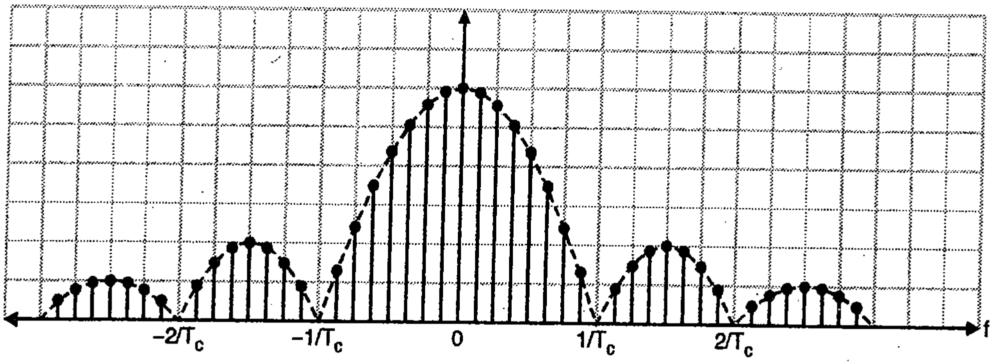
- This is as per the basic definition of the auto-correlation of a periodic signal.
- By solving the above expression we get,

$$R(\tau) = \begin{cases} 1 - \frac{N+1}{N T_c} \lvert \tau \rvert & \text{for} \lvert \tau \rvert < T_c \\ -\frac{1}{N} & \text{elsewhere} \end{cases} \quad ....(7.6.4)$$

- The auto-correlation function R (τ) of Equation (7.6.4) can be plotted as shown in Fig. 7.6.3. This shows that the auto-correlation R (τ) is a periodic function of time and it is a two valued function.



(E-476) Fig. 7.6.3 : Auto-correlation of a PN sequence



(E-477) Fig. 7.6.4 : Power spectral density of a PN sequence

7.6.5 Power Spectral Density of PN Sequence :

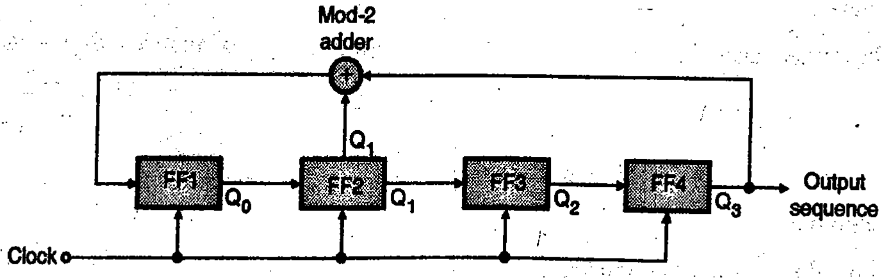
- The power spectral density of a PN sequence can be obtained from its auto-correlation function R (τ).
- We know that the power spectral density S (f) is the Fourier transform of the auto-correlation function R (τ). The power spectral density is as shown in Fig. 7.6.4.

Ex. 7.6.2 : A PN sequence is generated using a feedback shift register of length 4. Find the generated output sequence if the initial contents of the shift register are 1000. If the chip rate is 10^7 chips/sec., calculate the chip and PN sequence duration and period of output sequence. Draw its scheme arrangement.

May 2000. Dec. 06. May 08. May 09. 8 Marks. May 11. 4 Marks. Dec. 11. Dec. 15. 8 Marks

Soln. :

One of the possible schematic diagram of the PN generator is as shown in Fig. P. 7.6.2.



(E-490) Fig. P. 7.6.2 : A four stage shift register to generate PN sequence

The PN sequence generated by the above generator is shown in Table P. 7.6.2.

(E-575) Table P. 7.6.2 : PN sequence generated by the generator shown in Fig. P. 7.6.2

Clock Pulse Number	Shift register outputs				EX-OR gate output	PN sequence
	Q_3	Q_2	Q_1	Q_0	$Q_3 \oplus Q_2$	Q_3
0	0	0	0	1	$0 \oplus 0 = 0$	0
1	0	0	1	0	$0 \oplus 1 = 1$	0
2	0	1	0	1	$0 \oplus 0 = 0$	0
3	1	0	1	0	$1 \oplus 1 = 0$	1
4	0	1	0	0	$0 \oplus 0 = 0$	0
5	1	0	0	0	$1 \oplus 0 = 1$	1
6	0	0	0	1	$0 \oplus 0 = 0$	0
7	0	0	1	0	$0 \oplus 1 = 1$	0
8	0	1	0	1	$0 \oplus 0 = 0$	0
9	1	0	1	0	$1 \oplus 1 = 0$	1
10	0	1	0	0	$0 \oplus 0 = 0$	0
11	1	0	0	0	$1 \oplus 0 = 1$	1
12	0	0	0	1	$0 \oplus 0 = 0$	0
13	0	0	1	0	$0 \oplus 1 = 1$	0
14	0	1	0	1	$0 \oplus 0 = 0$	0
15	1	0	1	0	$1 \oplus 1 = 0$	1
The sequence repeats after this						

Chip duration :

The chip rate $R_c = 1 \times 10^7$ chips/sec. Hence the chip duration T_c is given by,

$$T_c = \frac{1}{R_c} = \frac{1}{1 \times 10^7}$$

$$= 0.1 \ \mu\text{sec.} \qquad \text{..Ans.}$$

Length of the PN sequence :

The length of the PN sequence is given by,

$$N = 2^m - 1$$

$$\text{But } m = 4 \qquad \therefore N = 2^4 - 1 = 15 \text{ digits } \text{...Ans.}$$

Duration of the PN sequence :

The duration of the PN sequence is given by,

$$T_b = N\,T_c$$

$$= 15 \times 0.1 \ \mu S$$

$$= 1.5 \ \mu S \qquad \text{...Ans.}$$

Ex. 7.6.3 : A PN sequence generator makes use of eight shift registers and has a chip rate of 10 MHz. Sketch the waveforms for autocorrelation function and power spectral density for PN sequence.

Dec. 01. 6 Marks. Dec. 06. 8 Marks

Soln. :

Given : Number of flip-flops m = 8

$\therefore$ Period of the maximum length sequence = $N = 2^m - 1$

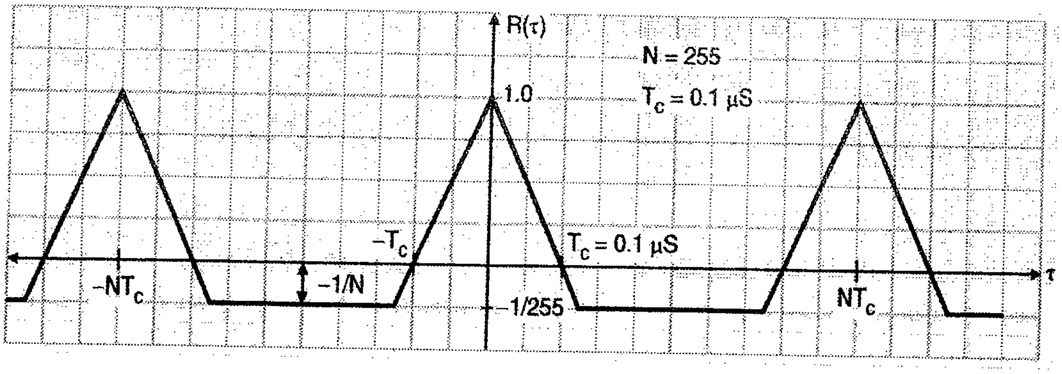
$$\therefore N = 2^8 - 1 = 255$$

$$\text{Chip rate } R_c = 10 \text{ MHz.}$$

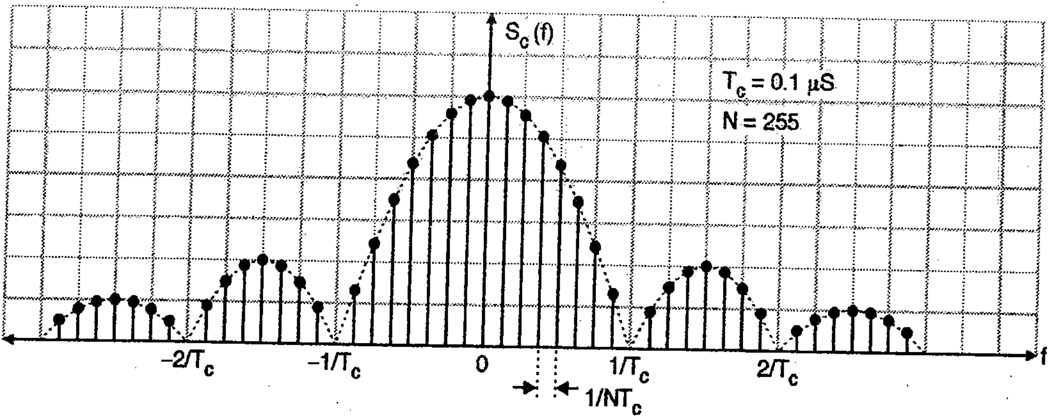
$$\therefore \text{ Chip duration } T_c = 1 / R_c$$

$$= \frac{1}{10 \times 10^6} = 0.1 \ \mu S.$$

The waveforms of the autocorrelation function and the power spectral density are as shown in Figs. P. 7.6.3(a) and (b) respectively.



(a) Autocorrelation function



(b) Power spectral density

(E-492) **Fig. P. 7.6.3**

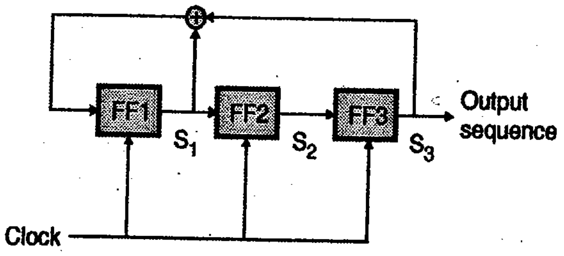
Ex. 7.6.4 : For a linear feedback shift register with three stages (m = 3) evaluate the maximum length PN sequence for feedback taps = [3, 1]. Draw the schematic arrangement and verify all the properties of PN sequence in generated output. Sketch the sequence, its auto-correlation function and PSD function if chip rate happens to be 10 MHz.

May 07, 10 Marks

Soln. : To obtain PN sequence

Sr. No.	State of shift register			$S_1 \oplus S_3$	PN sequence
	S_1	S_2	S_3		S_3
1	1	0	0	1	0
2	1	1	0	1	0
3	1	1	1	0	1
4	0	1	1	1	1
5	1	0	1	0	1
6	0	1	0	0	0
7	0	0	1	1	1
8	1	0	0	1	0 ← sequence repeats here

The generated PN sequence is 0 0 1 1 1 0 1 0.



(E-505) **Fig. P. 7.6.4(a) : A 3 stage feedback shift register**

To verify properties :

1. **Balance property :** In each period the number of 1's is always more than number of 0's. In this sequence there are 4 1's and 3 0's. This satisfies balance property.

2. **Run property :** m = 3, So $2^{3-1} = 4$ runs. These runs are given below.

$$C_n = \left\{ \underbrace{0\ 0}_{\text{Run 1}} \quad \underbrace{1\ 1\ 1}_{\text{Run 2}} \quad \underbrace{0}_{\text{Run 3}} \quad \underbrace{1}_{\text{Run 4}} \right\}$$

3. **Correlation property :** An autocorrelation of the sequence should be periodic and binary valued.

$$R_c(\tau) = 1 - \frac{N+1}{NT_c}|\tau| \quad \text{for } |\tau| < T_c$$

$$= -\frac{1}{N} \qquad \text{elsewhere}$$

Here length $N = 7$

$$R_c = \text{Chip rate} = 10 \text{ MHz}$$

$$T_c = \frac{1}{R_c} = 1 \times 10^{-7}$$

$$R_c(\tau) = \begin{cases} 1 - 11.428 \times 10^6 (|\tau|) & |\tau| < 1 \times 10^{-7} \\ -\frac{1}{7} & \text{elsewhere} \end{cases}$$

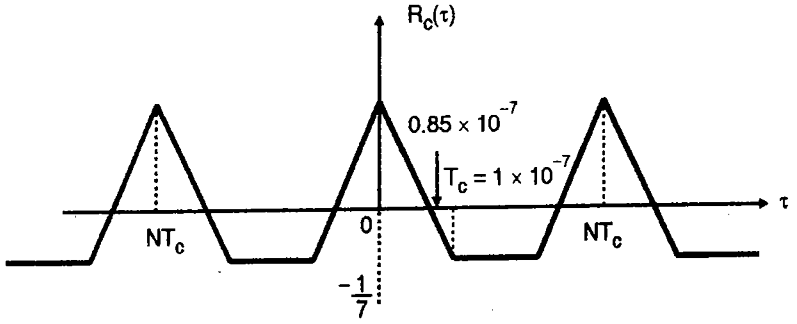
4. **Power spectral density :**

$$S_c(f) = \frac{1}{N^2}\delta(f) + \frac{1+N}{N^2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \text{sinc}^2\left(\frac{n}{N}\right) \delta\left(f - \frac{n}{NT_c}\right)$$

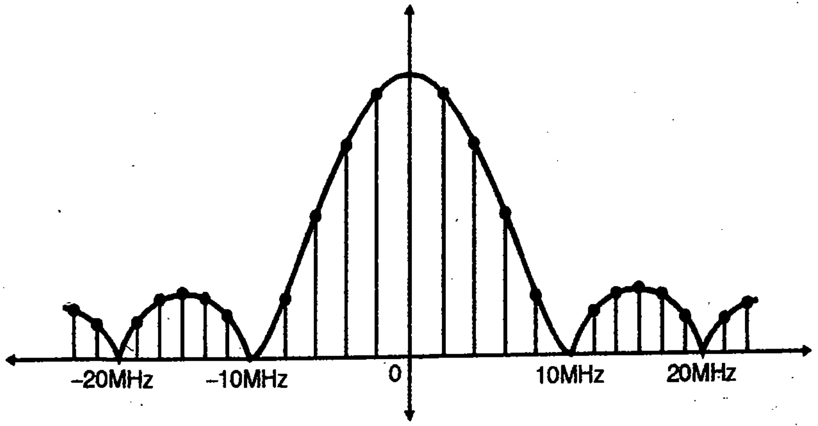
$$N = 7, \quad T_c = 1 \times 10^{-7}$$

$$\therefore \quad S_c(f) = \frac{1}{7^2}\delta(f) + \frac{1+7}{7^2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \text{sinc}^2\left(\frac{n}{7}\right) \delta\left(f - \frac{n}{7 \times 1 \times 10^{-7}}\right)$$

$$S_c(f) = 0.02\,\delta(f) + 0.163 \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \text{sinc}^2\left(\frac{n}{7}\right) \delta(f - 1.428 \times 10^6 n)$$



(E-506) **Fig. P. 7.6.4(b) : Autocorrelation of the sequence**



(E-507) **Fig. P. 7.6.4(c) : PSD of the PN sequence**

Ex. 7.6.5 : A PN sequence generator makes use of five shift registers and has a chip rate of 10 kHz. Draw a typical schematic for the generation of PN sequence. Sketch the waveforms for auto-correlation function and power spectral density for the PN sequence.

Dec. 03, 8 Marks

Soln. :

Number of flip flops m = 5

∴ Period of maximum length sequence

$$= N = 2^{5-1}$$

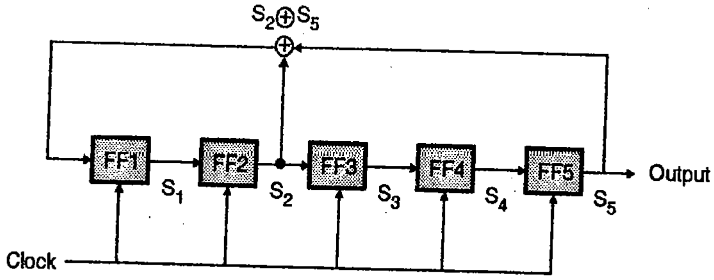
$$= 31 \text{ bits}$$

$$\text{Chip rate } R_c = 10 \text{ kHz}$$

$$\text{Chip duration } T_c = \frac{1}{R_c} = \frac{1}{10 \times 10^3}$$

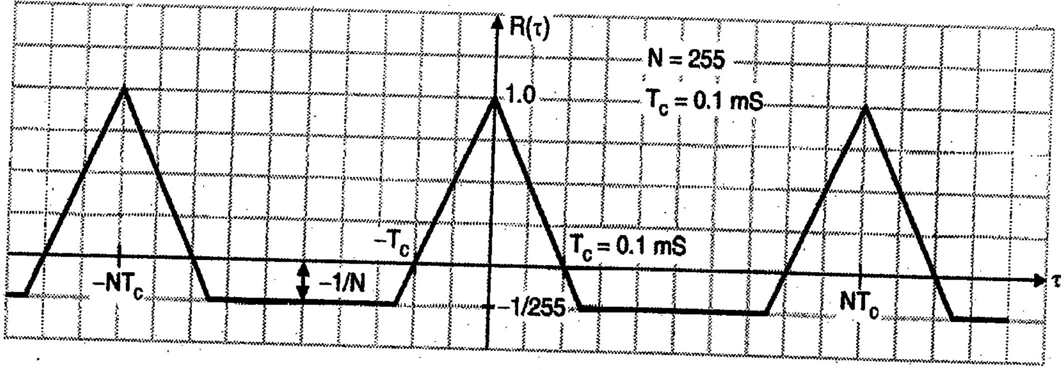
$$= 0.1 \text{ msec.}$$

Fig. P. 7.6.5(a) shows the block schematic of PN sequence generator.

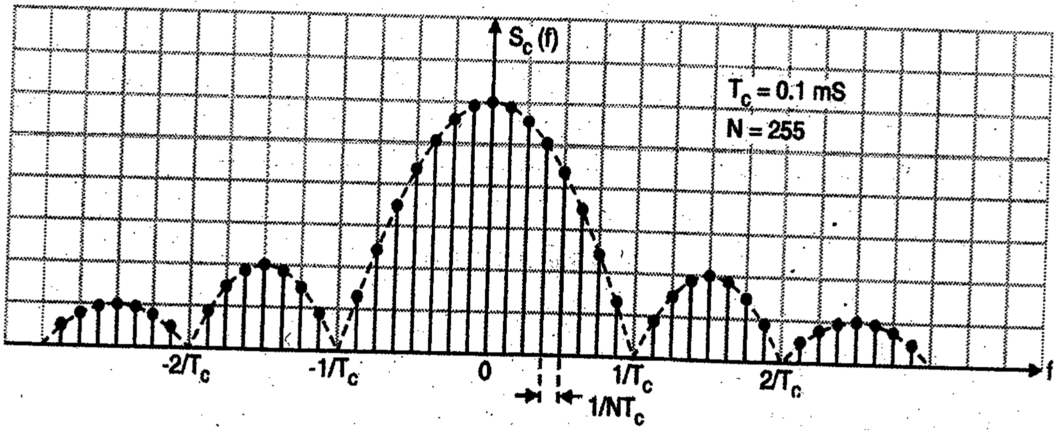


(E-508) Fig. P. 7.6.5(a)

Waveforms for auto-correlation function and power spectral density :



(E-509) Fig. P. 7.6.5(b) : Auto-correlation function

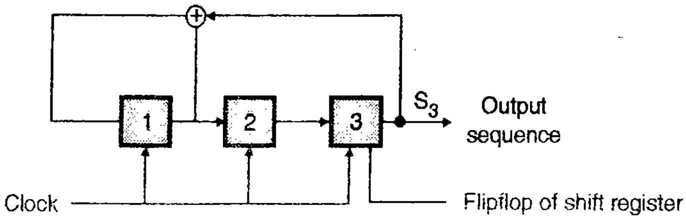


(E-510) Fig. P. 7.6.5(c) : Power spectral density

Ex. 7.6.6 : Design a three stage feedback shift register with proper taps to generate N = 7PN sequence. Draw the generator block and if the initial state of shift register is 100 (from left to right), find the output sequence.

Dec. 04, 6 Marks

Soln. :



(E-511) **Fig. P. 7.6.6**

State of flip -flop			Output PN sequence
$S_1 = S_1 \oplus S_3$	S_2	S_3	S_3
1	0	0	0
$1 \oplus 0 = 1$	1	0	0
1	1	1	1
$1 \oplus 1 = 0$	1	1	1
1	0	1	1
0	1	0	0
0	0	1	1
1	0	0	0

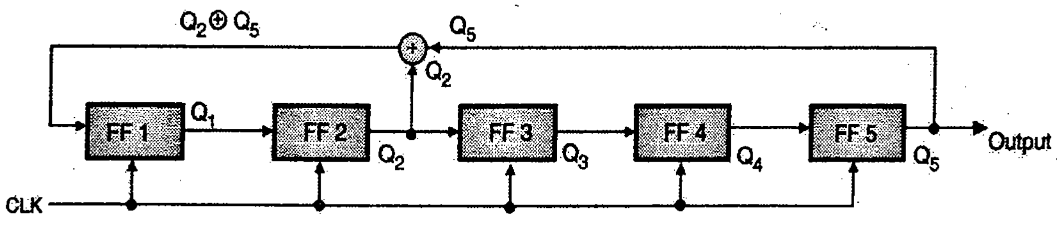
← Sequence repeats

(E-512)

Ex. 7.6.7 : Draw a 5 bit PN sequence generator and write the PN sequence generated by it. **May 06, 6 Marks**

Soln. :

Fig. P. 7.6.7(a) shows a 5 bit sequence generator and Table P. 7.6.7 shows the PN sequence generated by it.



(E-1379) **Fig. P. 7.6.7(a)**

Assume that the initial contents are,

$$Q_1\,Q_2\,Q_3\,Q_4\,Q_5 = 10000.$$

(E-1380) **Table P. 7.6.7**

Clock pulse number	Shift register outputs					EXOR output $Q_2 \oplus Q_5$	PN sequence Q_5
	Q_5	Q_4	Q_3	Q_2	Q_1		
0	0	0	0	0	1	$0 \oplus 0 = 0$	0
1	0	0	0	1	0	$0 \oplus 1 = 1$	0
2	0	0	1	0	1	$0 \oplus 0 = 0$	0
3	0	1	0	1	0	$0 \oplus 1 = 1$	0
4	1	0	1	0	1	$1 \oplus 0 = 1$	1
5	0	1	0	1	1	$0 \oplus 1 = 1$	0
6	1	0	1	1	1	$1 \oplus 1 = 0$	1
7	0	1	1	1	0	$0 \oplus 1 = 1$	0
8	1	1	1	0	1	$1 \oplus 0 = 1$	1
9	1	1	0	1	1	$1 \oplus 1 = 0$	1
10	1	0	1	1	0	$1 \oplus 1 = 0$	1
11	0	1	1	0	0	$0 \oplus 0 = 0$	0
12	1	1	0	0	0	$1 \oplus 0 = 1$	1
13	1	0	0	0	1	$1 \oplus 0 = 1$	1
14	0	0	0	1	1	$0 \oplus 1 = 1$	0

(Contd...)

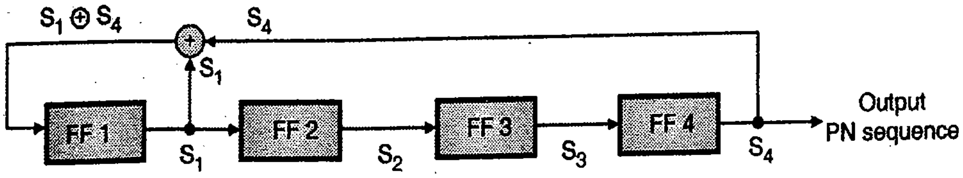
Clock pulse number	Shift register outputs					EXOR output $Q_2 \oplus Q_5$	PN sequence Q_5
	Q_5	Q_4	Q_3	Q_2	Q_1		
15	0	0	1	1	1	$0 \oplus 1 = 1$	0
16	0	1	1	1	1	$0 \oplus 1 = 1$	0
17	1	1	1	1	1	$1 \oplus 1 = 0$	1
18	1	1	1	1	0	$1 \oplus 1 = 0$	1
19	1	1	1	0	0	$1 \oplus 0 = 1$	1
20	1	1	0	0	1	$1 \oplus 0 = 1$	1
21	1	0	0	1	1	$1 \oplus 1 = 0$	1
22	0	0	1	1	0	$0 \oplus 1 = 1$	0
23	0	1	1	0	1	$0 \oplus 0 = 0$	0
24	1	1	0	1	0	$1 \oplus 1 = 0$	1
25	1	0	1	0	0	$1 \oplus 0 = 1$	1
26	0	1	0	0	1	$0 \oplus 0 = 0$	0
27	1	0	0	1	0	$1 \oplus 1 = 0$	1
28	0	0	1	0	0	$0 \oplus 0 = 0$	0
29	0	1	0	0	0	$0 \oplus 0 = 0$	0
30	1	0	0	0	0	$1 \oplus 0 = 1$	1
31	0	0	0	0	1	$0 \oplus 0 = 0$	0

The sequence repeats after this

Ex. 7.6.8 : For a 4 stage shift with feedback combination of (4, 1) demonstrate the balance property and run property of PN sequence, also calculate and plot the autocorrelation function of PN sequence produced by this shift register. **Dec. 07, May 15. 8 Marks**

Soln. :

Step 1 : Draw the 4 stage shift register :



(E-1382) **Fig. P. 7.6.8(a) : 4 stage shift register**

Step 2 : Obtain the PN sequence :

Assume that initially $S_1\ S_2\ S_3\ S_4 = 1000$. Table P. 7.6.8 shows the PN sequence generated by this circuit.

(E-1383) **Table P. 7.6.8**

Clock number	Shift register state				MOD2 adder output - $S_1 \oplus S_4$	PN sequence S_4
	S_4	S_3	S_2	S_1		
0	0	0	0	1	$0 \oplus 1 = 1$	0
1	0	0	1	1	$0 \oplus 1 = 1$	0
2	0	1	1	1	1	0
3	1	1	1	1	0	1
4	1	1	1	0	1	1
5	1	1	0	1	0	1
6	1	0	1	0	1	1
7	0	1	0	1	1	0
8	1	0	1	1	0	1
9	0	1	1	0	0	0
10	1	1	0	0	1	1
11	1	0	0	1	0	1
12	0	0	1	0	0	0
13	0	1	0	0	0	0
14	1	0	0	0	1	1
15	0	0	0	1	1	0

Repeat after this

$\therefore$ PN sequence = [000 1111 01 01 1001]

Step 3 : Properties of PN sequence :

1. **Balance property :** In one period of PN sequence there are seven 0s and eight 1s. As number of 1s is greater than number of 0s the balance property is satisfied.

2. **Run property :**

- As per this property there should be 2^{m-1} runs where m = Number of stages (4 here).

$\therefore$ Number of runs $= 2^{4-1} = 2^3 = 8$

- In the actual PN sequence generated the runs are identified as follows :

$$\text{PN sequence : } \underbrace{0\ 0\ 0}_{1}\ \underbrace{1\ 1\ 1\ 1}_{2}\ \underbrace{0}_{3}\ \underbrace{1}_{4}\ \underbrace{0}_{5}\ \underbrace{1\ 1}_{6}\ \underbrace{0\ 0}_{7}\ \underbrace{1}_{8}$$

Run : 1 2 3 4 5 6 7 8

(E-1384)

Thus the run property also is verified.

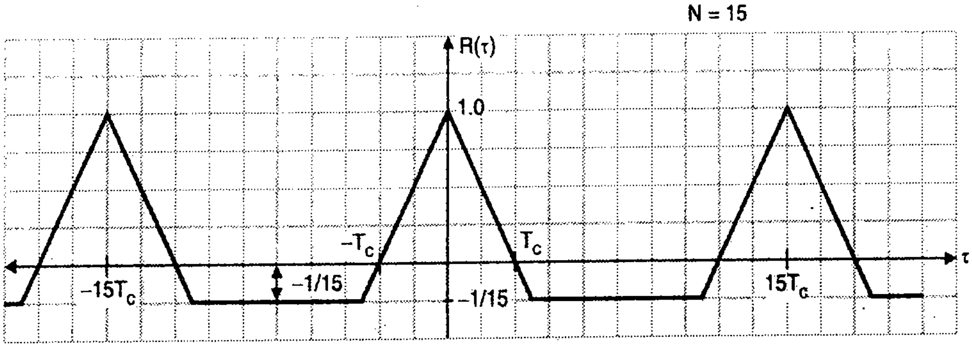
Step 4 : Autocorrelation function :

$$R(\tau) = \begin{cases} 1 - \dfrac{N+1}{N\,T_c} |\tau| & \text{for } |\tau| < T_c \\ -\dfrac{1}{N} & \text{elsewhere} \end{cases}$$

$$\text{Here } N = 2^m - 1 = 2^4 - 1 = 15$$

$$\therefore \quad R(\tau) = \begin{cases} 1 - \dfrac{16 |\tau|}{15\,T_c} & \ldots\text{for } |\tau| < T_c \\ -\dfrac{1}{15} & \ldots\text{elsewhere} \end{cases}$$

Fig. P. 7.6.8(b) shows the autocorrelation function.



(E-1388) **Fig. P. 7.6.8(b) : Auto-correlation of a PN sequence**

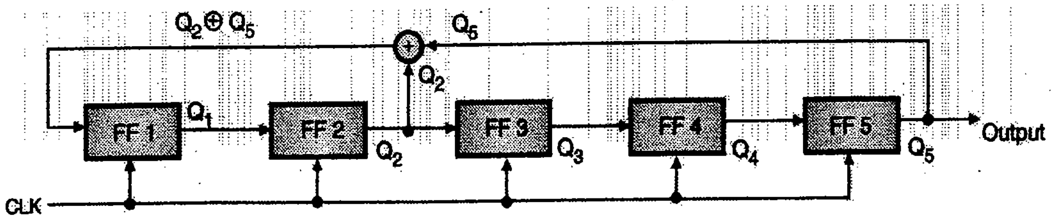
Ex. 7.6.9 : Draw a neat circuit diagram to generate maximum length sequence using linear feedback shift register of length m = 5 with feedback taps [5, 2]. Find generated output sequences if initial contents of SR are [10000]. If chip rate is 10^7 chips/sec, calculate chip and PN sequence duration and period of output sequence.

Dec. 09. 8 Marks

Soln. :

Step 1 : To generate the output PN sequence :

The required shift register is as shown in Fig. P. 7.6.9.



(E-1385) **Fig. P. 7.6.9**

Assume that the initial contents are $Q_1\ Q_2\ Q_3\ Q_4\ Q_5$ = 10000. The PN sequence is generated as shown in Table P. 7.6.9.

(E-1386) Table P. 7.6.9

Clock pulse number	Shift register outputs					EXOR output $Q_2 \oplus Q_5$	PN sequence Q_5
	Q_5	Q_4	Q_3	Q_2	Q_1		
0	0	0	0	0	1	$0 \oplus 0 = 0$	0
1	0	0	0	1	0	$0 \oplus 1 = 1$	0
2	0	0	1	0	1	$0 \oplus 0 = 0$	0
3	0	1	0	1	0	$0 \oplus 1 = 1$	0
4	1	0	1	0	1	$1 \oplus 0 = 1$	1
5	0	1	0	1	1	$0 \oplus 1 = 1$	0
6	1	0	1	1	1	$1 \oplus 1 = 0$	1
7	0	1	1	1	0	$0 \oplus 1 = 1$	0
8	1	1	1	0	1	$1 \oplus 0 = 1$	1
9	1	1	0	1	1	$1 \oplus 1 = 0$	1
10	1	0	1	1	0	$1 \oplus 1 = 0$	1
11	0	1	1	0	0	$0 \oplus 0 = 0$	0
12	1	1	0	0	0	$1 \oplus 0 = 1$	1
13	1	0	0	0	1	$1 \oplus 0 = 1$	1
14	0	0	0	1	1	$0 \oplus 1 = 1$	0

(Contd...)

Clock pulse number	Shift register outputs					EXOR output $Q_2 \oplus Q_5$	PN sequence Q_5
	Q_5	Q_4	Q_3	Q_2	Q_1		
15	0	0	1	1	1	$0 \oplus 1 = 1$	0
16	0	1	1	1	1	$0 \oplus 1 = 1$	0
17	1	1	1	1	1	$1 \oplus 1 = 0$	1
18	1	1	1	1	0	$1 \oplus 1 = 0$	1
19	1	1	1	0	0	$1 \oplus 0 = 1$	1
20	1	1	0	0	1	$1 \oplus 0 = 1$	1
21	1	0	0	1	1	$1 \oplus 1 = 0$	1
22	0	0	1	1	0	$0 \oplus 1 = 1$	0
23	0	1	1	0	1	$0 \oplus 0 = 0$	0
24	1	1	0	1	0	$1 \oplus 1 = 1$	1
25	1	0	1	0	0	$1 \oplus 0 = 1$	1
26	0	1	0	0	1	$0 \oplus 0 = 0$	0
27	1	0	0	1	0	$1 \oplus 1 = 0$	1
28	0	0	1	0	0	$0 \oplus 0 = 0$	0
29	0	1	0	0	0	$0 \oplus 0 = 0$	0
30	1	0	0	0	0	$1 \oplus 0 = 1$	1
31	0	0	0	0	1	$0 \oplus 0 = 0$	0

The sequence repeats after this

Step 2 : Chip duration :

$$\text{Chip rate, } R_C = 10^7 \text{ chips/sec.} \qquad \text{...given}$$

$$\therefore \quad \text{Chip duration } T_C = \frac{1}{R_C} = \frac{1}{10^7}$$

$$= 0.1 \ \mu\text{sec} \qquad \text{...Ans.}$$

Step 3 : Duration of PN sequence :

$$N = 2^m - 1$$

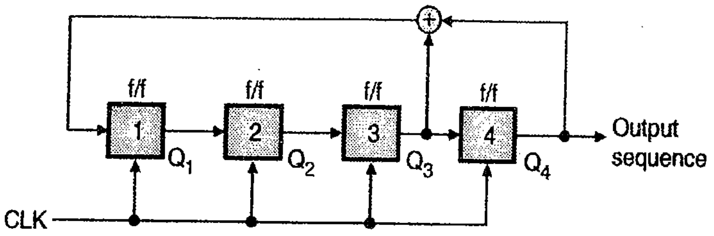
$$= 2^5 - 1 = 31 \text{ digits} \qquad \text{...Ans.}$$

Step 4 : Period of output sequence :

$$T_b = N \times T_C$$

$$= 31 \times 0.1 \ \mu S = 3.1 \ \mu S \qquad \text{...Ans.}$$

Ex. 7.6.10 : For the shift register given in problem, demonstrate the balance property of PN sequence. Also calculate and plot auto-correlation function of the PN sequence produced by this shift register. **May 13, 10 Marks**



(E-1360) **Fig. P. 7.6.10**

Soln. :

Step 1 : To obtain the PN sequence :

The PN sequence generated by the given shift register is as shown in Table P. 7.6.10. Assume the initial output of shift register is $Q_4\,Q_3\,Q_2\,Q_1 = 0001$.

(E-1362) **Table P. 7.6.10**

Clock pulse number	Shift register outputs Q_4	Q_3	Q_2	Q_1	EXOR gate output $Q_4 \oplus Q_3$	PN sequence Q_4
0	0	0	0	1	0 $(0 \oplus 0)$	0
1	0	0	1	0	0	0
2	0	1	0	0	1	0
3	1	0	0	1	1	1
4	0	0	1	1	0	0
5	0	1	1	0	1	0
6	1	1	0	1	0	1
7	1	0	1	0	1	1
8	0	1	0	1	1	0
9	1	0	1	1	1	1
10	0	1	1	1	1	0
11	1	1	1	1	0	1
12	1	1	1	0	0	1
13	1	1	0	0	0	1
14	1	0	0	0	1	1
15	0	0	0	1	0	0

The sequence repeats after this

Step 2 : Verification of properties :

1. **Balance property :**

In this PN sequence there are 8 1s and 7 0s. Thus the number of 1s is greater than the number of 0s. This satisfies the balance property.

2. Autocorrelation :

$$\text{Assume } R_c = \text{Chip rate} = 10 \text{ MHz.}$$

$$\therefore T_c = \frac{1}{R_c} = 1 \times 10^{-7}$$

$$\text{Length } N = 15.$$

An autocorrelation of the PN sequence is given by

$$R_c(\tau) = 1 - \frac{N+1}{N T_c} |\tau| \qquad \text{...for } |\tau| < T_c$$

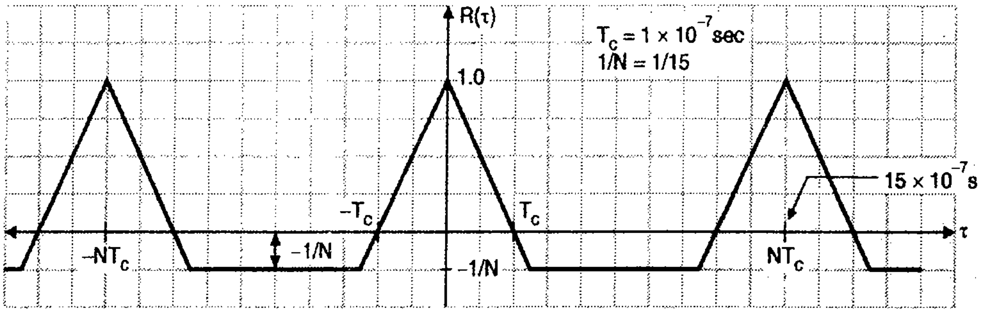
$$= -\frac{1}{N} \qquad \text{...elsewhere}$$

Substituting the values we get

$$R_c(\tau) = 1 - \frac{14}{15 \times 10^{-7}} (|\tau|) \text{ ...} |\tau| < 1 \times 10^{-7}$$

...elsewhere

The autocorrelation function is plotted in Fig. P. 7.6.10(a).



(E-1363) **Fig. P. 7.6.10(a) : Auto-correlation of a PN sequence**

7.7 A Notion of Spread Spectrum :

- The most important advantage of spread spectrum modulation is that it provides protection against externally generated interfering signals. Such signals are called as jamming signals.
- In the spread spectrum technique the signal containing information is spread over the frequency spectrum. That means it is made to occupy a bandwidth which is much larger than the minimum bandwidth required for its normal transmission without spreading.
- This will make the signal to appear like a noise because noise is spread over the entire band and get mixed into the background.
- Thus the spread spectrum is a method of **camouflaging** the message signal.
- One of the methods to used for widening the bandwidth of the data sequence is to use the modulation.
- Refer Fig. 7.7.1 which shows the transmitter, channel and receiver of an idealized model of baseband spread spectrum system.

Definition of jamming :

An externally generated interfering signal is called as a jamming signal and such an intentional interference is called as jamming.

7.7.1 Operation of the Encoder (Transmitter) :

- The input data sequence is denoted by d(t). This data sequence is first converted into an NRZ sequence b (t) by the NRZ encoder.

- The NRZ signal b (t) and the pseudonoise signal c (t) are applied to the two inputs of a product modulator.
- At the output of the product modulator, we obtain the spread spectrum signal. The spectrum of this signal is quite spread out as compared to the spectrum of b (t) which is a narrow band signal.
- Thus a data sequence b (t) is used to modulate a wideband pseudo-noise (PN) sequence c (t) by applying these two sequences to the product modulator or multiplier. Both sequences b (t) and c (t) are in polar form.
- According to the fourier transform theory the multiplication of signals in the time domain results in convolution of their frequency domain i.e.

$$c(t) \times b(t) \overset{F}{\longleftrightarrow} C(f) * B(f) \qquad ...(7.7.1)$$

- Hence if the data sequence b(t) is narrowband and the PN sequence c(t) is a wideband sequence, then the product sequence m (t) = c (t) × b (t) will have a spectrum M(f) which will be nearly the same as that of the PN sequence c (t).

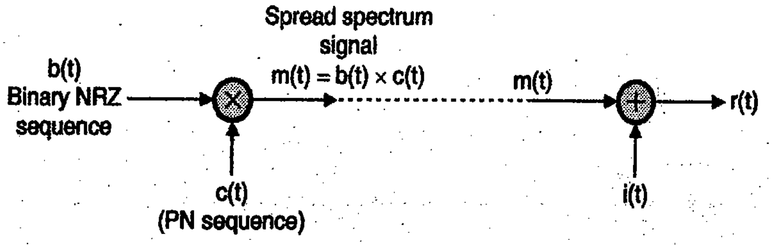
Thus the narrowband signal b (t) will be spread over the wideband and the PN sequence performs the role of a spreading code.

- Note that the transmitted signal m (t) is a baseband signal.
- The spread spectrum signal m (t) is transmitted over the channel where an additive interference i (t) is added to it.
- The signal is received by the receiver. The received signal is therefore expressed as,

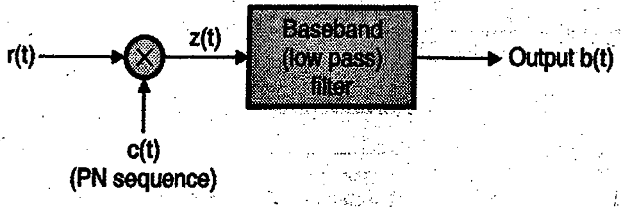
$$r(t) = m(t) + i(t) \qquad ...(7.7.2)$$

$$= c(t)b(t) + i(t) \qquad ...(7.7.3)$$

- The waveforms at different points in the transmitter are as shown in Fig. 7.7.2.



(a) Transmitter **(b) Channel**



(c)

(E-478) **Fig. 7.7.1 : Idealized model of DSSS system**

7.7.2 Receiver :

- To recover the original sequence b (t), the received signal r (t) is applied to a demodulator as shown in Fig. 7.7.1(c).
- The demodulator consists of a multiplier followed by a low- pass filter.
- The multiplier is supplied with a locally generated PN sequence which is an exact "replica" of the PN sequence used at the transmitter.
- The receiver needs to operate in perfect "synchronization" with the transmitter.
- The demodulated signal is given by,

$$z(t) = c(t) \times r(t) \quad ...(7.7.4)$$

- Substituting the expression for r (t) i.e.

$$r(t) = c(t)b(t)+i(t)$$

We get, $z(t) = c(t)[c(t)b(t)+i(t)]$

$$\therefore \quad z(t) = c^2(t)b(t)+c(t)i(t) \quad ...(7.7.5)$$

- In Equation (7.7.5) note that the desired signal b(t) has been multiplied by $c^2(t)$. We know that,

$$c(t) = \pm 1$$

$\therefore \quad c^2(t) = +1$ for all the values of t.

Hence Equation (7.7.5) gets converted to,

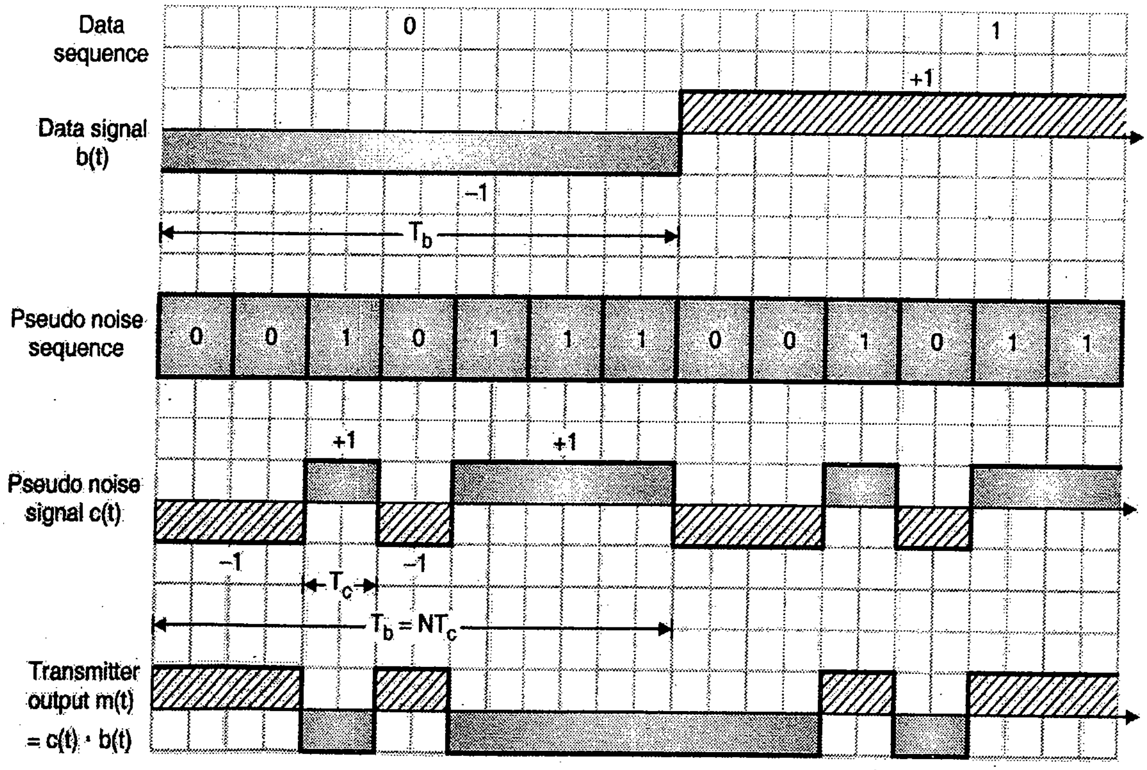
$$\mathbf{z(t) = b(t)+c(t)i(t)} \quad ...(7.7.6)$$

Thus the multiplier output in a receiver contains the desired signal b (t) and the product of the PN sequence c (t) and interference signal i (t).

- Due to the multiplication, the product signal $c(t) \cdot i(t)$ becomes a wideband signal whereas b(t) is a narrow band signal.
- Hence by applying the multiplier output to a baseband (low pass) filter we can pass only signal b(t) and attenuate the interference signal c (t) i (t) heavily.
- Thus the effect of interference signal is reduced to a great extent.

Summary :

- Thus the use of a spreading code in the transmitter produces a wideband signal that appears like noise for any receiver, which does not know anything about the spreading code.
- As the period of the spreading code is made longer, the transmitted signal becomes truly random.
- The price paid for the improved protection against the interference is the increased bandwidth, increased complexity and increased processing delay.



(E-479) Fig. 7.7.2 : Waveforms of DS SS transmitter

7.8 Direct Sequence Spread Spectrum with Coherent BPSK :

SPPU : Dec. 05, Dec. 07, May 09, May 11, Dec. 14

University Questions

Q. 1 Draw the block diagram of DSSS system transmitter and receiver. Write functional names inside the blocks and input output signals for each block. **(Dec. 05, 6 Marks)**

Q. 2 With the help of mathematical expression and block diagram explain direct sequence spread spectrum. **(Dec. 07, 10 Marks)**

Q. 3 Explain the working of direct sequence spread spectrum transmitter and receiver. **(May 09, 8 Marks)**

Q. 4 Explain DSSS in detail and state the applications of the same. **(May 11, 6 Marks)**

Q. 5 Draw block diagram of DSSS- PSK transmitter. **(Dec. 14, 2 Marks)**

- The spread spectrum technique discussed in the previous section is called as **"Direct Sequence Spread Spectrum"** (DS-SS) technique.
- The DS-SS technique can be used in practice for transmission of signai over a bandpass channel such as the satellite channel.
- For such an application, the coherent binary phase shift keying (BPSK) is used as modulation scheme in the transmitter and receiver. The transmitter is as shown in Fig. 7.8.1(a).

7.8.1 DS-BPSK Transmitter :

SPPU : May 12

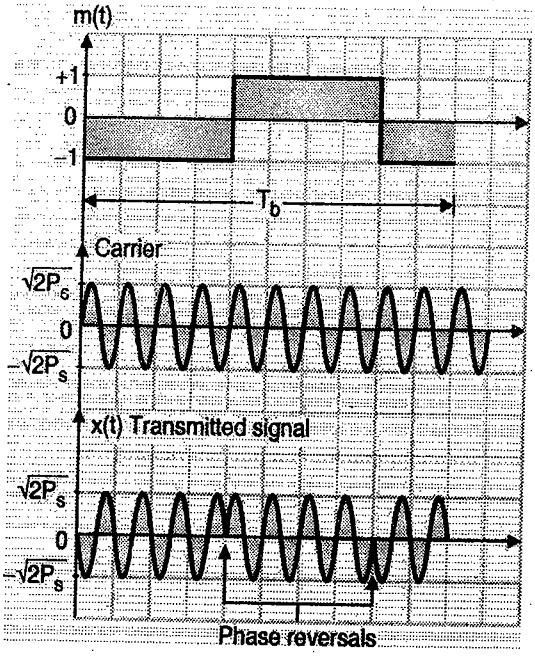
University Questions

Q. 1 Explain DS - SS BPSK transmitter and receiver with suitable block diagram and derive the power spectral density of the same. **(May 12, 8 Marks)**

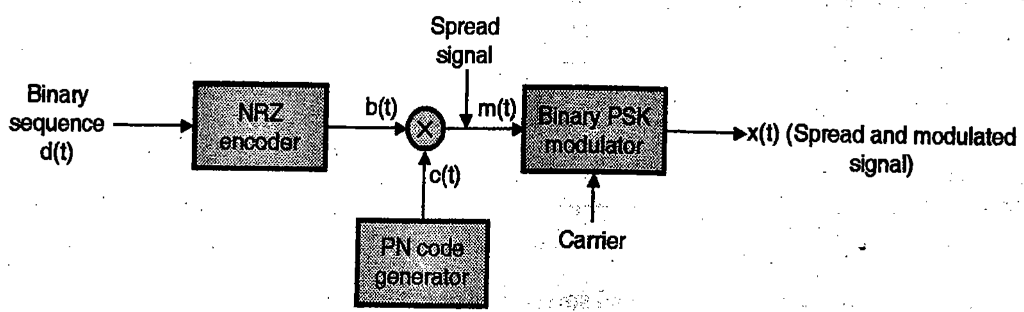
The DS-BPSK transmitter is shown in Fig. 7.8.1(a).

Operation :

- The binary sequence $d(t)$ is converted into NRZ signal $b(t)$ by applying $d(t)$ to the NRZ encoder.
- The NRZ signal $b(t)$ at the output of NRZ encoder is then used to modulate the PN sequence $c(t)$ generated by the PN code generator.
- The transmitter of Fig. 7.8.1(a) uses two stages of modulation. The first stage uses a product modulator or multiplier with $b(t)$ and $c(t)$ as its inputs and the second stage consists of a BPSK modulator.
- The modulated signal at the output of the product modulator i.e. $m(t)$ is the spread version of the original input and its used to modulate the carrier for BPSK modulation. Thus the BPSK modulator will modulate the spread signal (SS) to produce a DS-SS BPSK signal.
- The transmitted signal $x(t)$ is thus a direct sequence spread BPSK i.e. DS-BPSK signal. The waveforms for the DS-BPSK transmitter are shown in Fig. 7.8.1(b).
- As shown in Fig. 7.8.1(b) the product modulator output signal $m(t) = b(t) \times c(t)$ is an NRZ signal with amplitudes ± 1V. It is multiplied with the BPSK carrier.



(E-481) **Fig. 7.8.1(b) : Waveforms of DS-BPSK transmitter**



(E-480) **Fig. 7.8.1(a) : Direct sequence spread spectrum coherent PSK transmitter**

- The carrier signal applied to the BPSK modulator is given by,

$$V_{carrier}(t) = \sqrt{2P_s}\sin(2\pi f_c t) \quad ...(7.8.1)$$

- The output of BPSK modulator i.e. x(t) is transmitted. x(t) is given mathematically as -

$$x(t) = m(t) \times V_{carrier}(t)$$
$$= m(t) \times \sqrt{2P_s}\sin(2\pi f_c t)$$

But $m(t) = \pm 1$

$$\therefore \quad x(t) = \pm\sqrt{2P_s}\sin(2\pi f_c t) \quad ...(7.8.2)$$
$$= +\sqrt{2P_s}\sin(2\pi f_c t) \quad \text{...Positive m (t)}$$
$$= -\sqrt{2P_s}\sin(2\pi f_c t)$$

...Negative m (t)

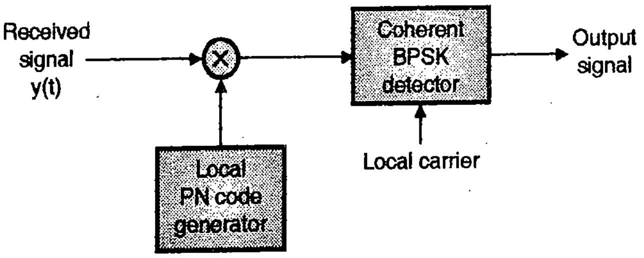
- Thus the phase shift of x(t) is 0° corresponding to a positive m (t) and it is 180° corresponding to a negative m (t).

7.8.2 DS-BPSK Receiver : SPPU : May 12

University Questions

Q. 1 Explain DS - SS BPSK transmitter and receiver with suitable block diagram and derive the power spectral density of the same. **(May 12, 8 Marks)**

The DS-BPSK receiver is as shown in Fig. 7.8.2.



(E-482) **Fig. 7.8.2 : The DS-BPSK receiver**

Operation :

- At the receiver we have to generate the replica of the original PN-sequence used at the transmitter.
- The received signal y(t) and the locally generated replica of the PN - sequence are applied to a multiplier. This is the first stage of multiplication.
- The multiplier performs the de-spreading operation. Output of multiplier is then applied to a coherent BPSK detector with a locally generated synchronous carrier applied to it.
- At the output of the coherent BPSK detector we get back the original data sequence i.e. d (t).

Synchronization :

- For proper operation, the spread spectrum system requires a local PN sequence at the receiver to by **synchronized** with the PN sequence at transmitter.
- The synchronization is carried out in two parts :

1. Acquision and 2. Tracking.

7.8.3 Performance Parameters of a DS-SS System : SPPU : Dec. 06

University Questions

Q. 1 Explain the performance parameters of DS-SS system. **(Dec. 06, 8 Marks)**

Some of the important performance parameters of a direct sequence spread spectrum system are as follows :

1. Processing gain
2. Probability of error
3. Jamming margin

7.8.3.1 Processing Gain :

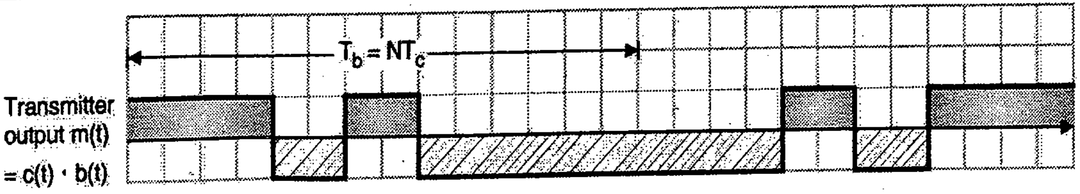
- The processing gain of a DS-SS system represents the extent of spreading in the frequency domain applied to an unspread signal.
- We can define the processing gain PG as the ratio of the bandwidth of the spread spectrum signal to the bandwidth of the unspread signal.

$$\therefore \text{ Processing gain } = \frac{\text{BW of spread spectrum signal}}{\text{BW of unspread signal}} \quad ...(7.8.3)$$

- Thus higher the value of PG more is the spreading of signal in the frequency domain.
- Now let us obtain the bandwidths of the spread spectrum signal and that of the unspread data signal b (t).

Bandwidth of spread signal m (t) :

As shown in Fig. 7.8.3 the spread signal m (t) is obtained by multiplying the NRZ signals b (t) and PN sequence c (t).



(E-483) **Fig. 7.8.3 : Spread spectrum signal m (t) = b (t) × c (t)**

- The one bit period of the spread signal m (t) is given by " T_c". The bandwidth of a NRZ signal is equal to the reciprocal of its one bit period.

$$\therefore \quad \text{BW of spread signal} = \frac{1}{T_c} \quad ...(7.8.4)$$

Bandwidth of the unspread signal :

- The unspread signal b (t) is an NRZ signal and the bandwidth of the NRZ signal is reciprocal of the bit period i.e. the reciprocal of the one bit period of the input signal d(t).

$$\therefore \quad \text{BW (unspread signal)} = 1/T_b \quad ...(7.8.5)$$

- Now substitute Equations (7.8.4) and (7.8.5) into Equation (7.8.3) to get,

$$\text{Processing Gain} = \frac{1/T_c}{1/T_b} = \frac{T_b}{T_c} \quad ...(7.8.6)$$

- The processing gain (PG) should be sufficiently high. We can increase PG by reducing the value of T_c. And we can reduce T_c by making the PN sequence longer (i.e. if the chip time T_c is reduced).

7.8.3.2 Probability of Error :

- We have already proved that the error probability P_e of a coherent BPSK system is given by,

$$P_e = \frac{1}{2} \operatorname{erfc} \sqrt{E_b/N_0} \quad ...(7.8.7)$$

Where E_b = Energy per bit

and $N_0/2$ = Power spectral density of white noise.

- In a direct-sequence spread binary PSK system the interference may be treated as a wideband noise signal with a power spectral density of $N_0/2$.

$$\therefore \quad \frac{N_0}{2} = \frac{J\,T_c}{2} \quad ...(7.8.8)$$

$$\text{OR} \quad N_0 = J\,T_c \quad ...(7.8.9)$$

where J = Average interference power

and T_c = Chip duration

- Substituting this expression for N_0 into Equation (7.8.7) we get error probability as,

$$P_e = \frac{1}{2} \operatorname{erfc} \sqrt{\frac{E_b}{J\,T_c}} \quad ...(7.8.10)$$

- This is the required expression for the error probability. The complementary error function is a monotonically decreasing function. Hence the error probability decreases as the value of $\sqrt{E_b/J\,T_c}$ increases.
- But P_e increases if $\sqrt{E_b/J\,T_c}$ decreases due to increase in the average interference power J or the chip period T_c.

7.8.3.3 Antijam Characteristics (Jamming Margin) :

SPPU : Dec. 12, May 15

University Questions

Q. 1 Explain in brief : Jamming margin

(Dec. 12, May 15, 2 Marks)

- Since the energy per bit i.e. $E_b = P_s\,T_b$ where, P_s is the average signal power and T_b is the bit duration, we can express the bit energy to noise density ratio as follows :

$$\frac{E_b}{N_0} = \frac{P_s\,T_b}{N_0} \quad ...(7.8.11)$$

But $N_0 = J\,T_c$ ...Referring to Equation (7.8.9)

$$\therefore \quad \frac{E_b}{N_0} = \frac{P_s\,T_b}{J\,T_c}$$

$$\frac{E_b}{N_0} = \left(\frac{T_b}{T_c}\right)\left(\frac{P_s}{J}\right) \quad ...(7.8.12)$$

- In Equation (7.8.6) we have defined the process gain PG as,

$$PG = \frac{T_b}{T_c} \qquad \therefore \frac{E_b}{N_0} = PG\left(\frac{P_s}{J}\right)$$

$$\therefore \quad \frac{J}{P_s} = \frac{PG}{E_b/N_0} \quad ...(7.8.13)$$

Jamming margin :

- The ratio J/P_s is called as the **jamming margin.** Hence the jamming margin is defined as the ratio of average interference power J and the signal power P_s. The jamming margin has an ideal value equal to zero and practically it should be as small as possible.
- If the jamming margin and the process gain both are expressed in dB then,

$$(\text{Jamming margin})\,\text{dB} = (\text{Processing Gain})\,\text{dB} - 10 \log_{10} [E_b/N_0]_{\min} \quad ...(7.8.14)$$

- Where $(E_b/N_0)_{\min}$ is the minimum bit energy to noise density ratio needed to support a prescribed average error probability.

Ex. 7.8.1 : A spread spectrum communication system is characterised by the following parameters. Duration of each information bit, T_b = 4.095 mS. Chip duration of a PN sequence, T_c = 1 μS. Calculate the processing gain and jamming margin if $(E_b / N_0) = 10$ and the average probability of error $P_e = 0.5 \times 10^{-5}$. **Dec. 16, 6 Marks**

Soln. : It has been given that,

$$T_b = 4.095 \text{ mS}, \quad T_c = 1 \ \mu\text{S}, (E_b / N_0) = 10,$$

$$P_e = 0.5 \times 10^{-5}$$

Processing gain PG :

$$\text{P.G.} = \frac{T_b}{T_c} = \frac{4.095 \times 10^{-3}}{1 \times 10^{-6}} = 4095 \qquad \text{...Ans.}$$

We know that $T_b = N\,T_c$.

$$PG = \frac{N\,T_c}{T_c} = N \qquad \therefore \ N = 4095$$

Jamming margin :

$$(\text{Jamming Margin})_{dB} = (PG)_{dB} - 10 \log_{10} [E_b / N_0]$$

$$\therefore (\text{Jamming Margin})_{dB} = 10 \log_{10} 4095 - 10 \log_{10} [10]$$

$$= 36.1 - 10 = 26.1 \text{ dB} \qquad \text{...Ans.}$$

7.8.4 Advantages of DS-SS System :

1. This system has a very high degree of discrimination against the multipath signals. Therefore the interference caused by the multipath reception is minimized successfully.
2. The performance of DS-SS system in presence of noise is superior to other systems such as FH-SS system.
3. This system combats the intentional interference (jamming) most effectively.

7.8.5 Disadvantages of DS-SS System :

1. With the serial search system, the acquisition time is too large. This makes the DS-SS system slow.
2. The sequence generated at the PN code generator output must have a high rate. The length of such a sequence needs to be long enough to make the sequence truely random.
3. The channel bandwidth required, is very large. But this bandwidth is less than that of a FH-SS system.
4. The synchronization is affected by the variable distance between the transmitter and receiver.

7.8.6 Features of DSSS :

Some of the important features of DSSS are as follows :

1. It provides good security against potential jamming or interpretation.
2. The DSSS is extremely effective against narrowband jamming signals.
3. The narrowband communication signals can coexist with the DSSS signals.
4. The DSSS signal is not very effective against broadband interference.

7.8.7 Applications of DS-SS System :

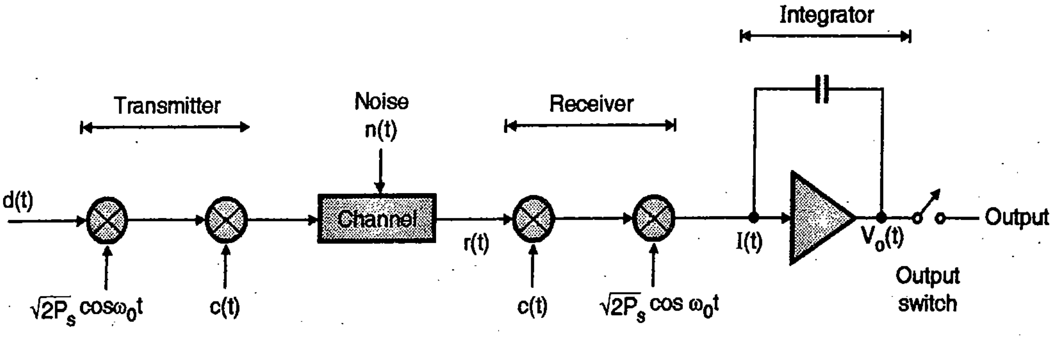
Some of the important applications of the DS-SS system are as follows :

1. To combat the intentional interference (jamming)
2. To reject the unintentional interference
3. To minimize the self interference due to multipath propagation
4. In the Low Probability of Intercept (LPI) signal
5. In obtaining the message privacy
6. Code division multiple access with DS-SS.

Ex. 7.8.2 : A single tone jammer, $V_j = \sqrt{2 P_j} \cos(\omega_0 t + \theta)$ is applied to direct sequence spread spectrum signal. With the help of suitable derivations, show that effective jamming power depends on bitrate of signal to be transmitted, pseudo-random noise sequence and the normalised power of interfering signal. **May 2000, 8 Marks**

Soln. :

Fig. P. 7.8.2 shows a spread spectrum transmitter with BPSK technique.



(E-491) **Fig. P. 7.8.2 : BPSK system with spread spectrum technique**

- Assume that the DS-SS signal interferes with a sinusoidal signal of normalized power P_j and carrier f_0. Hence the noise in Fig. P. 7.8.2 is replaced by the interference signal (single tone jammer).

$$\therefore \ n(t) = V_j = \sqrt{2P_j}\cos(\omega_0 t + \theta) \qquad ...(1)$$

- Therefore the input of the receiver is given by,

$$r(t) = \sqrt{2P_s}\cos\omega_0 t \cdot d(t) \cdot c(t)$$

$$+\sqrt{2P_j}\cos(\omega_0 t + \theta) \qquad ...(2)$$

- As the pseudo-random sequence c(t) has only two values ± 1 hence $c^2(t) = 1$. Therefore the signal at the input of the integrator is

$$\begin{aligned} I(t) &= r(t) \times c(t) \times \sqrt{2P_s}\cos\omega_0 t \\ &= [\sqrt{2P_s} \times \cos\omega_0 t \times d(t) \times c(t) \\ &\quad + \sqrt{2P_j}\cos(\omega_0 t + \theta)] \\ &\quad \times c(t)\sqrt{2P_s}\cos\omega_0 t \\ &= 2P_s\, d(t)\, c(t) \times \cos^2(\omega_0 t) \\ &\quad + 2P_j P_s\, c(t)\cos(\omega_0 t + \theta)\cos\omega_0 t \end{aligned}$$

$$\text{But } c^2(t) = 1$$

$$\begin{aligned} \therefore I(t) &= P_s d(t)[1 + \cos 2\omega_0 t] \\ &\quad + P_j P_s c(t)[\cos(2\omega_0 t + \theta) \\ &\quad + \cos\theta] \\ &= P_s d(t)(1 + \cos 2\omega_0 t) \\ &\quad + P_j P_s c(t)[\cos 2\omega_0 t \cos\theta - \sin 2 \\ &\qquad \omega_0 t \sin\theta + \cos\theta] \\ &= P_s d(t)(1 + \cos 2\omega_0 t) \\ &\quad + P_j P_s c(t)[\cos\theta(\cos 2\omega_0 t + 1) \\ &\quad - \sin 2\omega_0 t \sin\theta] \\ &= P_s d(t)(1 + \cos 2\omega_0 t) \\ &\quad + P_j P_s c(t)(\cos 2\omega_0 t + 1)\cos\theta \\ &\quad - P_j P_s c(t)(\sin 2\omega_0 t \sin\theta) \qquad ...(3) \end{aligned}$$

The integrator output will be zero for all those terms involving sine and cosine which are multiples of ω_0 t.

$\therefore$ Integrator output $V_0 t \propto P_s d(t) + P_j c(t) \cdot \cos\theta$ ...(4)

- The power spectral density of the jammer is given by,

$$S_j(f) = \frac{P_j \cos^2\theta}{2f_c}\left[\frac{\sin \pi f/f_c}{\pi f/f_c}\right] \qquad ...(5)$$

- Since $f_b << f_c$, the power spectral density given by Equation (5) remains constant in the range ± f_b.

$$\therefore S_j(f) = \frac{P_j \cos^2\theta}{2f_c} \ ; \ |f| \leq f_b \qquad ...(6)$$

- If the interfering signal is White Gaussian noise with a spectral density of $N_0/2$ then the error probability of integrate and dump receiver shown in Fig. P. 7.8.2 is given by

$$P_e = \frac{1}{2}\text{erfc}\left[\sqrt{E_b/N_0}\right] \qquad ...(7)$$

- But here the interfering signal is sinusoidal single frequency signal. Hence we can use Equation (7) with $N_0/2 = S_j(f)$.

$$\therefore \ P_e = \frac{1}{2}\text{erfc}\left[\sqrt{\frac{E_b f_c}{P_j \cos^2\theta}}\right]$$

$$\text{But } E_b = P_s T_b.$$

$$\therefore \ P_e = \frac{1}{2}\text{erfc}\left[\sqrt{\frac{P_s T_b f_c}{P_j \cos^2\theta}}\right]$$

$$= \frac{1}{2}\text{erfc}\,[(P_s/P_j)(f_c/f_b)(1/\cos^2\theta)]^{1/2} \qquad ...(8)$$

- The angle θ is the phase angle of the single tone jamming signal and it have no correlation with the signal phase. Hence all the values of θ can be assumed to be equally likely. Hence $\cos^2\theta = 1/2$. Substituting this value we get,

$$P_e = \frac{1}{2}\text{erfc}\,[2(P_s/P_j)(f_c/f_b)]^{1/2}$$

$$= \frac{1}{2}\text{erfc}\left[\frac{P_s}{P_j/2(f_c/f_b)}\right]^{1/2} \qquad ...(9)$$

- The term $\frac{P_j}{2(f_c/f_b)}$ is called as the effective jamming power. This power in comparison with the signal power P_s decides the error probability due to jammer.

$$\therefore \text{Effective jamming power} = \frac{P_j}{2(f_c/f_b)}$$

$$= \frac{P_j \times f_b}{2f_c}$$

$$= \frac{P_j}{2T_b f_c} \qquad ...(10)$$

This shows that the effective jamming power depends on the bit rate of the signal T_b, normalized jamming power P_j and the P.N. sequence.

Ex. 7.8.3 : The power spectrum of DS-SS signal shows that f = 0, power is 1 mW and spectrum goes through zero at 20.47 MHz away from carrier of 1000 MHz. If the spacing between spectral lines is 0.01 MHz, determine the received power, the chip rate and number of shift registers to generate PN sequence.

May 07, 8 Marks

Soln. :

1. To obtain chip rate :

$$R_c = \frac{1}{T_c} = 20.47 \text{ MHz}$$

2. **To obtain number of shift registers :**

The spacing between two components in PSD is $\frac{1}{NT_c}$

$$\therefore \quad \frac{1}{NT_c} = 0.01 \text{ MHz}$$

$$\therefore \quad N = \frac{1}{T_c} \times \frac{1}{0.01 \times 10^6}$$

$$\therefore \quad N = 2047 = 2^m - 1$$

$$\therefore \quad m = 11$$

A shift register with 11 stages will be required to generate the PN sequence.

3. **To obtained the received power :**

$$S_c(f) = \frac{1}{N^2}\delta(f) + \frac{1+N}{N^2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \text{sinc}^2\left(\frac{n}{N}\right) \delta\left(f - \frac{n}{NT_c}\right)$$

$$S_c(f) = \frac{1}{(2047)^2}\delta(f) + \frac{1+2047}{(2047)^2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \text{sinc}^2\left(\frac{n}{2047}\right) \delta(f - 0.01 \times 10^6 n)$$

$$P = \int_{-\infty}^{\infty} S_c(f)\, df$$

$$P = \int_{-\infty}^{\infty} 2.3865 \times 10^{-7} \delta(f)\, df + \int_{-\infty}^{\infty} 4.8875 \times 10^{-4} \sum_{n=-\infty}^{\infty} \text{sinc}^2\left(\frac{n}{2047}\right) \delta(f - 0.01 \times 10^6 n)$$

$$P = 2.3865 \times 10^{-7} + 4.8875 \times 10^{-4} \sum_{n=-\infty}^{\infty} \text{sinc}^2\left(\frac{n}{2047}\right) \qquad ...(1)$$

We know that,

$$x(n) = \frac{\sin(W_n)}{\pi_n}$$

$$\text{Diff}\{x(n)\} = X(\omega) = \begin{cases} 1 & |\omega| \leq W \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{Sinc}x = \frac{\sin \pi x}{\pi x}$$

$$\text{sinc}\left(\frac{n}{2047}\right) = \frac{\sin\left(\frac{\pi n}{2047}\right)}{\frac{\pi n}{2047}}$$

$$= 2047 \frac{\sin\left(\frac{\pi n}{2047}\right)}{\pi n}$$

$$\therefore \quad W = \frac{\pi}{2047}$$

$$\therefore \quad \sum_{n=-\infty}^{\infty} \text{sinc}^2\left(\frac{n}{2047}\right) = (2047)^2 \sum_{n=-\infty}^{\infty} \frac{\sin^2\left(\frac{\pi n}{2047}\right)}{(\pi n)^2}$$

$$= (2047)^2 \sum_{n=-\infty}^{\infty} x^2(n) \qquad ...(2)$$

Parseval's theorem states that,

$$\sum_{n=-\infty}^{\infty} x^2(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X^2(\omega)\, d\omega$$

Equation (2) becomes,

$$\sum_{n=-\infty}^{\infty} \text{sinc}^2\left(\frac{n}{2047}\right) = (2047)^2 \frac{1}{2\pi} \int_{-\pi}^{\pi} X^2(\omega)\, d\omega$$

$$= (2047)^2 \times \frac{1}{2\pi} \int_{-\omega}^{\omega} (1)^2\, d\omega \text{ as } X(\omega)$$

$$= \begin{Bmatrix} 1 & \text{for} & |\omega| \leq W \\ 0 & & \text{elsewhere} \end{Bmatrix}$$

$$= (2047)^2 \frac{W}{\pi}$$

Substituting $W = \frac{\pi}{2047}$

$$\therefore \quad \sum_{n=-\infty}^{\infty} \text{sinc}^2\left(\frac{n}{2047}\right) = (2047)^2 \times \frac{\pi}{2047} \times \frac{1}{\pi}$$

$$\sum_{n=-\infty}^{\infty} \text{sinc}^2\left(\frac{n}{2047}\right) = (2047)$$

Substituting in Equation (1)

$$P = 2.3865 \times 10^{-4} + 4.8875 \times 10^{-4} \times 2047$$

$$P = 1.0004888 \text{ W}$$

Ex. 7.8.4 : A DSS-BPSK system has r_b = 3 kbps, $N_0 = 10^{-10}$ W/Hz and is received with $P_e = 10^{-7}$. Calculate the processing gain needed for the system to achieve $P_e = 10^{-6}$ in the presence of a single tone jammer whose received power is ten times larger than the correct signal. **Dec. 05, 10 Marks**

Soln. :

Given : $r_b = 3$ kbps, $N_0 = 10^{-10}$ W / Hz, $P_e = 10^{-7}$

Desired $P_e = 10^{-6}$ When $J = 10^{-6}$

To find : Processing gain.

The error probability is given by,

$$P_e = \frac{1}{2}\,\text{erfc}\sqrt{\frac{E_b}{J\,T_c}}$$

But $J/P_s = 10$ given and $E_b = P_s\,T_b$

$$\therefore \quad P_s/J = 0.1$$

$$\therefore \quad P_e = \frac{1}{2}\,\text{erfc}\sqrt{\frac{0.4\,T_b}{T_c}}$$

$$= \frac{1}{2}\,2Q\sqrt{2}\times\sqrt{\frac{0.1T_b}{T_c}}$$

$$\text{But} \quad T_b/T_c = \text{P.G.}$$

$$\therefore \quad 10^{-6} = 2\,Q\,[\sqrt{0.2\,\text{P.G.}}\,]$$

$$\therefore \quad 5\times10^{-7} = Q\,[\sqrt{0.2\,\text{P.G.}}\,]$$

$$\therefore \quad 0.2\,\text{P.G.} = (5.6)^2$$

$$\therefore \quad \text{P.G.} = \frac{31.36}{0.2} = 156.8$$

Ex. 7.8.5 : A direct sequence BPSK spread spectrum system has a processing gain of 500. What is interference margin against a continuous tone interference if desired error probability is 10^{-5}. **May 06, 6 Marks**

Soln. :

Given : P.G. = 500, $P_e = 10^{-5}$

To find : Interference margin.

1. Calculate E_b/N_0 :

$$P_e = \frac{1}{2}\,\text{erfc}\sqrt{E_b/J\,T_c}$$

$$\text{But}\ J\,T_c = N_0$$

$$\therefore \quad P_e = \frac{1}{2}\,\text{erfc}\sqrt{E_b/N_0}$$

$$= \frac{1}{2}\times 2Q\,[\sqrt{2E_b/N_0}\ ]$$

$$\therefore \quad 10^{-4} = Q\,[\sqrt{2E_b/N_0}\ ]$$

$$\therefore \quad 3.6 = \sqrt{2E_b/N_0}$$

$$\therefore \quad E_b/N_0 = 6.48$$

2. Jamming margin :

$$\text{Jamming margin}\ (J/P_s) = \frac{PG}{E_b/N_0} = \frac{500}{6.48} = 77.16$$

$$\text{Jamming margin (dB)} = 10\log 77.16$$

$$= 18.87\ \text{dB.}$$

Ex. 7.8.6 : The information bit duration in DS-BPSK spread spectrum communication system is 4 mS while the chipping rate is 1 MHz. Assuming an average error probability of 10^{-5} for proper defection of message signal, calculate the jamming margin. Interpret your result. Given Q (4.25) = 10^{-5} **Dec. 10, 6 Marks, May 16, 9 Marks**

Soln. :

Given :

1. Chipping rate = 1 MHz

$$\therefore \quad \text{Chip duration}\ T_c = \frac{1}{1\times10^6} = 1\,\mu\,\text{sec.}$$

2. Information bit duration T_b = 4 mS

3. $P_e = 10^{-5}$

To find : Jamming margin

1. Processing gain PG :

$$PG = \frac{T_b}{T_c} = \frac{4\times10^{-3}}{1\times10^{-6}} = 4000$$

$$P_e = \frac{1}{2}\,\text{erfc}\sqrt{E_b/N_o}$$

$$= \frac{1}{2}\times 2Q\,[\sqrt{2E_b/N_o}\,]$$

$$10^{-5} = Q\,[\sqrt{2E_b/N_o}\,]$$

$$\therefore \quad 4.25 = [\sqrt{2E_b/N_o}\,]$$

$$\therefore \quad E_b/N_o = 9.03$$

2. Jamming margin :

$$\text{Jamming margin}\ (J/P_s) = \frac{PG}{E_b/N_o} = \frac{4000}{9.03}$$

$$= 442.90$$

$$\text{Jamming margin dB} = 10\log 442.90$$

$$= 26.463\ \text{dB} \qquad \text{...Ans.}$$

Ex. 7.8.7 : The information bit duration is DS-BPSK. Spread spectrum communication system is 10 mS while the chipping rate is 1 MHz. Assuming an average error probability is 10^{-6} for proper detection of message signal, calculate the Jamming margin.

May 11, 6 Marks

Soln. :

Given : $T_b = 10 \text{ mS} = 10 \times 10^{-3} \text{ s}$

and $T_c = \frac{1}{R_C} = \frac{1}{1 \times 10^6} = 1 \times 10^{-6} \text{ s}$

$P_e = 10^{-6}$

Step 1 : Find (E_b/N_o) :

For DS/BPSK system,

$$P_e = \frac{1}{2} \text{erfc} \sqrt{E_b/N_o}$$

$$\therefore \quad 10^{-6} = \frac{1}{2} \text{erfc} \sqrt{E_b/N_o}$$

$$\therefore \quad 2 \times 10^{-6} = \text{erfc} \sqrt{E_b/N_o}$$

It is observed that erfc $(3.3) \approx 2 \times 10^{-6}$

$$\therefore \quad \sqrt{E_b/N_o} = 3.3$$

$$\therefore \quad \frac{E_b}{N_o} = (3.3)^2 = 10.89$$

Step 2 : Calculate PG and jamming margin :

$$PG = \frac{T_b}{T_c} = \frac{10 \times 10^{-3}}{1 \times 10^{-6}} = 10{,}000$$

$$\text{Jamming margin} = \frac{PG}{(E_b/N_o)} = \frac{10{,}000}{10.89} = 918.3$$

$$(\text{J.M.})_{dB} = 10 \log_{10} (918.3) = 29.63 \text{ dB}$$ **...Ans.**

Ex. 7.8.8 : In a DSSS-BPSK system, the feedback shift register used to generate the PN sequence of length 15. The system is required to have an average probability of symbol error as 10^{-5}. Calculate :

1. Processing gain
2. Antijam margin

Given :

Dec. 13, 8 Marks

X	erfc(x)
3.01	0.00002074
3.02	0.00001947
3.03	0.00001827
3.04	0.00001714

Soln. :

Given : DSSS-BPSK system, $m = 15$ $P_e = 10^{-5}$

To find : 1. Processing gain 2. Antijam margin.

Step 1 : Find N :

For the PN sequence, period N is given by,

$$N = 2^m - 1 = 2^{15} - 1 = 32767$$

Step 2 Find processing gain :

For a DS-BPSK system,

Processing gain = N = 32767 **...Ans.**

Step 3 : Jamming margin :

$$(\text{Jamming margin})_{dB} = (\text{Processing gain})_{dB} - 10 \log_{10} [E_b / N_0]$$

$$= 10 \log_{10} 32767 - 10 \log_{10} [E_b / N_0]$$

$$= 45.15 \text{ dB} - 10 \log_{10} [E_b / N_0] \quad ...(1)$$

But we do not know the value of $[E_b / N_0]$.

$$P_e = \frac{1}{2} \text{erfc} \sqrt{E_b / N_0}$$

$$\therefore \quad 2 \times 10^{-5} = \text{erfc} \sqrt{E_b / N_0}$$

$$0.00002 = \text{erfc} \sqrt{E_b / N_0}$$

From the given table, we get,

$$x = \sqrt{E_b / N_0} = 3.01$$

$$\therefore \quad E_b / N_0 = (3.01)^2 = 9.06 \quad ...(2)$$

Substituting in Equation (1), we get,

$$(\text{Jamming margin})_{dB} = 45.15 - 10 \log_{10} (9.06)$$

$$= 45.15 - 9.57 = 35.58 \text{ dB}$$ **...Ans.**

Ex. 7.8.9 : A spread spectrum system has the following parameters :

Information bit duration $T_b = 4.095$ m sec.

PN chip duration $T_c = 1\ \mu$ sec.

Find the processing gain. What is the number of shift registers required ? Also find the jamming margin if the $\frac{E_b}{N_o} = 10$ for the BPSK scheme.

Dec. 14, 6 Marks

Soln. :

For processing gain and jamming margin refer Ex. 7.8.1.

Number of shift registers (m) :

$$N = 2^m - 1$$

$$4095 = 2^m - 1$$

$$\therefore m = 12 \qquad \text{...Ans.}$$

7.9 PSD of DS Spread Spectrum :

SPPU : May 12

University Questions

Q. 1 Explain DS - SS BPSK transmitter and receiver with suitable block diagram and derive the power spectral density of the same. **(May 12, 8 Marks)**

- We assume that the type of modulation used alongwith DS is QPSK. This QPSK signal corresponding to the n^{th} data symbol is shaped with the amplitude shaping pulse to obtain the DS/QPSK signal.
- If the n^{th} data symbol has a zero mean value and if it is uncorrelated symbol, then the power spectral density of the complex envelope is given by,

$$S(f) = \frac{A^2}{T} \sigma_x^2 \, |H_a(f)|^2 \qquad \text{...(7.9.1)}$$

where A = Amplitude, T = Symbol duration, σ_x = Variance.

$H_a(f)$ = F.T. of $h_a(t)$ i.e. the amplitude shaping pulse.

- The amplitude shaping pulse is given by,

$$h_a(t) = \sum_{k=0}^{N-1} a_k h_c(t - kT_c) \qquad \text{...(7.9.2)}$$

- Take the Fourier transform of $h_a(t)$ to get,

$$F[h_a(t)] = H_a(f)$$

$$= H_c(f) \sum_{k=0}^{N-1} a_k e^{-j2\pi fkT_c} \qquad \text{...(7.9.3)}$$

And $$|H_a(f)|^2 = |H_c(f)|^2 \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a_k a_l^* \, e^{-j2\pi f(k-l)T_c} \qquad \text{...(7.9.4)}$$

- This expression can be converted in the form of DTFT (Discrete Time Fourier Transform) as follows :

$$|H_a(f)|^2 = |H_c(f)|^2 \, 2N \phi_{k,k}(f) \qquad \text{...(7.9.5)}$$

Where $\phi_{k,k}(f)$ = DTFT and is defined as,

$$\phi_{k,k}(f) = \sum_{n=-N+1}^{N-1} \phi^a_{k,k}(n) \, e^{-j2\pi fnT_c} \qquad \text{...(7.9.6)}$$

- Substitute $T = n\,T_c$ and $\sigma_x^2 = \frac{1}{2} E[\,|x_i|^2\,] = 1/2$ we get,

$$S(f) = \frac{A^2}{T} |H_c(f)|^2 \phi_{k,k}(f) \qquad \text{...(7.9.7)}$$

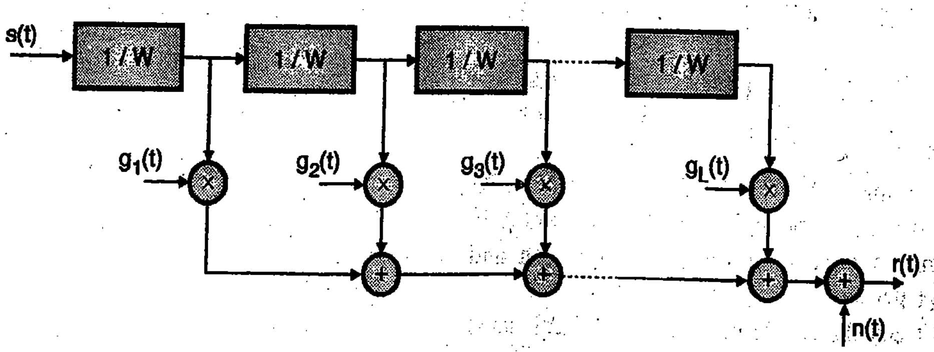
This is the required expression for the PSD.

7.10 DS Spread Spectrum on Frequency Selective Fading Channels :

- The frequency selective fading means, signal fading will take place only at certain particular values of frequency. But fading will not take place for signals having other frequencies than these particular frequencies.
- Such a channel can be modelled a tapped delay line as shown in Fig. 7.10.1. This tapped delay line has L-taps as shown.

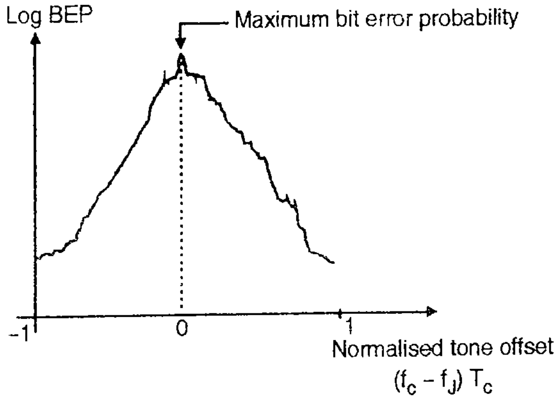
$g_1(t)$, $g_2(t)$ ... etc. are the tap gains.

- Such a model is very convenient as it simplifies analysis to a great extent. s(t) is the DS spread spectrum signal that travels over the tapped delay line which acts as frequency selective fading channel.
- The noise n(t) is added at the end to obtain the received signal r(t).
- Since the channel is frequency selective fading type, we have to take the Fourier transform of s(t) to analyze its performance in the frequency domain.



(E-488) **Fig. 7.10.1 : Tapped delay line acts as frequency selective fading channel**

- This indicates that the bit error probability does not remain constant. Instead it is a function of frequency as shown in Fig. 7.10.2.



(E-489) **Fig. 7.10.2 : Performance of SS on frequency selective fading channel**

- Fig. 7.10.2 indicates that the maximum bit error probability corresponds to $(f_c - f_J)\, T_c = 0$. That means when $f_J = f_c$.

  Where f_J corresponds to the interference frequency.

7.11 Frequency Hop Spread Spectrum (FH-SS) Signals :

SPPU : Dec. 10, May 11, Dec. 14, May 15

University Questions

Q. 1 Explain Frequency Hop Spread Spectrum System (FHSS). How is FHSS advantageous over DSSS ? **(Dec. 10, 6 Marks)**

Q. 2 State classification of spread spectrum and explain FHSS in detail. **(May 11, 7 Marks)**

Q. 3 Draw and explain FHSS spread spectrum system with transmitter and receiver section. **(Dec. 14, 8 Marks)**

Q. 4 Draw and explain FHSS system with transmitter and receiver section. **(May 15, 8 Marks)**

- In the DS-SS system discussed in the previous section, the NRZ data sequence b (t) modulates the PN sequence.
- The product signal $[\, m(t) = b(t) \times c(t)]$ is spread instantaneously in the frequency domain due to this process.
- The capacity of DS-SS system to reject the intentional interference (jamming) is dependent on the "processing gain PG".
- The processing gain $PG = (T_b / T_c)$. Hence PG increases if the chip period T_c is decreased which in turn permits a greater transmission bandwidth and more chips per bit.

  Problem : But the capabilities of the physical used for the generation of PN spread spectrum signal put a practical limitation on the maximum value of "processing gain" and hence on the capability to combat jamming.
- Under some operating conditions the maximum attainable processing gain $PG_{(max)}$ is not sufficiently high for combating the jamming.
- Under such conditions an alternative system called Frequency Hop (FH) spread spectrum.

Principle of operation of (FH – SS) system :

- In this system the data is used to modulate a carrier. The data modulated carrier is then **randomly hopped** from one frequency to the other.
- Due to this, the spectrum of transmitted signal is spread sequentially rather than instantaneously.

Types of modulation :

A common modulation technique used is the M-ary frequency shift keying (MFSK). The combination of frequency hopping (FH) and MFSK is known as FH / MFSK.

Note : It is important to understand that the frequency hopping does not cover the entire spread spectrum instantaneously. Rather it covers the entire spectrum sequentially. Therefore we have to consider the rate at which the frequency hops occur. Based on the rate of hopping, the FH/MFSK system has been classified into two categories.

7.11.1 Types of Frequency Hopping :

- Depending on the rate of frequency hopping, the FH/MFSK systems are classified into two categories :
  1. Slow frequency hopping.
  2. Fast frequency hopping.

1. **Slow frequency hopping :**
   - In slow frequency hopping the **symbol rate R_s** of the MFSK signal is an integer multiple of the hop rate R_h.
   - That means several symbols are transmitted corresponding to each frequency hop.

   ∴ **Each frequency hop ⇒ Several symbols.**

   i.e. frequency hopping takes place slowly.
2. **Fast frequency hopping :**
   - In the fast frequency hopping the hop rate R_h is an integer multiple of the MFSK **symbol rate R_s.**
   - That means during the transmission of one symbol, the carrier frequency will hop several times.

   ∴ **Each symbol transmission ⇒ Several frequency hops.**

   - Thus the frequency hopping takes place at a fast rate.

7.12 Slow Frequency Hopping :

SPPU : May 08, May 15, Dec. 16

University Questions

Q. 1 Explain the following frequency hop spread spectrum systems, with the help of relevant diagram.

1. Slow frequency hopping.
2. Fast frequency hopping. **(May 08, 8 Marks)**

Q. 2 Explain in brief : Slow Frequency Hopping. **(May 15, 2 Marks)**

Q. 3 Draw the block diagram of FH-SS systems transmitter and receiver. Write the functional names inside the blocks and input output signals of each block. **(Dec. 16, 6 Marks)**

- Fig. 7.12.1 shows the block diagram of a slow-frequency hopping FH/MFSK transmitter.

7.12.1 Operation of the FH/FMSK Transmitter :

SPPU : Dec. 05, May 06, Dec. 07, Dec. 09, Dec. 14, May 15, Dec. 16

University Questions

Q. 1 Draw the block diagram of FH-SS system transmitter and receiver. Write the functional names inside the blocks and input output signals of each block. **(Dec. 05, 6 Marks)**

Q. 2 Draw the block diagram of FHSS transmitter and receiver. **(May 06, 4 Marks)**

Q. 3 With a neat block diagram explain working of frequency hopped transmitter and receiver combination. **(Dec. 07, 8 Marks)**

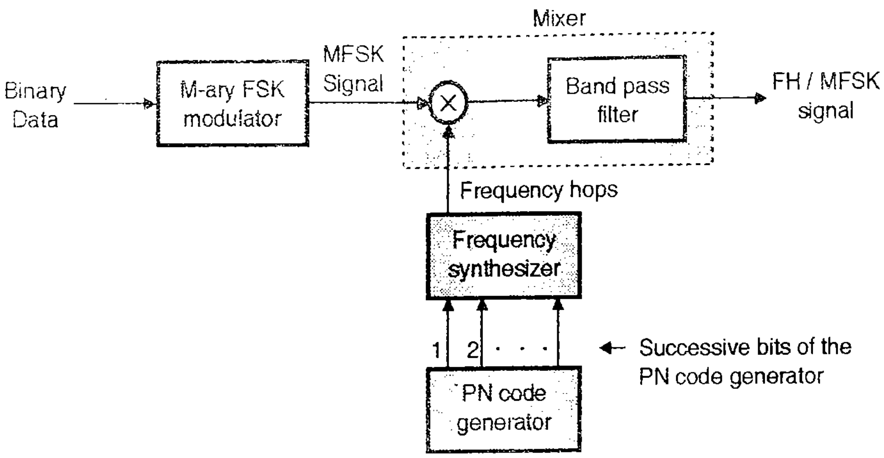
Q. 4 With neat block schematic, explain working of FHSS transmitter and receiver. Comment on the commercial applications of FHSS. **(Dec. 09, 8 Marks)**

Q. 5 Draw and explain FHSS spread spectrum system with transmitter and receiver section. **(Dec. 14, 8 Marks)**

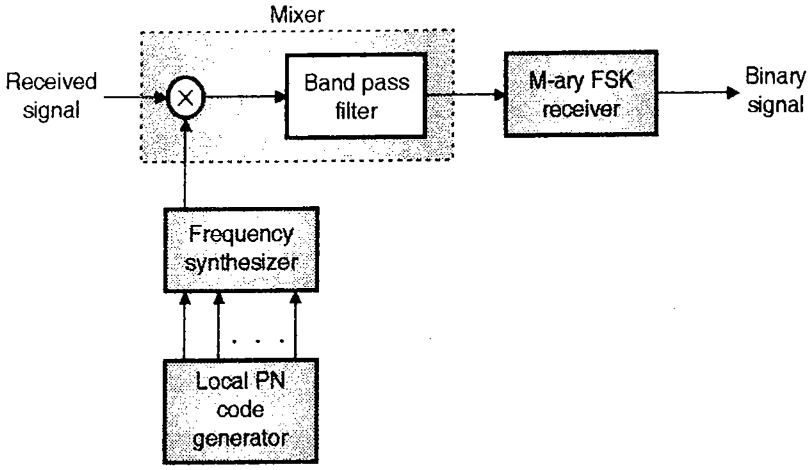
Q. 6 Draw and explain FHSS system with transmitter and receiver section. **(May 15, 8 Marks)**

Q. 7 Draw the block diagram of FH-SS systems transmitter and receiver. Write the functional names inside the blocks and input output signals of each block. **(Dec. 16, 6 Marks)**

- The binary data sequence b (t) is applied to the M-ary FSK modulator the output of which goes to the input of the mixer.
- The other input to the mixer block is obtained from a digital frequency synthesizer. The mixer consists of a multiplier followed by a band pass filter.
- At the multiplier output we get the two input frequencies, their sum and their difference frequency components.
- The bandpass filter is designed to select only the sum frequency component rejecting all other components. This sum components of frequency is then transmitted.
- Successive K-bits of the input binary data sequence will form one symbol. M such symbols can be transmitted using the M-ary FSK system with $M = 2^k$.
- The M-ary FSK modulator will assign a distinct frequency for each of these M symbols.
- Thus the frequency of mixer input obtained from MFSK modulator is changing continuously.
- The other input to the mixer is obtained from the digital frequency synthesizer. The synthesizer output at a given instant of time is the "frequency hop".
- Each frequency hop is mixed with the MFSK signal to produce the transmitted signal as explained earlier.
- The frequency hops at the output of the synthesizer are controlled by the successive bits at the output of the PN code generator.
- The output bits of the PN generator change randomly. Therefore the synthesizer output frequency will also change randomly.
- Hence the frequency hops produced will vary in a random manner.
- If the number of successive bits at the output of PN generator is "n", then the total number of frequency hops will be 2^n. The total bandwidth of the transmitted FH/MFSK signal is equal to the sum of all the frequency hops.
- Therefore the bandwidth of the transmitted FH/MFSK signal is very large of the order of few GHz.



(E-484) Fig. 7.12.1 : Frequency hop spread M-ary FSK transmitter



(E-485) Fig. 7.12.2 : An FH-MFSK receiver

7.12.2 FH / MFSK Receiver :

SPPU : Dec. 05, May 06, Dec. 07, Dec. 09, Dec. 14, May 15, Dec. 16

University Questions

Q. 1 Draw the block diagram of FH-SS system transmitter and receiver. Write the functional names inside the blocks and input output signals of each block. **(Dec. 05, 6 Marks)**

Q. 2 Draw the block diagram of FHSS transmitter and receiver. **(May 06, 4 Marks)**

Q. 3 With a neat block diagram explain working of frequency hopped transmitter and receiver combination. **(Dec. 07, 8 Marks)**

Q. 4 With neat block schematic, explain working of FHSS transmitter and receiver. Comment on the commercial applications of FHSS. **(Dec. 09, 8 Marks)**

Q. 5 Draw and explain FHSS spread spectrum system with transmitter and receiver section. **(Dec. 14, 8 Marks)**

Q. 6 Draw and explain FHSS system with transmitter and receiver section. **(May 15, 8 Marks)**

Q. 7 Draw the block diagram of FH-SS systems transmitter and receiver. Write the functional names inside the blocks and input output signals of each block. **(Dec. 16, 6 Marks)**

- Due to the large bandwidth occupied by the FH/MFSK signal, the coherent detection of this signal is possible within each hop.
- This is because for coherent detection, the phase synchronization of the locally generated carrier with the transmitted carrier is essential.
- But the frequency synthesizers used in FH/MFSK receiver are unable to maintain this phase coherence over successive hops.
- Therefore most frequency hop spread spectrum systems use the non-coherent M-ary modulation schemes.
- The block diagram of an FH-MFSK receiver is as shown in Fig. 7.12.2.

Operation of FH / MFSK receiver :

- The received signal is applied to a mixer. The other input to the mixer comes from a digital frequency synthesizer.
- This digital synthesizer is driven by a PN code generator which is synchronized with the PN code generator at the transmitter and generates the same code sequence.

- Therefore the frequency hops produced at the synthesizer output will be identical to those at the synthesizer output at the transmitter.
- At the output of the multiplier we get the input signals, their sum and difference (as far as frequency is concerned).
- Out of these frequency components, the difference frequency component is selected by the bandpass filter that follows the multiplier.
- This difference signal is the MFSK signal. Thus the mixer removes the frequency hopping.
- The MFSK signal at the mixer output is then applied to a non-coherent MFSK demodulator. At the output of the MFSK detector we obtain the digital modulating signal b (t).
- The non-coherent M-ary FSK detector can be implemented by using a bank of M, non coherent matched filters.
- Each matched filter is matched to one of the tones of the MFSK signal. The largest output out of the M available outputs of filters is selected to obtain the digital modulating signal.

7.12.3 Chip Rate (R_c) of FH/MFSK System :

- Each distinct tone (frequency) of shortest duration is defined as a "chip" in the FH/MFSK system. This is completely different from the definition of chip in DS/BPSK system. The "chip rate" R_c for an FH/MFSK system is defined as :

$$\textbf{Chip rate, } R_c = \max(R_h, R_s) \quad ...(7.12.1)$$

where R_h = Hop rate and R_s = Symbol rate.

- Thus chip rate R_c is equal to either R_h or R_s whichever is higher for the slow hopping FH/MFSK signal, let the R_b represent the bit rate of the incoming signal, R_s be the symbol rate of the MFSK signal and R_h be the hop rate. Then the relation between them is given by,

$$\textbf{Chip rate } R_c = R_s = \frac{R_b}{K} \geq R_h \quad ...(7.12.2)$$

where K = $\log_2$ M or M = 2^k (M-ary FSK, K bit message).

- Equation (7.12.2) tells us that the chip rate is equal to the symbol rate R_s, because the symbol rate is higher than the hop rate for a slow hopping system.

$\therefore R_s \geq R_h$ and each symbol is made of K bits.

$$\therefore R_s = \frac{R_b}{K}.$$

7.12.4 Processing Gain PG :

SPPU : May 15

University Questions

Q. 1 Explain in brief : Processing Gain.

(May 15, 2 Marks)

- As already defined, the processing gain is given by :

$$PG = \frac{\text{BW (spread spectrum signal)}}{\text{BW (unspread signal)}} \quad ...(7.12.3)$$

This definition is same as that for DH-SS system.

7.12.5 Bandwidth of Spread Signal :

- Let the number of bits at the output of PN code generator be "n". Therefore the number of combinations will be 2^n . Hence the number of frequency hops will be equal to " 2^n ".

$$\therefore \text{ Number of frequency hops } = 2^n \quad ...(7.12.4)$$

- Let the symbol frequency be given by f_s. Therefore the bandwidth of the FH/MFSK signal is given by,

$$\text{BW(Spread spectrum signal)} = 2^n \times f_s \quad ...(7.12.5)$$

7.12.6 BW of Unspread Signal and Processing Gain :

The bandwidth of the unspread signal is equal to the symbol frequency f_s. Substituting the bandwidths into Equation (7.12.3) we get,

$$\text{Processing gain} = \frac{2^n f_s}{f_s} = 2^n \quad ...(7.12.6)$$

- We can increase the P.G. by increasing the number of bits "n" at the output of a PN code generator.

Ex. 7.12.1 : The FH/MFSK transmitter has the following parameters :

1. Number of bits per MFSK symbol : K = 2
2. Number of MFSK tones : M = $2^K = 2^2 = 4$
3. Length of PN sequence per hop : n = 3
4. Total number of frequency hops :
   $2^n = 2^3 = 8.$

Draw the FH/MFSK signal showing the variation in frequency with respect to time for one complete period of the PN sequence. Assume the PN sequence to be :

PN sequence	0	0	1	1	1	0	0	1	1	00	1	0	0	1

Therefore the period of one PN sequence is $2^4 - 1 = 15$ digits.

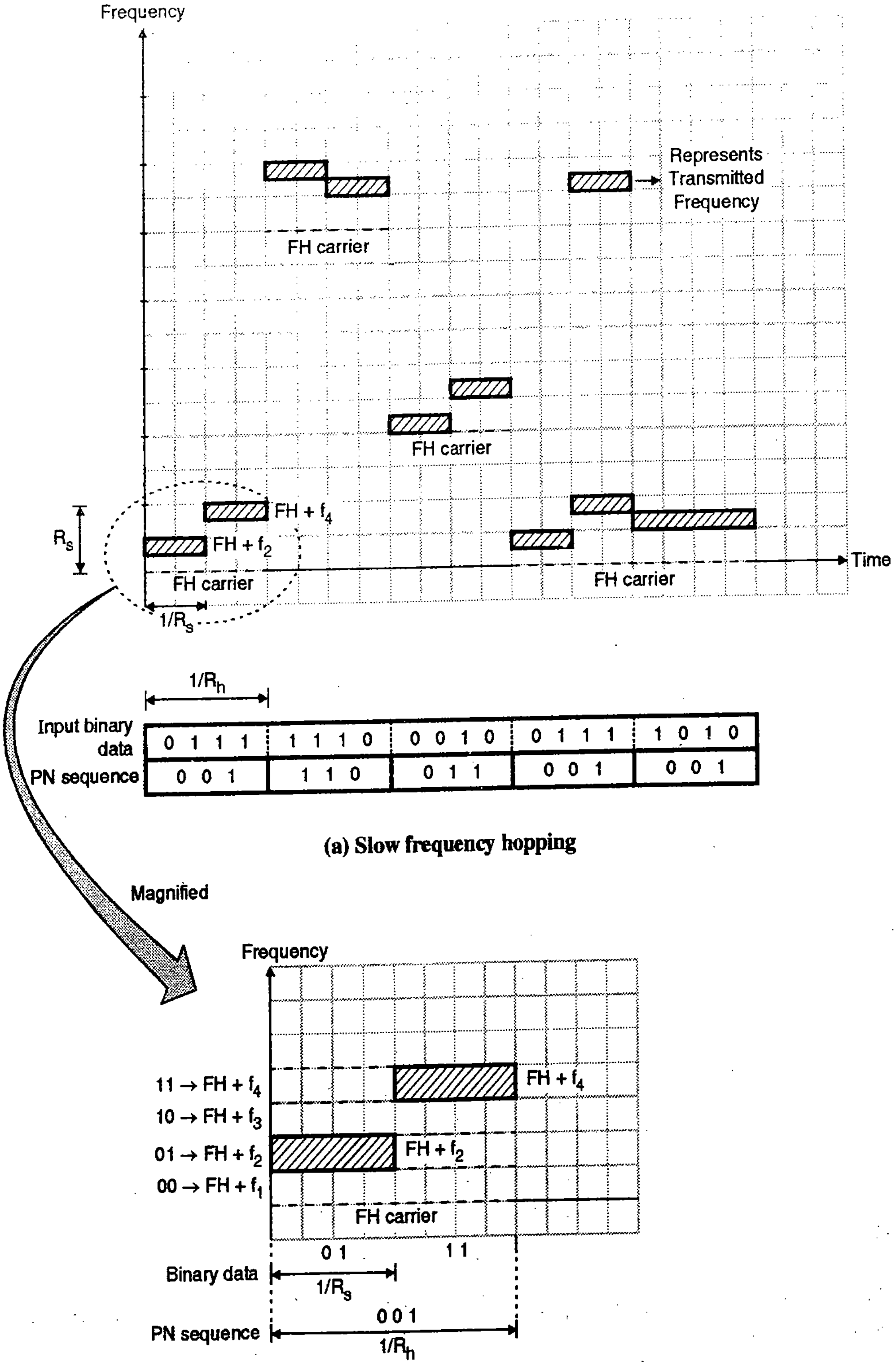
Soln. :

1. In this example as the number of bits per symbol is K = 2, the carrier is hopped to a new frequency after transmitting two symbols or four information bits.
2. As the length of PN sequence, n = 3 there are 2^n i.e. 8 different possible combinations. Therefore there should be 8 different hops at the output of the frequency synthesizer. But as shown in Fig. P. 7.12.1(a), only three FH carriers which are being used here because the combinations at the outputs of PN generator are 001, 110 and 011 only.

3. The number of symbols transmitted in one frequency hop is two. The total number of symbols is $M = 4$ i.e. 00, 01, 10, 11, corresponding to each symbol the MFSK modulator generates four frequencies i.e. f_1, f_2, f_3, f_4 respectively. Thus corresponding to each hop two of these frequencies will be transmitted as shown in Fig. P. 7.12.1(b). For example during the first hop period, the messages transmitted are :

Symbol	Transmitted frequency
0 1	$FH + f_2$
1 1	$FH + f_4$

where FH = carrier frequency for the hop which is controlled by the output of the PN code generator.



(a) Slow frequency hopping

(b) Magnified waveform for the period of first half

(E-486) **Fig. P. 7.12.1**

Ex. 7.12.2 : Consider a slow hop SS system with binary FSK that transmits two symbols per frequency hop and has a PN generator with k = 3 outputs. For a binary message sequence [01 10 11 01 10 00] draw the spectral output (output frequency versus data input). Determine the processing gain if $W_x = r_b = 3000$ and find the bit error probability in presence of white noise if $N_0 = 10^{-12}$ W/Hz, $S_R = 5.4 \times 10^{-8}$ W.

Dec. 05, 10 Marks

Soln. :

Slow hop SS system with BFSK,

2 symbols/frequency hop,

PN generator with k = 3 outputs.

Step 1 : To draw the spectral output :

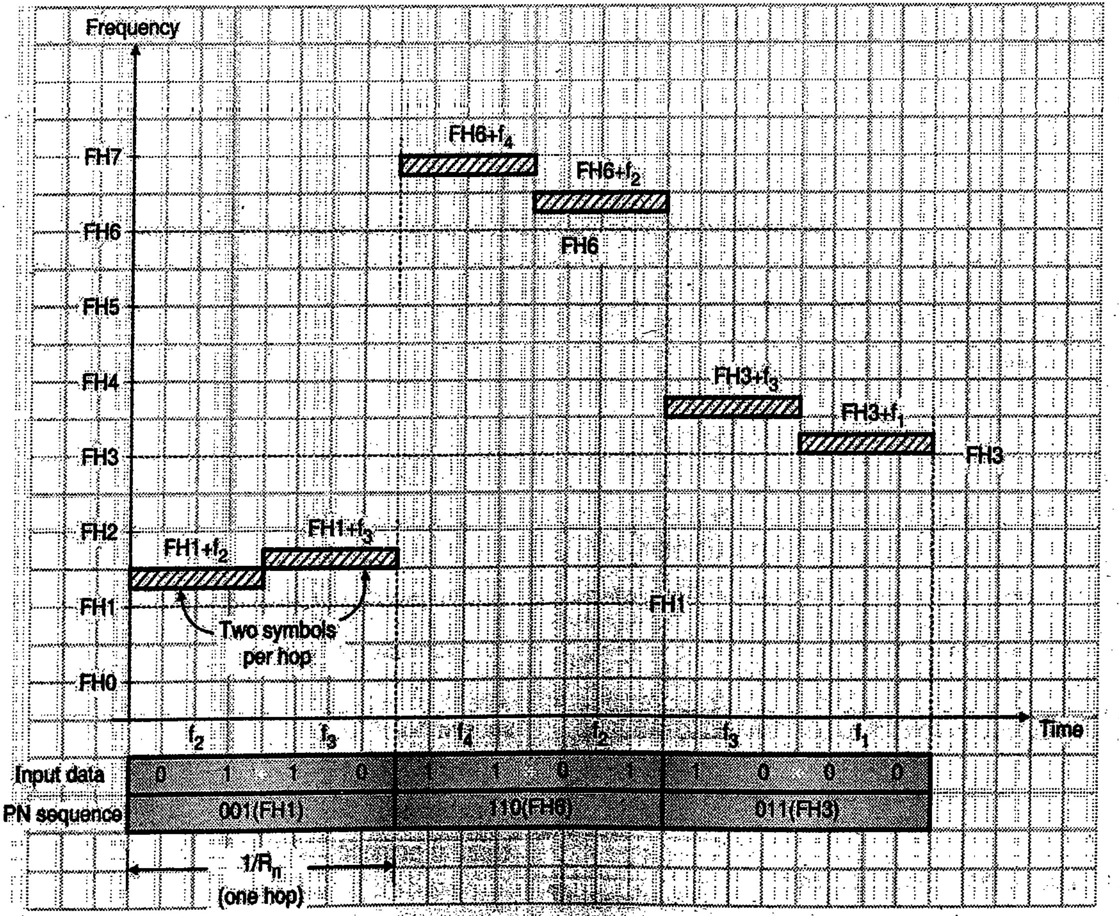
Assume the PN sequence to be

001 110 011 001 001 101 ...

- In this example as the number of bits per symbol is K = 2, the carrier is hopped to a new frequency after transmitting two symbols or four information bits.
- As the length of PN sequence, n = 3 there are 2^n i.e. 8 different possible combinations. Therefore there should be 8 different hops at the output of the frequency synthesizer.
- The number of symbols transmitted in one frequency hop is two. The total number of symbols is M = 4 i.e. 00, 01, 10, 11, corresponding to each symbol the MFSK modulator generates four frequencies i.e. f_1, f_2, f_3, f_4 respectively. Thus corresponding to each hop two of these frequencies will be transmitted as shown in Fig. P. 7.12.2. For example during the first hop period, the messages transmitted are :

Symbol	Transmitted frequency
0 1	FH 1 + f_2
1 0	FH 1 + f_3

Where FH 1 = Carrier frequency for the hop which is controlled by the output of the PN code generator. FH 1 corresponds to PN code of 001.



(E-1378) Fig. P. 7.12.2 : Spectral output of slow FH-SS systems

Step 2 : To find the processing gain (PG) :

$$PG = 2^{t}$$

$$\text{For this system } t = k = 3,$$

i.e. number of outputs of PN generator.

$$\therefore \quad PG = 2^{k} = 2^{3} = 8 \qquad \text{...Ans.}$$

Step 3 : To obtain the error probability (P_e) :

For FH-SS system

$$P_e = \frac{1}{2} e^{-\gamma_b/2} \qquad \text{...(1)}$$

$$\text{Where } \gamma_b = E_b/N_o$$

$$\therefore \quad P_e = \frac{1}{2} e^{-E_b/2N_o} \qquad \text{...(2)}$$

$$\text{But } E_b = P_s T_b, \text{ where } P_s = S_R = 5.4 \times 10^{-8}\text{ W}$$

$$\text{and } T_b = \frac{1}{\gamma_b} = \frac{1}{3000}$$

$$\therefore \quad E_b = \frac{5.4 \times 10^{-8}}{3000} = 1.8 \times 10^{-11}$$

$$\therefore \quad P_e = \frac{1}{2} e^{-1.8 \times 10^{-11}/2 \times 10^{-12}}$$

$$= 6.17 \times 10^{-5} \qquad \text{...Ans.}$$

Ex. 7.12.3 : A slow FH/MFSK system has the following parameters.

The number of bits / MFSK symbol = 4

The number of MFSK symbol per hop = 5

Calculate the processing gain of the system in decibels.

Dec. 02, 8 Marks, Dec. 07, 6 Marks, May 09, 4 Marks

Soln. :

- Let f_s denote the symbol frequency.
- The number of bits per symbol is 4. So the bandwidth of the unspreaded signal is equal to $f_s / 4$.
- The number of MFSK hops per symbol = 5. So the bandwidth of the spreaded signal is $5 f_s$.
- Hence the processing gain is given by,

$$P.G. = \frac{\text{BW of spreaded signal}}{\text{BW of unspreaded signal}} = \frac{5 f_s}{f_s / 4} = 20$$

Hence P. G. (dB) = $10 \log_{10}$ P. G.

$$= 10 \log_{10} 20 = 13 \text{ dB} \qquad \text{...Ans.}$$

Ex. 7.12.4 : Consider a slow hop spread spectrum system with binary FSK, two symbols per frequency hop, and a PN sequence generator with outputs with the binary message of 011011011000. The message is transmitted using the following PN sequence with k = 3 : {010, 110, 101, 100, 000, 101, 011, 001, 001, 111, 011, 001}, plot the output frequencies for the input message.

May 13, 8 Marks

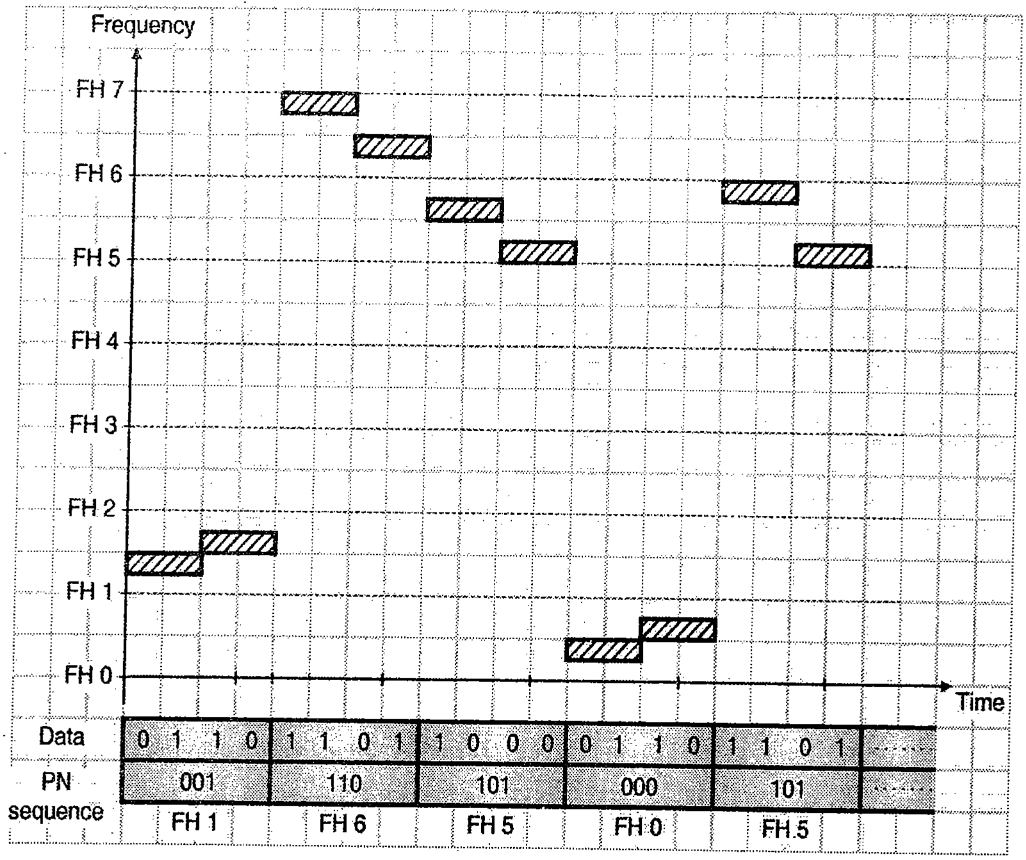
Soln. :

Given : Slow hop S.S. with BFSK, two symbols per frequency hop,

Data message : 011011011000, k = 3

- The plot of output frequencies versus data inputs is as shown in Fig. P. 7.12.4.
- The relation between data symbols, and PN sequence is as follows,

Data symbol	01 10	11 01	10 00	01 10	.....
PN sequence	001	110	101	000	.....



(E-1364) **Fig. P. 7.12.4**

7.13 Fast Frequency Hopping :

SPPU : May 08, May 15

University Questions

Q. 1 Explain the following frequency hop spread spectrum systems, with the help of relevant diagram.

1. Slow frequency hopping.
2. Fast frequency hopping. **(May 08, 8 Marks)**

Q. 2 Explain in brief : Fast Frequency Hopping. **(May 15, 2 Marks)**

- As explained earlier, the fast FH/MFSK system is different than the slow FH/MFSK system.
- Because in the fast FH/MFSK system, there are multiple hops for each M-ary symbol. Hence each hop is a "chip".

$\therefore$ Chip rate R_c = Rate of hopping R_h

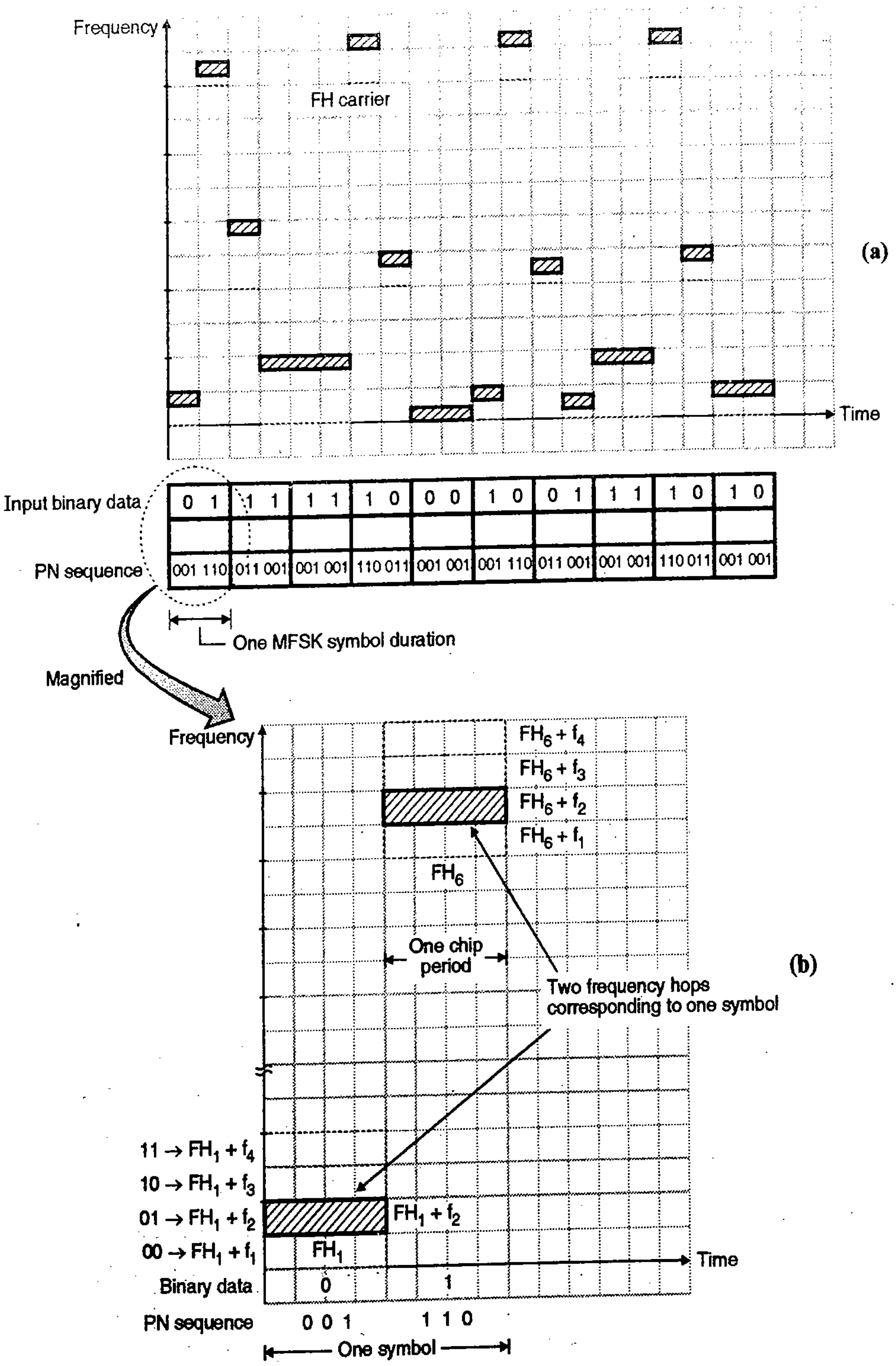
- The fast frequency hopping is used for defeating a smart jammer who tries to interfere the transmission.
- Before the jammer could understand the frequency band which is being used by the transmitter, the transmitted signal is hopped to a new carrier frequency.
- The principle of fast frequency hopping is illustrated in Fig. 7.13.1 (on next page).
- The data sequence used for the fast hopping is same as the one used for the slow hopping.
- The number of bits per MFSK symbol = K = 2. Therefore the number of MFSK tones = $2^K = 4$.
- The length of PN segment per hop i.e. n = 3. Therefore the total number of frequency hops = $2^3 = 8$.
- The PN sequence decides the hopping frequency (shown by dotted lines in Fig. 7.13.1(a). Two successive input binary bits 0 1 form the first symbol.
- During this symbol duration the PN sequence (3 digit) has two distinct values viz 001 and 110.
- Therefore one symbol duration corresponds to two frequency hops. As shown in Fig. 7.13.1(b) the frequency of the MFSK modulator for symbol 01 is f_2 and the outputs of the synthesizer corresponding to 001 and 110 outputs of the PN sequence generator are say FH_1 and FH_6.
- Therefore the transmitted frequencies are $(FH_1 + f_2)$ and $(FH_6 + f_2)$. The operation for the first symbol 01 is summarized below.

Summary of operation in the first symbol duration :

Symbol : 01	
Frequency of MFSK modulator = f_2	
Output of PN code generator	**Frequency of synthesizer (hop)**
001	FH_1
110	FH_6
Transmitted frequencies : $(f_2 + FH_1)$ and $(f_2 + FH_6)$	

7.13.1 Receiver used for Fast Hopping :

- For the recovery of the data, at the receiver, noncoherent detection is used.
- But the detection procedure is very much different from that used for a slow FH/MFSK system.
- In practice the following two procedures are considered :
  1. In this procedure, a separate decision is made on the K frequency-hop chips received. Then the estimation of the dehopped MFSK symbol is done based on the simple rule based on majority vote.
  2. In the second procedure, for each FH/MFSK symbol likelihood functions are computed, as functions of the total signal received over K chips and the largest one of them is selected.
- The receiver based on the second procedure minimizes the average probability of error. Hence practically it is preferred.



(E-487) **Fig. 7.13.1 : Waveforms of fast hopping system**

Ex. 7.13.1 : Represent variation of the frequency of a fast hop spread spectrum system with binary FSK, having following parameters :

Number of bits per MFSK symbol K = 2

Number of MFSK tones $M = 2^K = 4$

Length of PN segment per hop K = 3

Total number of frequency hops $2^K = 8$

For the binary message of 01111110001001111010

Generate the PN sequence for the message to be transmitted. The period of the PN sequence is $2^4 - 1 = 15$ with initial shift register content of 1100.

Dec. 13, May 15, 8 Marks, May 16, 9 Marks

Soln. :

Given : Fast hop spread spectrum with BFSK.

Number of bits/symbol, $K = 2$

Number of MFSK tones, $M = 2^K$

$= 4$

Length of PN segment per hop, $K = 3$

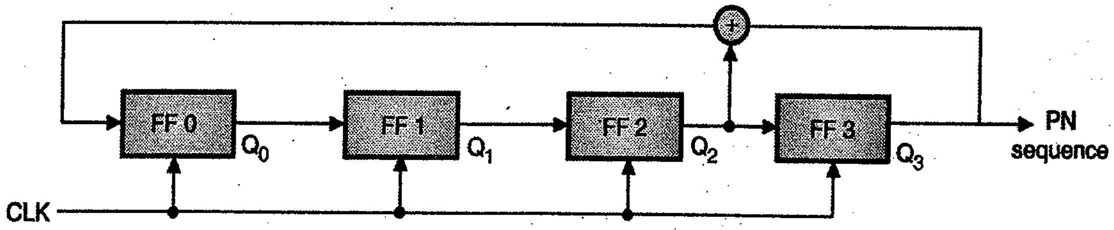
Number of frequency hops, $2^K = 8$

Binary message :

01 11 11 10 00 10 01 11 10 10

Step 1 : Generate the PN sequence :

The circuit diagram of the PN sequence generator is as shown in Fig. P. 7.13.1.



(E-1375) **Fig. P. 7.13.1 : PN sequence generator**

The initial state is given as,

$Q_3 Q_2 Q_1 Q_0$ =1 1 0 0

Table P. 7.13.1 summarizes the operation of PN generator.

(E-1376) Table P. 7.13.1 : PN sequence generator output

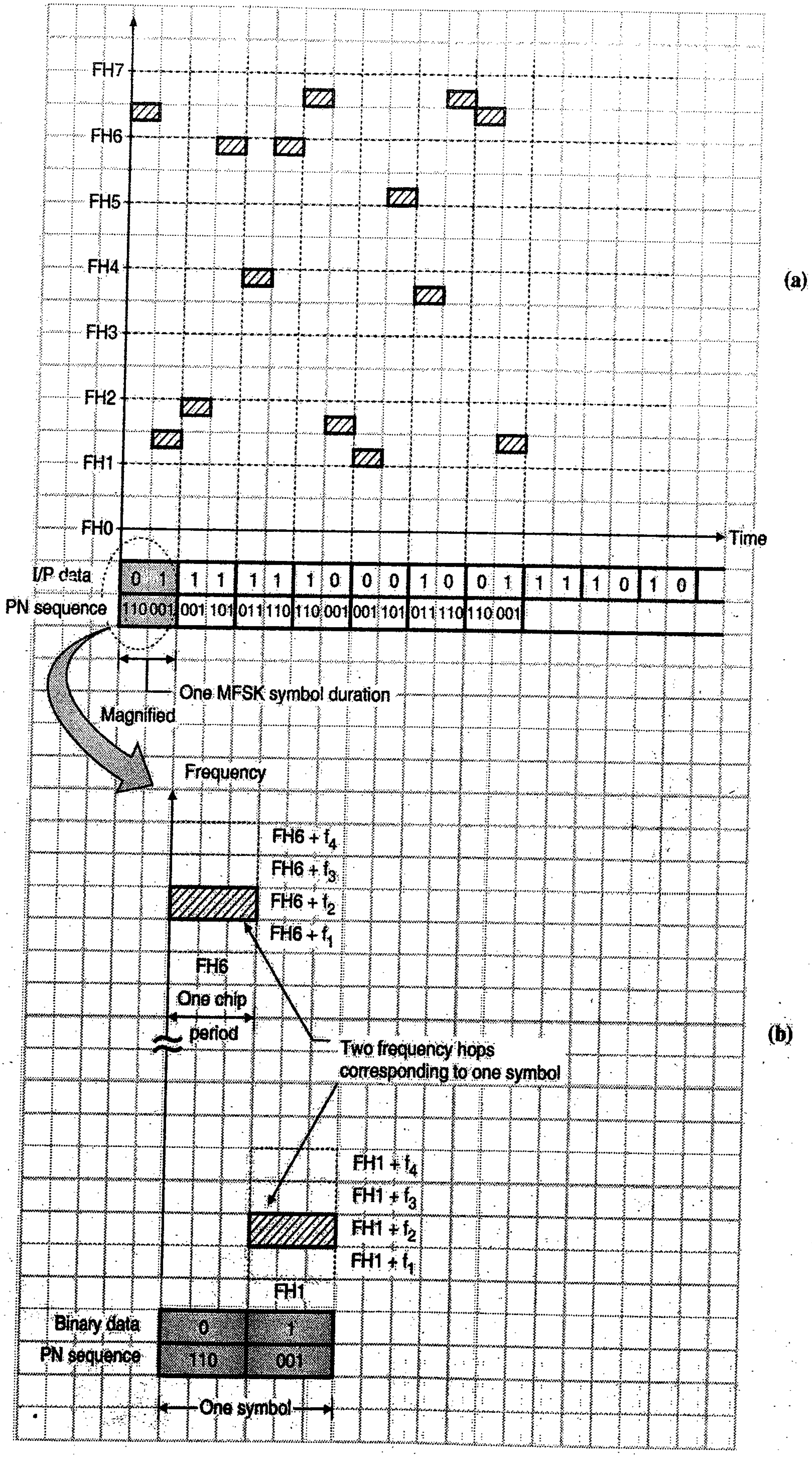
Clock pulse number	Shift register outputs				EXOR output $Q_3 + Q_2$	PN sequence Q_5
	Q_3	Q_2	Q_1	Q_0		
0	1	1	0	0	0	1
1	1	0	0	0	1	1
2	0	0	0	1	0	0
3	0	0	1	0	0	0
4	0	1	0	0	1	0
5	1	0	0	1	1	1
6	0	0	1	1	0	0
7	0	1	1	0	1	0
8	1	1	0	1	0	1
9	1	0	1	0	1	1
10	0	1	0	1	1	0
11	1	0	1	1	1	1
12	0	1	1	1	1	0
13	1	1	1	1	0	1
14	1	1	1	0	0	1
15	1	1	0	0	0	1

Thus the PN sequence generated is as follows :

110 001 001 101 011 110 ....

Step 2 : Represent variation in output frequency :

- The principle of fast frequency hopping is illustrated in Fig. P. 7.13.1(a).



(E-1377) **Fig. P. 7.13.1**

- The number of bits per MFSK symbol = K = 2. Therefore the number of MFSK tones = $2^K = 4$.
- The length of PN segment per hop i.e. n = 3. Therefore the total number of frequency hops = $2^3 = 8$.
- The PN sequence decides the hopping frequency (shown by dotted lines in Fig. P. 7.13.1(a). Two successive input binary bits 0 1 form the first symbol.
- During this symbol duration the PN sequence (3 digit) has two distinct values viz 110 and 001.
- Therefore one symbol duration corresponds to two frequency hops. As shown in Fig. P. 7.13.1(b) the frequency of the MFSK modulator for symbol 01 is f_2 and the outputs of the synthesizer corresponding to 110 and 001 outputs of the PN sequence generator are say FH_6 and FH_1 respectively.

Ex. 7.13.2 : The signal has the following parameters :

Number of bits per MFSK symbol k = 2

Number of MFSK tone M = $2^k = 4$

Length of PN segment per hop k = 3

Total number of frequency hopes $2^k = 8$

Sketch the output transmitted frequency of fast FH/MFSK signal.

Dec. 12, 8 Marks

Soln. : Similar to Ex. 7.13.1.

7.14 Applications of FHSS :

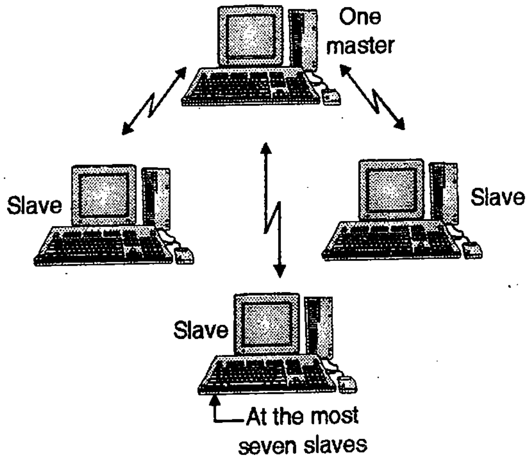
- FHSS is being used in a number of applications. The two most important of them are as follows :
  1. Wireless local area networks (WLAN) standard for Wi-Fi.
  2. Wireless personal area network (WPAN) standard of Bluetooth.
- IEEE 802.11 was the first Wi-Fi standard which was introduced in 1997. Then the improved standards i.e. 802.11a and 802.11b were introduced in 1999. They removed the FHSS option that was originally present in 802.11.
- Then the old standard 802.11 was revised and presented as a new product called **Bluetooth.**
- Bluetooth is the name given to a new technology which uses short-range radio links, that can replace the cable(s) connecting portable and/or fixed electronic devices.
- It is advantageous that it will allow the replacement of the many propriety cables that connect one device to another with one universal radio link.
- Its key features are robustness, low complexity, low power and low cost.
- Bluetooth has been designed to operate in noisy frequency environments. Therefore it uses a fast acknowledgement and **frequency-hopping** scheme to make the link robust.
- Bluetooth radio modules operate in the unlicensed ISM band at 2.4 GHz, and avoid interference from other signals by hopping to a new frequency after transmitting or receiving a packet.
- Compared with other systems in the same frequency band, the Bluetooth radio hops faster and uses shorter wavelengths.
- Thus Bluetooth is a wireless LAN technology designed to connect devices of different functions such as telephones, computers, printers, cameras, etc.
- A Bluetooth LAN is an Ad hoc network. It is possible to connect the Bluetooth LAN to the Internet.
- This technology is implemented using the IEEE 802.15 standard.

7.14.1 Architecture :

- Bluetooth defines two types of networks :
  1. Piconets and 2. Scatternets.

7.14.2 Piconets :

- The first type of a Bluetooth network is called as a **piconet** or a **small net.** It can have at the most eight stations. One of them is called as a **master** and all others are called as **slaves.**
- All the slave stations are synchronised in all aspects with the master.
- A piconet can have only one master station. Fig. 7.14.1 shows a piconet. A master can also be called as a primary station and slaves are secondary station.
- The communication between a master and slaves can be one-to-one or one-to-many. Note that the communication takes place between the master and slaves but no direct communication takes place between the slaves.



(G-388)**Fig. 7.14.1 : A piconet**

7.14.3 Scatternet :

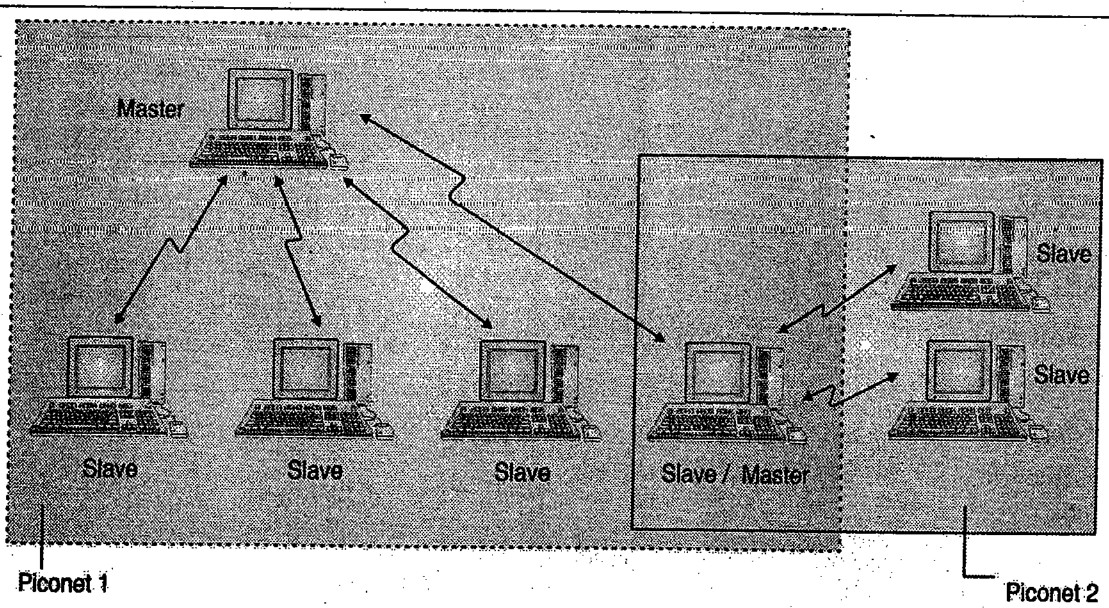
- A scatternet is obtained by combining piconets as shown in Fig. 7.14.2.
- Fig. 7.14.2 shows a scatternet consisting of two piconets. A slave in the first piconet can act as a master in the second piconet.
- It will receive the messages from the master in the first piconet by acting as a slave and then delivers the message to the slaves in the second piconet as shown in Fig. 7.14.2. So the same device acts as a slave in the first piconet and as master in the second piconet.

7.14.4 Bluetooth Devices :

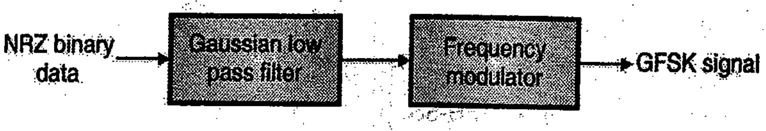
- Every Bluetooth device consists of a built in short range **radio transmitter.** The current data rate is 1 Mbps and the bandwidth is 2.4 GHz.
- So an interface between the IEEE 802.11 wireless LAN and Bluetooth LAN is possible.
- The Bluetooth specification standard defines a short-range (10-meter) radio link.
- The devices carrying Bluetooth enabled chips can easily transfer data at a rate of about 1 Mbps (Megabits per second) within 10 meters (33 feet) of range through walls, clothing and luggage bags.
- The interaction between devices occurs by itself without direct human intervention whenever they are within each other's range. In this process, the software technology embedded in the Bluetooth transceiver chip triggers an automatic connection to deliver and accept the data flow.
- Since Bluetooth is of short range with limited speed and low-power technology. It is less attractive to corporate wireless local area networks that are generally powered with the 802.11 wireless LAN technologies.
- Each Bluetooth-enabled device contains a 1.5-inch square transceiver chip operating in the ISM (industrial, scientific, and medical) radio frequency band of 2.40 GHz to 2.48 GHz.
- This frequency is generally available worldwide for free without any licensing restrictions. The ISM band is divided into 79 channels with each carrying a bandwidth of 1 MHz.

7.14.5 Modulation :

- The modulation of the basic rate Bluetooth signal is shown in Fig. 7.14.3.
- The binary signal is converted into GFSK signal before transmission. GSFK is a continuous phase FSK.
- Some of the important specifications of 802.11 FHSS and Bluetooth are as given below.



(G-389)**Fig. 7.14.2 : Scatternet**



(E-1357) **Fig. 7.14.3 : FHSS modulation in Bluetooth**

Table 7.14.1 : Specifications of 802.11 FHSS and Bluetooth

Sr. No.	Specification	802.11 FHSS	Bluetooth
1.	Frequency band	ISM 2.4-2.4835 GHz	ISM 2.4 -2.4835 GHz
2.	B.W. per channel	1 MHz	1 MHz
3.	Number of overlapping channels	79	79
4.	Minimum hopping distance	6	6
5.	Modulation	GFSK-2, GFSK-4	GFSK-2
6.	Hopping rate	2.5-160 Hz	1600 Hz
7.	Data rate	1 M bit/s and 2 M bit/s	723.1 kbits/s

7.15 Advantages and Disadvantages of FH-SS System :

SPPU : Dec. 10

University Questions

Q. 1 Explain Frequency Hop Spread Spectrum System (FHSS). How is FHSS advantages over DSSS ? **(Dec. 10, 6 Marks)**

Advantages :

1. The synchronization is not greatly dependent on the distance.
2. The serial search system with FH-SS needs shorter time for acquisition.
3. The processing gain PG is higher than that of DS-SS system.

Disadvantages :

1. The bandwidth of FH-SS system is too large (in GHz).
2. Complex and expensive digital frequency synthesizers are required to be used.

7.16 Comparisons :

7.16.1 Comparison of Slow and Fast Frequency Hopping :

Table 7.16.1 shows the comparison of FH-SS methods.

Table 7.16.1

Sr. No.	Slow frequency hopping	Fast frequency hopping
1.	More than one symbols are transmitted per frequency hop.	More than one frequency hops are required to transmit one symbol.
2.	Chip rate is equal to the symbol rate.	Chip rate is equal to the hop rate.
3.	Symbol rate is higher than hop rate.	Hop rate is higher than symbol rate.
4.	Same carrier frequency is used to transmit one or more symbols.	One symbol is transmitted over multiple carriers in different hops.
5.	A jammer can detect this signal if the carrier frequency in one hop is known.	A jammer can't detect this signal because one symbol is transmitted using more than one carrier frequencies.

7.16.2 Comparison of DSSS and FHSS :

SPPU : Dec. 15, May 16

University Questions

Q. 1 Compare DSSS with FHSS system. **(Dec. 15, 8 Marks)**

Q. 2 Compare DSSS with FHSS. **(May 16, 9 Marks)**

Sr. No.	Parameter	Direct sequence spread spectrum	Frequency hopping spread spectrum
1.	Definition	PN sequence of large bandwidth is multiplied with narrow band data signal.	Data bits are transmitted in different frequency slots which are changed by PN sequence.
2.	Chip rate	It is fixed $R_c = \frac{1}{T_c}$	$R_c = \max (R_h , R_s)$
3.	Modulation technique	BPSK	M-ary FSK
4.	Processing gain	$PG = \frac{T_b}{T_c} = N$	$PG = 2^t$
5.	Error probability	$P_e = \frac{1}{2} \text{erfc} \sqrt{\frac{E_b}{J T_c}}$	$P_e = \frac{1}{2} e^{-rb} R_c / 2$
6.	Acquisition time	Long	Short

Sr. No.	Parameter	Direct sequence spread spectrum	Frequency hopping spread spectrum
7.	Effect of distance	This system is distance relative	Effect of distance is less

7.17 Summary of Applications of Spread Spectrum Technique :

SPPU : May 11

University Questions

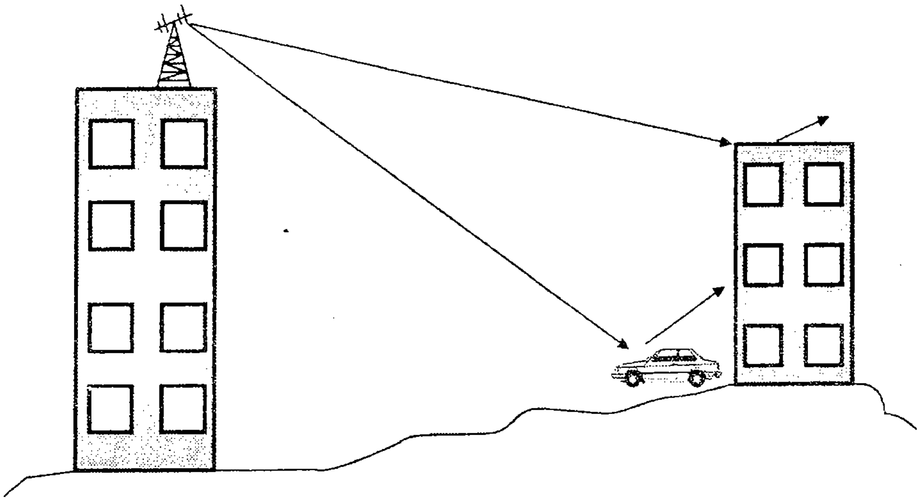
Q. 1 Explain DSSS in detail and state the applications of the same. **(May 11, 6 Marks)**

Some of the important applications of the spread spectrum technique are as follows :

1. It reduces the effect of intentional interference (jamming) : This can be used in the military applications as well as other commercial applications to avoid the intentional interference.
2. For reducing the unintentional interference as explained in section 7.3.
3. For suppressing the interference due to multipath reception as explained in section 7.3.
4. In the low probability of intercept (LPI) application as explained in section 7.3.
5. Due to large bandwidth of a spread spectrum signal, the "fading" (specially the frequency selective fading) does not affect the entire spectrum. Infact only a small portion of the complete spectrum is affected in the fading process. Thus fading does not affect the SS signal much. Therefore the spread-spectrum signals are used in the mobile communication.
6. Due to the use of pseudo-noise (PN) code sequence, the spread spectrum signal can be recognised only by the authorised receiver. All other receivers consider this signal as noise. Thus the SS communication is a "secured" communication.
7. The spread spectrum signals are used in the RADAR and other navigation systems for ranging or distance measurement. We know that wideband signals are "time limited". Therefore they can be used for the measurement of time delays very precisely.
8. The most important application of the spread spectrum technique is code division multiple access (CDMA). This is a multiuser communication system in which many users can access the available channel bandwidth simultaneously. To avoid them interfacing with each other, each user is allotted a particular code sequence, using the PN code generators. These code sequences are used by the receivers to detect their signals in presence of other unwanted signals. CDMA is used for the cellular mobile telephone systems.

7.18 Multiuser Radio Communication :

- Earlier the communication channels have been idealized which has a limited bandwidth and corrupted by additive white Gaussian noise. (AWGN).
- Such channels are physically present and they are well represent by the model mentioned above..
- A example of such a channel is a satellite communication channel. It is a multiuser channel.
- The satellite communication system uses the communication satellites which are placed in the geostationary orbit and it uses the line of sight communication for its operation.
- It uses the line of slight propagation for the operation of its uplink from an earth terminal to the transponder at the satellite and for the down link from transponder to another earth terminal.
- Another multiuser communication system is called as wireless communications which covers the mobile communication.
- The radio propagation channel which characterizes the wireless communication is different from the idealized AWGN model due to the multipath transmission involved.
- The multipath signal transmission is a non-gaussian type of signal dependent phenomenon which takes place due to reflections of transmitted signal from the fixed and moving objects as shown in Fig. 7.18.1.
- In the multiple access technique, different users are allowed to access a common channel.



(E-893) **Fig. 7.18.1 : Multipath signal transmission**

7.19 Multiple Access Techniques :

SPPU : Dec. 06, May 08, Dec. 08, May 09

University Questions

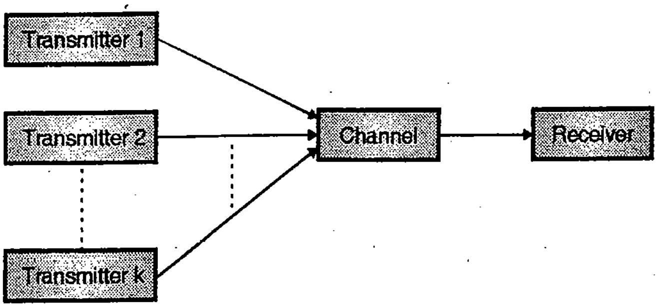
Q. 1 What is meant by Multiple Access ? Explain W-CDMA in detail. **(Dec. 06, 8 Marks)**

Q. 2 What is the multiple accessing techniques ? Explain the CDMA technique in detail. **(May 08, 8 Marks)**

Q. 3 What are the different multiple access techniques ? Explain in detail. **(Dec. 08, 6 Marks)**

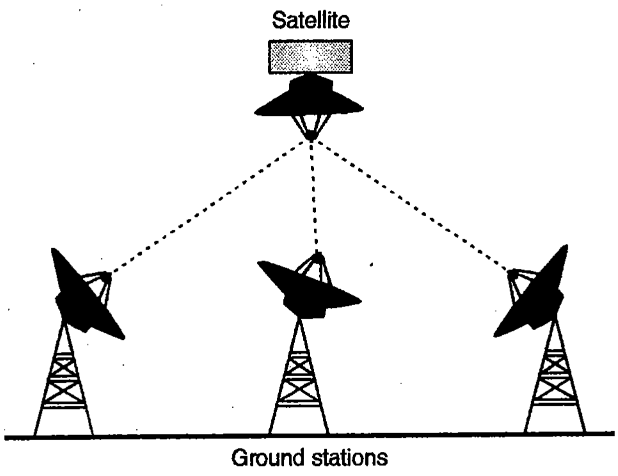
Q. 4 What is difference between multiplexing and multiple accesses technique ? What is multiple accesses technique ? Compare it with help of relevant diagram. **(Dec. 08, May 09, 8 Marks)**

- Till now we have discussed the communication systems which use only a single transmitter and a receiver. But in this section we will focus our attention on multiple users and multiple receiver communication links.
- Different ways in which multiple users can access a common channel for transmitting information are discussed in this section.

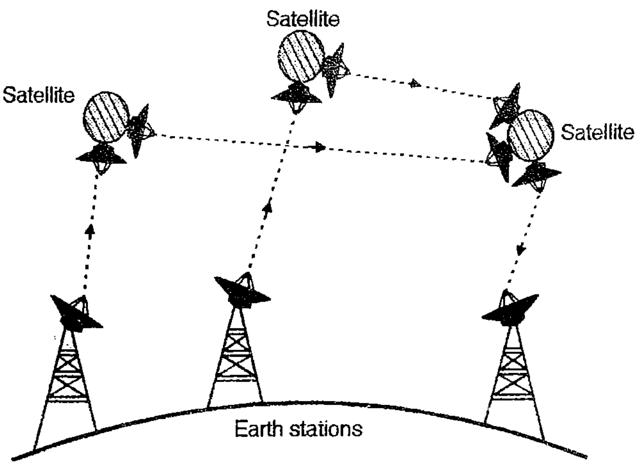


(E-894) **Fig. 7.19.1 : A multiple access system**

- The multiple access methods are used in the satellite networks, cellular and mobile communication networks and underwater acoustic networks.
- One type of multiple access system in which a large number of users sharing the same communication channel is shown in Fig. 7.19.1.
- The common channel can be the uplink in a satellite communication system or it can be a cable or it can be a frequency band.
- For example in a mobile communication system, the users of the network will be the transmitters and the receiver is in the base station.
- The second type of multiuser communication system is as shown in Fig. 7.19.2.
- In the second type multiuser system of Fig. 7.19.2 a single transmitter (satellite) sends information to multiple receivers. The other examples of such system are the radio and TV broadcast systems.



(E-895) **Fig. 7.19.2 : Second type of multiuser system**



(E-896) **Fig. 7.19.3 : A store and forward communication network with satellite relays**

- The multiple access and broadcast networks are the most common multiuser communication system. The third type of multiuser system is a store-and-forward network shown in Fig. 7.19.3.
- In this system the communication takes place in all the possible directions.
- The fourth type of system is shown in Fig. 7.19.4.

7.19.1 Difference between Multiple Access and Multiplexing :

SPPU : Dec. 08, May 09, Dec. 11

University Questions

Q. 1 What is difference between multiplexing and multiple accesses technique ? What is multiple accesses technique ? Compare it with help of relevant diagram. **(Dec. 08, May 09, 8 Marks)**

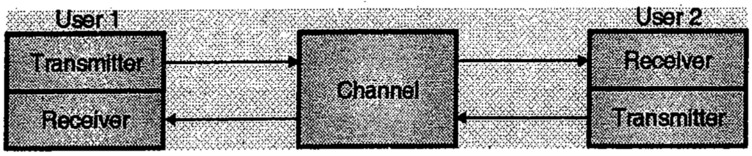
Q. 2 What is the difference between multiplexing and multiple access techniques ? Compare TDMA, FDMA and CDMA. **(Dec. 11, 8 Marks)**

There are very small differences between the multiple access and multiplexing. The differences are as follows :

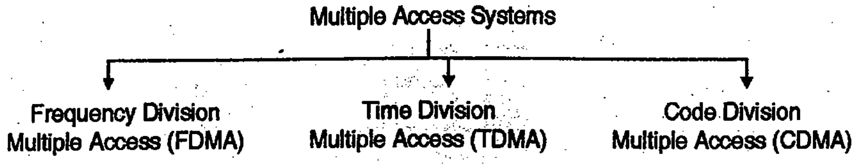
1. In multiple access the users will share the remote communication channels such as a satellite or radio channel. But in multiplexing the channel such as a telephone channel is shared by users confined to a local site.
2. In a multiplexed system, the requirements of users are generally fixed. But in a multiple access system the requirements of a user change dynamically with time. So the dynamic channel allocation is required to be provided.

7.19.2 Types of Multiple Access Systems :

- There are several methods in which the multiple users can send information through the communication channel to the receiver.
- There are four types of multiple access systems as shown in Fig. 7.19.5.



(E-897) **Fig. 7.19.4 : A two way communication channel**



(E-898) **Fig. 7.19.5 : Types of the multiple access systems**

7.20 Frequency Division Multiple Access (FDMA) : SPPU : Dec. 12

University Questions

Q. 1 Explain different types of multiple access techniques with the help of suitable diagram. **(Dec. 12, 8 Marks)**

- One of the simplest multiple access methods is the frequency division multiple access (FDMA). In this method, the channel bandwidth is subdivided into a number of subchannels as shown in Fig. 7.20.1.
- There are k non-overlapping subchannels, as shown in Fig. 7.20.1.
- This method is therefore called as frequency division multiple access and it is commonly used for the voice and data communication.

7.20.1 Features of FDMA :

Some of the important features of the FDMA are as follows :

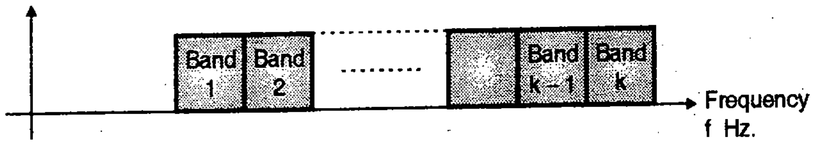
1. The overall channel bandwidth is being shared by the multiple users. Therefore a number of users can transmit their information simultaneously.
2. The adjacent frequency bands in the FDMA spectrum are likely to interfere with each other. Therefore it is necessary to include the guard bands between the adjacent frequency bands as shown in Fig. 7.20.2. The guard bands are also required because practically it is impossible to achieve the ideal filtering to separate the different users.
3. No code words and synchronization is not required.
4. Power efficiency is reduced.
5. FDMA is an old and proven system and is used for the analog signals.

7.20.2 Advantages of FDMA :

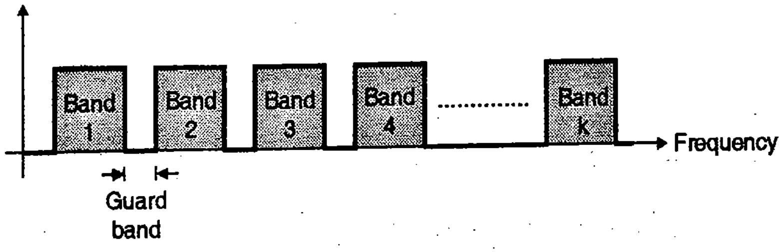
- All the stations can operate continuously all 24 hours without having to wait for their turn to come.
- The power required for transmission depends on the number of channels being transmitted.
- The signal to noise ratio is improved due to the use of FM.
- No synchronization is necessary.

7.20.3 Disadvantages of FDMA :

- Each channel or earth station can use only a part of the total satellite bandwidth.
- In spite of guard bands being provided, there is some adjacent channel interference present.
- As FM is used, it requires larger bandwidth, hence less number of channels will be accommodated in the bandwidth of a satellite.
- Due to the nonlinearity of companders, the intermodulation products are generated.



(E-899) Fig. 7.20.1 : Subdivision of channel into a number of subchannels



(E-1433) Fig. 7.20.2 : The guard bands included to avoid the interference

7.21 Time Division Multiple Access (TDMA) :

SPPU : Dec. 12

University Questions

Q. 1 Explain different types of multiple access techniques with the help of suitable diagram.

(Dec. 12, 8 Marks)

- Another method of creating multiple subchannels for multiple access is by subdividing the time duration T_f called as the frame duration, into say k non-overlapping subintervals each of duration T_f / k.
- After that each user who want to transmit information is assigned a particular slot, within each frame. This method is known as time division multiple access TDMA.

7.21.1 Features of TDMA :

SPPU : Dec. 12

University Questions

Q. 1 Explain different types of multiple access techniques with the help of suitable diagram.

(Dec. 12, 8 Marks)

- TDMA is used for the transmission of data and digital voice signals.
- It is necessary to include "guard times" between the adjacent channels as shown in Fig. 7.21.1.

Synchronization is necessary in TDMA.

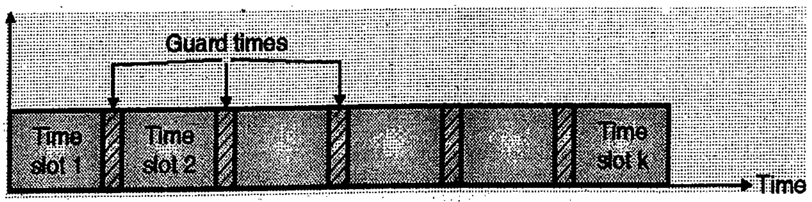
- Power efficiency of TDMA is better than that of the FDMA.
- TDMA is a method of time division multiplexing the digitally modulated carriers between various earth stations in a satellite network through a common statellite transponder.
- Each earth station transmits a **short burst** of digitally modulated carrier during the time slot assigned to it in the TDMA frame. Such a time slot is called as **epoch.**
- The burst of each earth station is synchronised so that at any instant of time, only one earth station's carrier is present in the transponder.
- The transponder receives this carrier, amplifies it and relays it back to all the earth stations. Thus each earth station receives the bursts from all other stations.

7.21.2 Advantages of TDMA :

1. At any instant of time, the carrier from only one station is present at the transponder. This reduces the intermodulation distortion.
2. TDMA is suitable for transmission of digital information.
3. It is possible to store the digital information, change the rate etc. in TDMA.

7.21.3 Disadvantages :

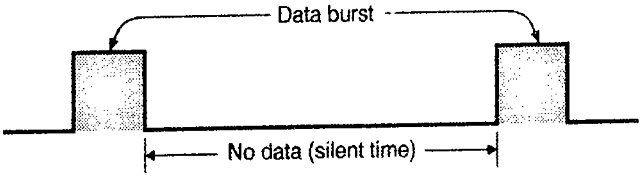
1. Precise synchronization is required.
2. Bit and frame timing must be achieved and maintained.



(E-500) **Fig. 7.21.1 : Guard times included in TDMA**

7.21.4 Problems with FDMA and TDMA :

- The problem with the FDMA and TDMA system is that, the channel is basically partitioned into independent single user subchannels.
- That means each subchannel in the FDMA is allotted to a single user and each time slot in TDMA has been allotted to a separate single user.
- The FDMA and TDMA systems however prove to be inefficient when the data from the users is bursty in nature as shown in Fig. 7.21.2.
- This type of data has low value of duty cycle. i.e. the time for which data is being transmitted is much shorter than the silent time.
- Under such circumstances where the transmission from different users is bursty and low duty cycle, the FDMA and TDMA system will be inefficient.
- This is because a large percentage of the available time or frequency slots do not convey any information.
- Such a type of data is observed in computer communication networks and to some extent, in the mobile cellular communication systems carrying digitized voice.



(E-501) **Fig. 7.21.2 : Bursty data signal with low duty cycle**

7.22 Code Division Multiple Access (CDMA) :

SPPU : Dec. 07, May 08, Dec. 09, Dec. 10, Dec. 12

University Questions

Q. 1 Explain CDMA in detail. **(Dec. 07, 6 Marks)**

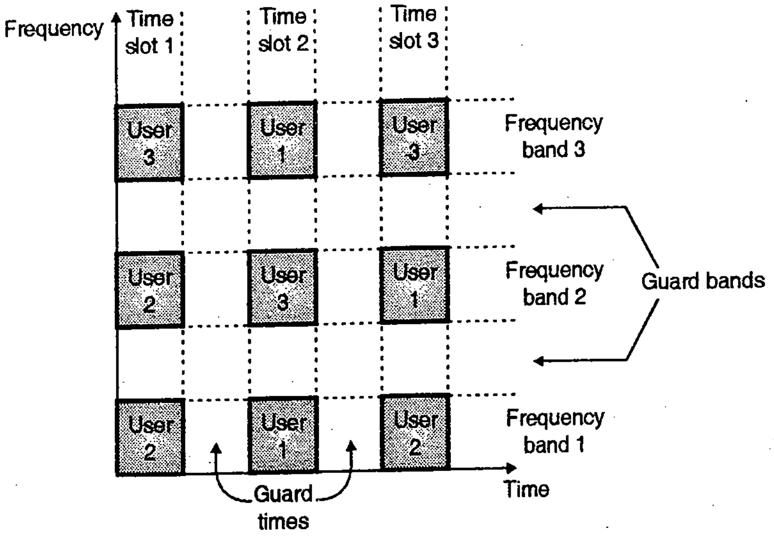
Q. 2 What is the multiple accessing techniques ? Explain the CDMA technique in detail. **(May 08, 8 Marks)**

Q. 3 Explain in detail operation of CDMA technique and compare performance parameters of FDMA, TDMA and CDMA. **(Dec. 09, 8 Marks)**

Q. 4 What is CDMA ? State its advantages and disadvantages. **(Dec. 10, 6 Marks)**

Q. 5 Explain different types of multiple access techniques with the help of suitable diagram. **(Dec. 12, 8 Marks)**

- An alternative to FDMA and TDMA is an another system called code division multiple access (CDMA). The most important feature of CDMA is as follows :
- In CDMA more than one user is allowed to share a channel or subchannel with the help of direct-sequence spread spectrum (DS-SS) signals.
- In CDMA each user is given a unique code sequence or signature sequence. This sequence allows the user to spread the information signal across the assigned frequency band.
- At the receiver the signal is recovered by using the same code sequence. At the receiver, the signals received from various users are separated by checking the cross-correlation of the received signal with each possible user signature sequence.
- In CDMA the users access the channel in a random manner. Hence the signals transmitted by multiple users will completely overlap both in time and in frequency.
- The CDMA signals are spread in frequency. Therefore the demodulation and separation of these signals at the receiver can be achieved by using the pseudorandom code sequence. CDMA is sometimes also called as Spread Spectrum Multiple Access (SSMA).



(E-907) **Fig. 7.22.1 : Structure of CDMA showing the guard bands and the guard times**

- In CDMA as the bandwidth as well as time of the channel is being shared by the users, it is necessary to introduce the guard times and guard bands as shown in Fig. 7.22.1.
- CDMA does not need any synchronization, but the code sequences or signature waveforms are required to be used.
- In CDMA one channel carries all the transmission simultaneously.

7.22.1 Analogy :

- In CDMA transmission takes place with different codes. It is analogus to a room full of people who talk different languages but only one pair knows English, another pair knows French, third pair knows Hindi, etc.
- So communication will take place only between the persons who know the same language.

7.22.2 Idea :

- Assume that there are four stations 1, 2, 3 and 4. Let the data from these stations be denoted by D_1, D_2, D_3 and D_4 respectively. Let the code assigned to this data be C_1, C_2, C_3 and C_4 respectively.
- The important properties of codes are as follows :

  1. These codes are orthogonal. That means if we multiply two different codes we get a 0.

  2. If any code is multiplied by itself we get 4 (i.e. the number of stations).
- Station - 1 performs multiplication $D_1 \cdot C_1$, station - 2 performs multiplication $D_2 \cdot C_2$ etc. as shown in Fig. 7.22.2(a).
- All the product terms are then added and the total signal is sent over the common channel.
- Each station receives the sum signal. If stations 2 and 3 are talking then station 3 will multiply the sum signal by the code of station - 2 i.e. C_2.
- At station 3 the multiplication takes place as follows :

$$\begin{aligned} \text{Sum signal} \times C_2 &= [C_1 D_1 + C_2 D_2 + C_3 D_3 + C_4 D_4] \\ &\quad \times C_2 \\ &= C_1 C_2 D_1 + C_2 C_2 D_2 + C_2 C_3 D_3 \\ &\quad + C_2 C_4 D_4 \\ &= 0 + 4 D_2 + 0 + 0 \end{aligned}$$

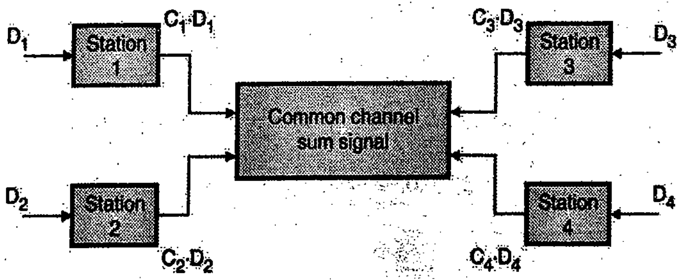
Station - 3 divides the result by 4 and gets D_2 i.e. data from station - 2.

7.22.3 Chips :

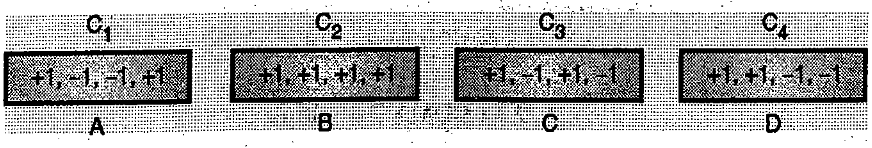
- In CDMA each station is assigned a code. It is a sequence of numbers called as a **chip.**
- Let there be four stations A, B, C and D and let the codes assigned to them be as shown in Fig. 7.22.2(b).
- These codes are called **orthogonal sequences** and have the following properties :
  1. Each sequence is made of N elements.
  2. If we multiply the sequence by a number then each element is that sequence will be multiplied by that number.

     e.g. $C_1 \times 3 = 3\,[+1, -1, -1, +1]$

     $= [+3, -3, -3, +3]$
  3. If two same sequences are multiplied with each other we get 4 as a result.
  4. If two different sequences are multiplied we get a 0.



(E-908) Fig. 7.22.2(a) : Idea of CDMA



(E-909) Fig. 7.22.2(b) : Chip sequence

7.22.4 Data Representation :

The encoding rules are as follows :

1. A bit 0 is encoded as – 1.
2. A bit 1 is encoded as + 1.
3. If a station is silent and no signal is to be transmitted, then this is encoded as 0.

7.22.5 CDMA Encoding :

- Let A, B and C be the three stations in a CDMA system. Let the unique codes assigned to them be as follows,

Table 7.22.1 : Codes assigned to different stations

Station	Code
A	+ 1, – 1, – 1, + 1
B	+ 1, + 1, + 1, + 1
C	+1, – 1, + 1, – 1

- Refer Fig. 7.22.3 which shows a CDMA multiplexer.
- We assume that station A wants to transmits a 0 bit, station B wants to remain silent and station C wants to transmit a 1.

Operation :

- The multiplexer receives one encoded number from each station (– 1, 0, + 1) as shown in Fig. 7.22.3.
- Each bit is multiplied with the code number of the corresponding station.
- The outputs of multipliers are added to obtain the multiplexed CDMA signal.
- The multiplexed CDMA signal is then transmitted through the common link.

7.22.6 CDMA Decoder :

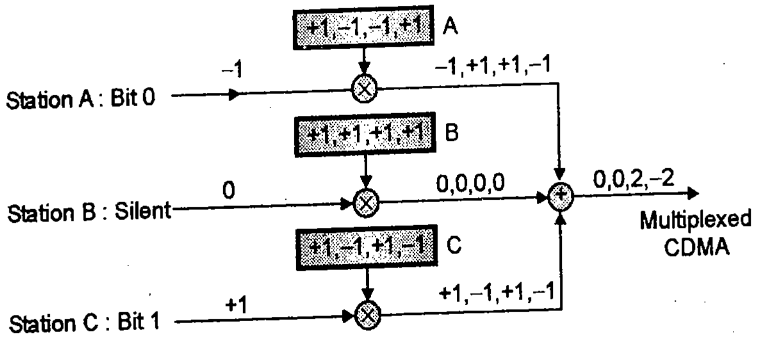
- Fig. 7.22.4 shows the block schematic of a CDMA demultiplexer.

Operation :

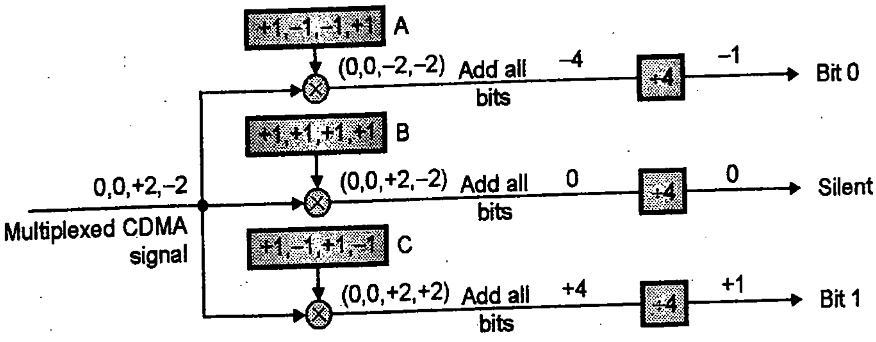
- The Multiplexed CDMA signal is applied to all the multipliers.
- It is multiplied with the code numbers assigned to the stations A, B and C.
- The bits available at the output of each multiplier are added together. This addition will always be either – 4, 0 or + 4.
- The result of addition at the output of each multiplier is divided by 4 to obtain the originally transmitted bits.

Orthogonal sequences :

- The chip sequences i.e. the codes assigned to various stations are not selected randomly. They are chosen carefully.
- The codes used in the multiplexer and demultiplexer are called as orthogonal sequences.



(E-910) **Fig. 7.22.3 : CDMA multiplexer**



(E-911) **Fig. 7.22.4 : CDMA Demultiplexer**

7.22.7 Comparison of FDMA, TDMA and CDMA :

SPPU : May 06, Dec. 06, May 07, Dec. 09, Dec. 11

University Questions

Q. 1 Compare CDMA and TDMA. **(May 06, Dec. 06, 4 Marks)**

Q. 2 Compare TDMA, FDMA and CDMA techniques. **(May 07, 6 Marks)**

Q. 3 Explain in detail operation of CDMA technique and compare performance parameters of FDMA, TDMA and CDMA. **(Dec. 09, 8 Marks)**

Q. 4 What is the difference between multiplexing and multiple access techniques ? Compare TDMA, FDMA and CDMA. **(Dec. 11, 8 Marks)**

Sr. No.	FDMA	TDMA	CDMA
1.	Overall bandwidth is shared among many stations.	Time sharing takes place.	Sharing of bandwidth and time both takes place.
2.	Due to nonlinearity of devices inter modulation products are generated due to interference between adjacent channels.	Due to incorrect synchronization there can be an interference between the adjacent time slots.	Both type of interferences will be present.
3.	Synchronization is not necessary.	Synchronization is essential.	Synchronization is not necessary.
4.	Codeword is not required.	Codeword is not required	Codewords are required.
5.	Guard bands between adjacent channels are necessary.	Guard times between adjacent time slots are necessary.	Guard bands and Guard times both are necessary.

7.22.8 CDMA Services :

SPPU : May 07, May 12

University Questions

Q. 1 Enumerate all the CDMA services. **(May 07, 2 Marks)**

Q. 2 Write short notes on : CDMA services. **(May 12, 18 Marks)**

Following are some of the important CDMA services :

1. Voice services
2. Data services
3. Circuit switched data
4. Packet switched data
5. Message services
6. CDMA radio
7. Locationbased ervices
8. CDMA radio channel

1. **Voice services :**
   - In this type of service two or more people can exchange information in the voice frequency band, through the network.
   - The IS-95 CDMA service provides the analog as well as digital voice service. FM is used for the analog voice service whereas digitizing and compression of audio signal is used in digital voice service.
   - The quality of service provided by IS-95 can vary depending on different factors. CDMA system can be used for various types of speech compression and the service provider can select the compression process.
2. **Data services :**
   - This service is used for transferring information in the form of data between two or more devices. This service can be provided inside or outside the audio frequency band.
   - If the audio band (upto 4 kHz) is used, then the data modem must be used on the analog radio channel. This modem converts the digital data to analog signals. (FSK, PSK, etc.).
   - When data signals are transmitted over CDMA, a data transfer adapter (DTA) is used. DTA converts the data bits into a format which is suitable for mobile device operating on a CDMA digital radio channel.
   - Each CDMA channel can provide upto 14.4 kbps. If higher data speeds are required, then multiple data channels (upto 8) are combined together.
3. **Circuit switched data :**
   - This method maintains a dedicated path between two devices.
   - A connection needs to be established before initiating communication. For this the address is sent first and a path is established.
   - After setting up this path, data is transferred continuously.
   - The connection is disconnected after the data transfer is complete.

4. **Packet switched data :**
   - The data is transferred from sender to receiver in the form of small packets. Packet switching principle is used.
   - Each packet contains the data and the destination address. Each packet can take different path to reach the destination.
   - CDMA packet switched data service is an **always on** type service.
5. **Messaging services :**

   This service allows the transfer of short messages between two persons. Each message contains a few hundred characters.
6. **Location based services (LBS) :**
   - These are the information or advertisement services which are based on the location of the user.
   - The CDMA allows LBSs through GSM (Global positioning system).
7. **CDMA radio :**
   - In IS-95 system, there are three types of radio channel used :
   1. Wideband digital radio channel.
   2. Narrowband voice (audio) channel.
   3. Narrowband audio control channel.
   - Some of the important characteristics of IS-95 CDMA radio system are, it uses a wide radio channel, multiple frequency bands, frequency reuse, channel multiplexing, full duplex radio channel pairs, RF power control, variable rate speech coding, discontinuous transmission for increased battery life.
8. **CDMA radio channel :**
   - The radio channel used by the IS-95 which uses a 1.23 MHz slot, is divided into 64 channels.
   - Some of these are used for traffic (voice or data), and some are used for control purposes.
   - The spread spectrum technique is used in order to ensure better quality of reception.

7.23 Code Division Multiple Access (CDMA) with DS-SS :

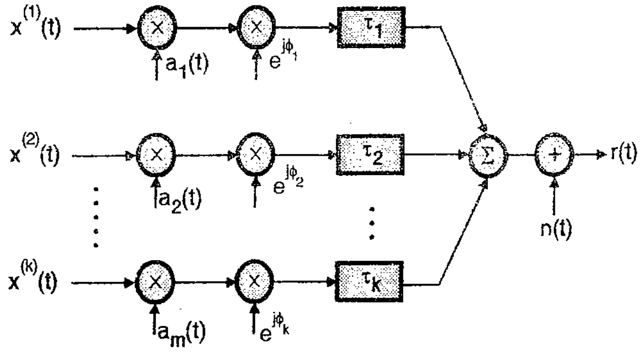
- An alternative to FDMA and TDMA is an another system called code division multiple access (CDMA). The most important feature of CDMA is as follows.
- In CDMA more than one user is allowed to share a channel or subchannel with the help of direct-sequence spread spectrum signals.
- In CDMA each user is given a unique code sequence or signature sequence. This sequence allows the user to spread the information signal across the assigned frequency band.
- At the receiver the signal is recovered by using the same code sequence. At the receiver, the signals received from various users are separated by checking the cross-correlation of the received signal with each possible user signature sequence.
- In CDMA the users access the channel in a random manner. Hence the signals transmitted by multiple users will completely overlap both in time and in frequency.
- The CDMA signals are spread in frequency. Therefore the demodulation and separation of these signals at the receiver can be achieved by using the pseudorandom code sequence. CDMA is sometimes also called as spread spectrum multiple access (SSMA).

7.23.1 A CDMA System Based on FH Spread Spectrum Signals :

- In section 7.23 we have discussed a CDMA system based on the DS- SS signals. It is also possible to have a CDMA system based on the FH-SS signals. Some of the important features of such a system are as follows :
- CDMA systems using FH-SS signal are used for the mobile users on the land, air and sea. This is because of the relaxed timing requirements as compared to those in the DS-SS systems.
- Each transmitter-receiver pair in FH-SS system is assigned its own pseudorandom FH (frequency hopping) pattern. The frequency hopping pattern for every transmitter receiver pair is unique and different from that of other pairs.
- Frequency hop over significantly larger bandwidths as compared to those possible with DS-SS are possible. Therefore the processing gain PG of FH-SS system is higher than that of a DS-SS system.
- The capacity of CDMA with FH-SS is higher than that of a DS-SS system.
- Note that all the advantages stated above are applicable only to the CDMA system based on FH-SS only.

7.23.2 Error Probability for DS CDMA on AWGN Channels :

- In the DS CDMA system, the multiple access capability is obtained by assigning a unique PN spreading sequence to each user.
- But the transmissions from different users are not synchronized. So they arrive at the receiver with different amplitudes, delays and phases.
- The DS CDMA signaling on AWGN channel is shownin Fig. 7.23.1.



(E-503) **Fig. 7.23.1 : DS CDMA signaling on an AWGN channel**

- The signals from various transmitters will arrive at the receiver with different power levels. But in DS-CDMA systems the power control is exercised such that all the signals arrive at the receiver with same power level.
- This is essential to reduce the far-near effect. That means to avoid the capture effect imposed by the strong signals.
- The receiver shown in Fig. 7.23.1 is a correlation receiver where the composite received signal is multiplied by a synchronized replica of the spreading sequence.
- The two sequences (original and its replica) will cancel each other and the desired data sequence is obtained at the output of the correlator.
- The exact error probability with a conventional correlator depends on the particular spreading sequence used and the random amplitudes, delays and phases of the signals arriving at the receiver.
- Unfortunately the exact error probability is difficult to derive and evaluate.

Review Questions

Q. 1 State the problems encountered by a communication system

Q. 2 How is SS signal different from the normal one ?

Q. 3 State true or false : The SS signal is pseudorandom in nature.

Q. 4 State some applications of SS modulation.

Q. 5 Name various types of SS systems.

Q. 6 Which modulation techniques are used in SS ?

Q. 7 Define a PN sequence.

Q. 8 What is meant by a maximum length sequence ?

Q. 9 State the important properties of the maximum length sequence.

Q. 10 State the balance property.

Q. 11 State the run property

Q. 12 Define chip duration.

Q. 13 What is chip rate ?

Q. 14 State the performance parameters of a DS-SS system.

Q. 15 Define the processing gain.

Q. 16 What is the BW of spread signal ?

Q. 17 Which factors decide the error probability ?

Q. 18 Describe a modulation technique that offer certain advantages over band limited, non-linear channels.

Q. 19 Explain with block diagram, direct sequence spread spectrum technique.

Q. 20 Describe a modulation technique that offer certain advantages over bandlimited non-linear channels.

Q. 21 Explain with block diagram, direct sequence spread spectrum technique.

Q. 22 Describe FH-FSK system in brief.

Q. 23 Draw the block diagram for FH-SS system and explain the working. Differentiate between slow frequency hopping and fast frequency hopping.

Q. 24 Explain the principle and working of FH-SS in detail. Differentiate between slow frequency hopping and fast frequency hopping.

Q. 25 Explain the principle of FDMA, TDMA.

Q. 26 Explain the principle of CDMA.

Q. 27 Compare FDMA, TDMA and CDMA.

Q. 28 Explain : 1. GPS 2. IS – 95.

7.24 University Questions and Answers :

Q. 1 Explain various wireless standards for Wi - Fi and Wi - Max. **(Dec. 2014, May 16, 4 Marks)**

Ans. : For Wi-Fi refer section 7.14.

Wi-Max :

- The long form of Wi-Max is Worldwide Interoperability for Microwave Access. It is a wireless communication standard which can provide data rates upto 1 Gbps.
- Wi-Max refers to interoperable implementations of IEEE 802.16 family of standards.

Uses/Applications :

- The Wi-Max can be used in the following applications :
  1. To provide portable mobile broadband connectivity.
  2. It can be used as an alternative to cable, Digital Subscriber Line (DSL) for providing a broadband access.
  3. To provide services such as voice on IP (VoIP).
  4. For providing a source of Internet connectivity.

Internet access :

- Wi-Max is capable of providing at home or mobile internet access to anyone across the whole city or even whole country.
- It is relatively cheap to use Wi-Max to provide Internet access to the remote locations.

Connecting :

- We can use a Wi-Max USB MODEM for mobile Internet Devices that can provide connectivity to a Wi-Max network they are known as subscriber stations (SS).
- Portable units of Wi-Max are in the form of handsets similar to smart phones, PC USB dongles and embedded devices in laptops.

Mobile phones based on Wi-Max :

- HTC launched the first Wi-Max enabled mobile phone called the Max 4G in 2008 only for Russia.
- Its next generation was made available in America in March 2010. This mobile is capable of carrying out data and mobile sessions simultaneously.

Spectrum allocation :

There is no uniform global licenced spectrum for Wi-Max. However the three licenced spectrum profiles published by the Wi-Max forum are : 2.3 GHz, 2.5 GHz and 3.5 GHz.

Spectral efficiency :

One of the biggest advantage of Wi-Max is its high spectral efficiency. This is due to the multiple reuse and smart antenna technologies.

Limitations :

An inherent problem with Wi-Max is that it cannot operate at higher bit rates over long distances. We can get either higher bit rates or longer distances but not both at a time.

Q. 2 The signal has the following parameter number of bits per MFSK symbol K = 2 number of MFSK tone $M = 2^K = 4$ length of PN sequence per hop K = 3 total number frequency hops $2^K = 8$. Sketch the output transmits frequency of fast FH /MFSK signals. **(Dec. 2015, 8 Marks)**

Ans. : Similar to Ex. 7.13.1.

Q. 3 What are multiple access techniques ? Explain WCDMA in detail. **(Dec. 2016, 6 Marks)**

Ans. :

For multiple access technique refer section 7.19.

The W-CDMA is based on the network fundamentals of GSM. Out of many 3G technologies the W-CDMA and CDMA 2000 are the most popular systems.

3G W-CDMA(UMTS) :

W-CDMA stands for wideband CDMA. Whereas UMTS means Universal Mobile Telecommunication System. This system has evolved since 1996 and it has been developed by European Telecom Standards Institute (ETSI) and European carriers, manufactures, government regulators, collectively. UMTS in 1998 was submitted to ITU for consideration as a world standard and it was named as IMT-2000. UMTS or W-CDMA is designed to have a backward compatibility with the second generation systems such as GSM, IS-136 and PDC TDMA technologies as well as the 2.5G TDMA systems. The W-CDMA retains the network structure and bit level packaging of GSM data and adds capacity and bandwidth to it with the use of the new CDMA air interface. The 3G W-CDMA air interface standard supports the **always on** packet based wireless service. This will allow the computers, entertainment devices and mobile telephones to connect themselves to simultaneously share the wireless network. Due to the always on service they can be connected to Internet anytime, anywhere. The packet data rates upto 2.048 Mbps per user (for a stationary user) are being supported by W-CDMA.

This allows users to get a high quality data, multimedia, streaming audio, streaming video and broadcast type services from the Internet. In the future version of W-CDMA the packet data rates for a stationary user will be increased upto 8 Mbps per user. The W-CDMA is designed to support the public and private network features, video conferencing and Virtual Home Entertainment (VHE). With W-CDMA, a user can do the broadcasting, mobile commerce, games, interactive video, all over the world, using a small handset. The minimum spectrum requirement of W-CDMA is 5 MHz, which is different from the other 3G standards. Due to this wider band-width the W-CDMA system has to use a completely different RF equipment at each base station (as compared to the 2G equipments). With W-CDMA, it is possible to support the data rates from 8 kbps to 2 Mbps. Each channel can carry 100 to 350 simultaneous voice calls at any given instant of time. Generally speaking using W-CDMA will result in at least six times increase in the spectral efficiency as compared to the GSM system. As W-CDMA needs to use new base station equipment, which takes time and money the installation of W-CDMA is going to take place gradually.

Features of W-CDMA (UMTS) :

The key features of W-CDMA or UMTS are as follows :

1. It is a wideband DS-CDMA system
2. Backward compatibility with GSM
3. Packet Data Rate on downlink : 2.048 Mbps
4. Minimum forward channel bandwidth : 5 MHz.
5. Frame structure : 16 slots per frame.

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Appendix A

Table A-1 : Properties of Fourier Transform

Sr. No.	Property	Mathematical expression
1.	Linearity or superposition	$[a_1 x_1 (t) + a_2 x_2 (t)] \leftrightarrow [a_1 X_1 (f) + a_2 X_2 (f)]$ a_1 and a_2 are constants.
2.	Time scaling	$x(\alpha t) \overset{F}{\leftrightarrow} \frac{1}{\lvert\alpha\rvert} x(f/\alpha)$. α is constant.
3.	Duality or symmetry	If $x(t) \overset{F}{\leftrightarrow} X(f)$ then $X(t) \overset{F}{\leftrightarrow} x(-f)$
4.	Time shifting	$(t - t_d) \overset{F}{\leftrightarrow} e^{-j2\pi f t_d} X(f)$
5.	Area under x (t)	$\int_{-\infty}^{\infty} x(t)\, dt = X(0)$
6.	Area under X (f)	$\int_{-\infty}^{\infty} X(f)\, df = x(0)$
7.	Frequency shifting	$e^{j2\pi f_c t} x(t) \overset{F}{\leftrightarrow} X(f - f_c)$
8.	Differentiation in time domain	$\frac{d}{dt} x(t) \overset{F}{\leftrightarrow} j2\pi f X(f)$
9.	Integration in time domain	$\int_{-\infty}^{t} x(\lambda)\, d\lambda \overset{F}{\leftrightarrow} \frac{1}{j2\pi f} X(f)$
10.	Conjugate functions	If $x(t) \overset{F}{\leftrightarrow} X(f)$ then $x^*(t) \overset{F}{\leftrightarrow} X(-f)$
11.	Multiplication in time domain	$x_1(t)\, x_2(t) \overset{F}{\leftrightarrow} \int_{-\infty}^{\infty} X_1(\lambda)\, X_2(f - \lambda)\, d\lambda$
12.	Convolution in time domain	$x_1(t) * x_2(t) \overset{F}{\leftrightarrow} X_1(f)\, X_2(f)$

Table A-2 : Fourier Transform Pairs

Sr. No.	Signal	Mathematical representation	Fourier transform
1.	Rectangular pulse of amplitude A and duration T.	$x(t) = A \text{ rect}\left(\frac{t}{T}\right)$	$X(f) = AT \text{ sinc}(fT)$
2.	Sinc pulse	$x(t) = A \text{ sinc}(2Wt)$	$X(f) = \frac{A}{2W} \text{ rect}\left(\frac{f}{2W}\right)$
3.	Decaying exponential signal for t > 0	$x(t) = e^{-\alpha t} u(t) \cdot a > 0$	$\frac{1}{\alpha + j2\pi f}$
4.	Rising exponential pulse for t < 0	$x(t) = e^{\alpha t} u(-t)$	
5.	Double exponential pulse	$x(t) = e^{-\alpha\|t\|}, a > 0$	$X(f) = \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$
6.	Unit impulse	$\delta(t)$	$X(f) = 1$
7.	DC signal	$x(t) = 1$	$X(f) = \delta(f)$
8.	Cosine signal	$x(t) = \cos(2\pi f_0 t)$	$\frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$
9.	Sine signal	$x(t) = \sin(2\pi f_0 t)$	$\frac{1}{2j}[\delta(f - f_0) + \delta(f + f_0)]$
10.	Signum function	$x(t) = \text{sgn}(t)$	$X(f) = \frac{1}{j\pi f}$
11.	Unit step	$x(t) = u(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$

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Appendix B

Trignometric identities :

1. $e^{j\theta} = \cos\theta + j\sin\theta$

2. $\cos\theta = \frac{1}{2}\,[e^{j\theta} + e^{-j\theta}]$

3. $\sin\theta = \frac{1}{2j}\,[e^{j\theta} - e^{-j\theta}]$

4. $\sin^2\theta + \cos^2\theta = 1$

5. $\sin(2\theta) = 2\sin\theta\cos\theta$

6. $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

7. $\cos^2\theta = \frac{1}{2}\,[1 + \cos 2\theta]$

8. $\sin^2\theta = \frac{1}{2}\,[1 - \cos 2\theta]$

9. $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

10. $\cos(A \pm B) = \cos A\cos B \pm \sin A\sin B$

11. $\tan(A \pm B) = \frac{\tan A + \tan B}{1 \pm \tan A\tan B}$

12. $\sin A\sin B = \frac{1}{2}\,[\cos(A - B) - \cos(A + B)]$

13. $\cos A\cos B = \frac{1}{2}\,[\cos(A + B) + \cos(A - B)]$

14. $\sin A\cos B = \frac{1}{2}\,[\sin(A - B) + \sin(A + B)]$

15. $\cos^3\theta = \frac{1}{4}\,(3\cos\theta + \cos 3\theta)$

16. $\sin^3\theta = \frac{1}{4}\,(3\sin\theta - \sin 3\theta)$

17. $a\cos\theta + b\sin\theta = C\cos(\theta + \phi)$, where $C = (a^2 + b^2)^{1/2}$ and $\phi = \tan^{-1}(-b/a)$

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Appendix C

Series Expansions, Summations and Complex Numbers

1. Expansions :

Taylor series :

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

$$\text{where } f^{(n)}(a) = \left.\frac{d^n f(x)}{dx^n}\right|_{x=a}$$

MacLaurin series :

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}\cdot x^n + \ldots$$

$$\text{where } f^{(n)}(0) = \left.\frac{d^n f(x)}{dx^n}\right|_{x=0}$$

Binomial series :

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots \mid nx \mid < 1$$

Exponential series :

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$$

Logarithmic series :

$$ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots$$

Trignometric series :

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots$$

$$\sin^{-1} x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \ldots$$

$$\text{sinc } x = 1 - \frac{1}{3!}(\pi x)^2 + \frac{1}{5!}(\pi x)^4 + \ldots$$

2. Summations :

Arithmetic series :

$$\sum_{n=1}^{N} n = \frac{N(N+1)}{2}$$

Geometric series :

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$$

3. Complex Numbers :

$$e^{\pm j\pi/2} = \pm j$$

$$e^{\pm jn\pi} = \begin{cases} 1 & \text{... n even} \\ -1 & \text{... n odd} \end{cases}$$

$$a + jb = re^{j\theta} \quad \text{where } r = (a^2 + b^2)^{1/2} \text{ and } \theta = \tan^{-1}(b/a)$$

$$(re^{j\theta})^k = r^k e^{jk\theta}$$

$$(r_1 e^{j\theta_1})(r_2 e^{j\theta_2}) = r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

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Appendix D

Integrals

Indefinite integrals :

1. Integration by parts :

$$\int u\,dv = uv - \int v\,du$$

$$\int f\,(x)\,g'\,(x)\,dx = f\,(x)\,g\,(x) - \int f'\,(x)\,g\,(x)dx$$

2. $\int \sin ax\,dx = -\frac{1}{a}\cos ax$

3. $\int \cos ax\,dx = \frac{1}{a}\sin ax$

4. $\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2\,ax}{4\,a}$

5. $\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2\,ax}{4\,a}$

6. $\int x \sin ax\,dx = \frac{1}{a^2}(\sin ax - ax \cos ax)$

7. $\int x \cos ax\,dx = \frac{1}{a^2}(\cos ax + ax \sin ax)$

8. $\int x^2 \sin ax\,dx = \frac{1}{a^3}(2\,ax \sin ax + 2\cos ax - a^2 x^2 \cos ax)$

9. $\int x^2 \cos ax\,dx = \frac{1}{a^3}(2\,ax \cos ax - 2\sin ax + a^2 x^2 \sin ax)$

10. $\int \sin ax \sin bx\,dx = \frac{\sin[(a-b)x]}{2\,(a-b)} - \frac{\sin[(a+b)x]}{2\,(a+b)}\quad a^2 \neq b^2$

11. $\int \sin ax \cos bx\,dx = -\left[\frac{\cos(a-b)\,x}{2\,(a-b)} + \frac{\cos(a+b)\,x}{2\,(a+b)}\right]\quad a^2 \neq b^2$

12. $\int \cos ax \cos bx\,dx = \frac{\sin[(a-b)\,x]}{2\,(a-b)} + \frac{\sin(a+b)\,x}{2\,(a+b)}\quad a^2 \neq b^2$

13. $\int e^{ax}\,dx = \frac{1}{a}e^{ax}$

14. $\int x\,e^{ax}\,dx = \frac{e^{ax}}{a^2}(ax - 1)$

15. $\int x^2 e^{ax}\,dx = \frac{e^{ax}}{a^3}(a^2 x^2 - 2\,ax - 2)$

16. $\int x\,e^{ax^2}\,dx = \frac{1}{2a}e^{ax^2}$

17. $\int e^{ax}\sin bx\,dx = \frac{1}{a^2 + b^2}e^{ax}[\,a \sin(bx) - b \cos(bx)\,]$

18. $\int e^{ax}\cos bx\,dx = \frac{1}{a^2 + b^2}e^{ax}[\,a \cos(bx) + b \sin(bx)\,]$

19. $\int \frac{dx}{a^2 + b^2 x^2} = \frac{1}{ab} \tan^{-1}\left(\frac{bx}{a}\right)$

20. $\int \frac{x\,dx}{x^2 + a^2} = \frac{1}{2} \ln (x^2 + a^2)$

21. $\int \frac{x^2\,dx}{a^2 + b^2 x^2} = \frac{x}{b^2} - \frac{a}{b^3} \tan^{-1}\left(\frac{bx}{a}\right)$

22. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$

Definite integrals :

23. $\int_0^\infty \frac{x \sin (ax)}{b^2 + x^2}\, dx = \frac{\pi}{2} e^{-ab}, \qquad a > 0 \text{ and } b > 0$

24. $\int_0^\infty \frac{\cos (ax)}{b^2 + x^2}\, dx = \frac{\pi}{2b} e^{-ab}, \qquad a > 0 \text{ and } b > 0$

25. $\int_0^\infty \frac{\cos (ax)}{(b^2 - x^2)^2}\, dx = \frac{\pi}{4b^3} [\sin (ab) - ab \cos (ab)]$

26. $\int_0^\infty \text{sinc } x\, dx = \int_0^\infty \text{sinc}^2 x\, dx = \frac{1}{2}$

27. $\int_0^\infty e^{-ax^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}, \qquad a > 0$

28. $\int_0^\infty x^2 e^{(-ax^2)}\, dx = \frac{1}{4a}\sqrt{\frac{\pi}{a}}, \qquad a > 0$

29. $\int_0^\infty \frac{x^{m-1}}{1 + x^n}\, dx = \frac{\pi/n}{\sin (m\pi/n)}, \quad n > m > 0$

30. $\int_0^\infty \frac{\sin x}{x}\, dx = \int_0^\infty \frac{\tan x}{x}\, dx = \frac{\pi}{2}$

31. $\int_0^\infty \frac{\sin x \cos ax}{x}\, dx = \begin{cases} \pi/2 & a^2 < 1 \\ \frac{\pi}{4} & a^2 = 1 \\ 0 & a^2 > 1 \end{cases}$

32. $\int_0^\pi \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}$

33. $$\int_{0}^{\pi} \frac{\cos n x}{1 + x^2} \, dx = \frac{\pi}{2} e^{-|n|}$$

34. $$\int_{0}^{\infty} \sin (x^2) \, dx = \int_{0}^{\infty} \cos (x^2) \, dx = \frac{1}{2} \sqrt{\frac{\pi}{2}}$$

35. $$\int_{0}^{\infty} x^n e^{-ax} \, dx = \frac{n\,!}{a^{n+1}} \qquad \ldots n \geq 1, a > 0$$

36. $$\int_{0}^{\infty} e^{-a^2 x^2} \, dx = \frac{1}{2a} \sqrt{\pi} \qquad \ldots a > 0$$

37. $$\int_{0}^{\infty} x^2 e^{-x^2} \, dx = \frac{1}{4} \sqrt{\pi}$$

38. $$\int_{0}^{\infty} e^{-ax} \cos x \, dx = \frac{a}{1 + a^2} \qquad \ldots a > 0$$

39. $$\int_{0}^{\infty} e^{-ax} \sin x \, dx = \frac{1}{1 + a^2} \qquad \ldots a > 0$$

40. $$\int_{0}^{\infty} e^{-a^2 x^2} \cos bx \, dx = \frac{1}{2a} \sqrt{\pi} \, e^{-(b/2a)^2}$$

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Appendix E

Error Function

1. The error function erf (u) is defined as,

$$\text{erf}(u) = \frac{2}{\sqrt{\pi}} \int_0^u e^{-z^2} dz$$

$$\begin{aligned} \text{Note that } \text{erf}(0) &= 0 \\ \text{erf}(\infty) &= 1 \\ \text{erf}(-u) &= -\text{erf}(u) \end{aligned}$$

2. The complementary error function erfc (u) is defined as,

$$\text{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^\infty e^{-z^2} dz$$

3. These two functions are related as erfc (u) = 1 – erf (u)

u	erf (u)	erfc (u)	u	erf (u)	erfc (u)
0.00	0.0000	1.0000	1.10	0.88021	0.11979
0.05	0.05637	0.94363	1.15	0.89612	0.10388
0.10	0.11246	0.88754	1.20	0.91031	0.08969
0.15	0.16800	0.83200	1.25	0.92290	0.07710
0.20	0.22270	0.77730	1.30	0.93401	0.06599
0.25	0.27633	0.72367	1.35	0.94376	0.05624
0.30	0.32863	0.67137	1.40	0.95229	0.04771
0.35	0.37983	0.62017	1.45	0.95970	0.04030
0.40	0.42839	0.57161	1.50	0.96611	0.03389
0.45	0.47548	0.52452	1.55	0.97162	0.02838
0.50	0.52050	0.47950	1.60	0.97635	0.02365
0.55	0.56332	0.43668	1.65	0.98038	0.01962
0.60	0.60386	0.39614	1.70	0.98379	0.01621
0.65	0.64203	0.35797	1.75	0.98667	0.01333
0.70	0.67780	0.32220	1.80	0.98909	0.01091
0.75	0.71116	0.28884	1.85	0.99111	0.00889
0.80	0.74210	0.25790	1.90	0.99279	0.00721
0.85	0.77067	0.22933	1.95	0.99418	0.00582
0.90	0.79691	0.20309	2.00	0.99532	0.00468
0.95	0.82089	0.17911	2.50	0.99959	0.00041
1.00	0.84270	0.15730	3.00	0.99998	0.00002
1.05	0.86244	0.13756	3.30	0.999998	0.000002

Appendix F

The sinc Function

Numerical values of sinc x = (sin πx) /πx and its square are tabulated below for x = 0 to 3.9.

x	sinc x	sinc^2 x	x	sinc x	sinc^2 x
0.0	1.000	1.000	2.0	0.000	0.000
0.1	0.984	0.968	2.1	0.047	0.002
0.2	0.935	0.875	2.2	0.085	0.007
0.3	0.858	0.737	2.3	0.112	0.013
0.4	0.757	0.573	2.4	0.126	0.016
0.5	0.637	0.405	2.5	0.127	0.016
0.6	0.505	0.255	2.6	0.116	0.014
0.7	0.368	0.135	2.7	0.095	0.009
0.8	0.234	0.055	2.8	0.067	0.004
0.9	0.109	0.012	2.9	0.034	0.001
1.0	0.000	0.000	3.0	0.000	0.000
1.1	−0.089	0.008	3.1	−0.032	0.001
1.2	−0.156	0.024	3.2	−0.058	0.003
1.3	−0.198	0.039	3.3	−0.078	0.006
1.4	−0.216	0.047	3.4	−0.089	0.008
1.5	−0.212	0.045	3.5	−0.091	0.008
1.6	−0.189	0.036	3.6	−0.084	0.007
1.7	−0.151	0.023	3.7	−0.070	0.005
1.8	−0.104	0.011	3.8	−0.049	0.002
1.9	−0.052	0.003	3.9	−0.025	0.001

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G

APPENDIX

Sampling

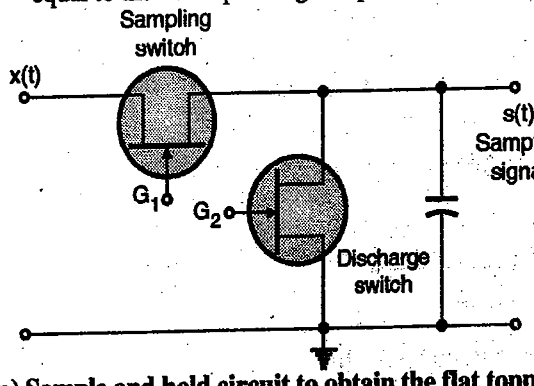
G.1 Flat Top Sampling or Rectangular Pulse Sampling :

SPPU : Dec. 14

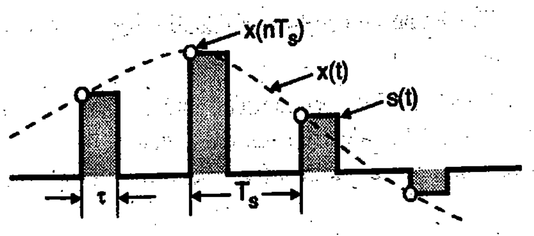
University Questions

Q. 1 Explain the flat-top sampling with functional diagram. Draw spectral diagram for the flat-top sampled signal and aperture effect. **(Dec. 14, 8 Marks)**

- The natural sampling is rarely employed in practice. Instead the other practical sampling technique called flat top sampling is employed in practice.
- In the flat top sampling technique, the analog signal x (t) is sampled instantaneously at the rate $f_s = \frac{1}{T_s}$ and the duration of each sample is lengthened to a duration "τ" as shown in Fig. G.1.1(b).
- Thus the amplitudes of these pulses are constant and equal to the corresponding sampled values.



(a) Sample and hold circuit to obtain the flat topped samples



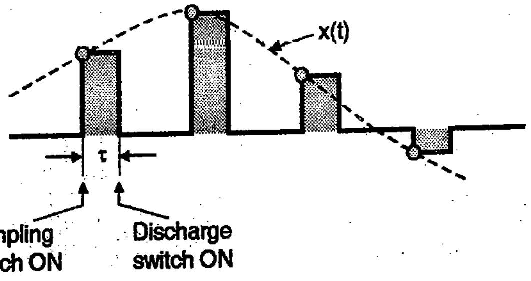
(b) Flat top sampled signal

(D-433) **Fig. G.1.1**

- The flat top pulses can be obtained by using the sample and hold circuit shown in Fig. G.1.6(a).

Operation of the sample and hold circuit :

- The sample and hold circuit consists of two FET switches and a capacitor as shown in Fig. G.1.1(a). The analog signal x (t) is applied at the input of this circuit and the sampled signal s (t) is obtained across the capacitor.
- A gate pulse will be applied to gate G_1 at the instant of sampling for a very short time. The sampling switch will turn on and the capacitor charges through it to the sample value x (nT_s).
- The sampling switch is then turned off. Both the FETs will remain OFF for a duration of "τ" seconds and the capacitor will hold the voltage across it constant for this period. Thus the pulse is stretched to "τ" seconds.
- At the end of the pulse interval (τ), a pulse is applied to G_2 i.e. gate terminal of discharge FET. This will turn on the discharge FET and short circuit the capacitor. The output voltage then reduces to zero. This is as shown in Fig. G.1.2.



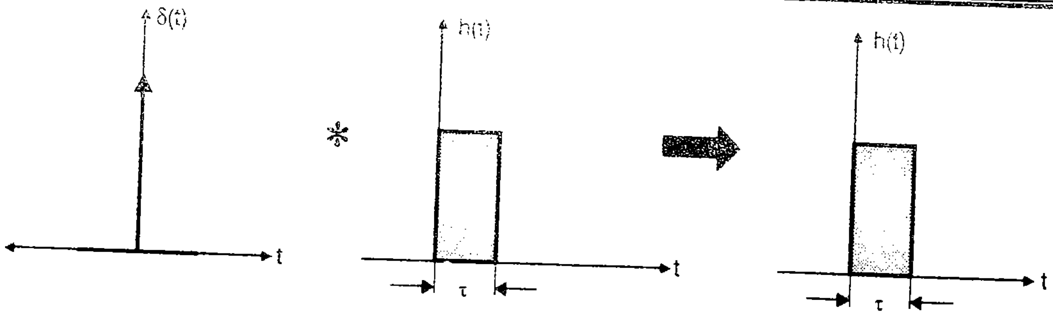
(D-434) **Fig. G.1.2 : Operation of sample and hold circuit**

Principle of generating the flat top sampled pulses :

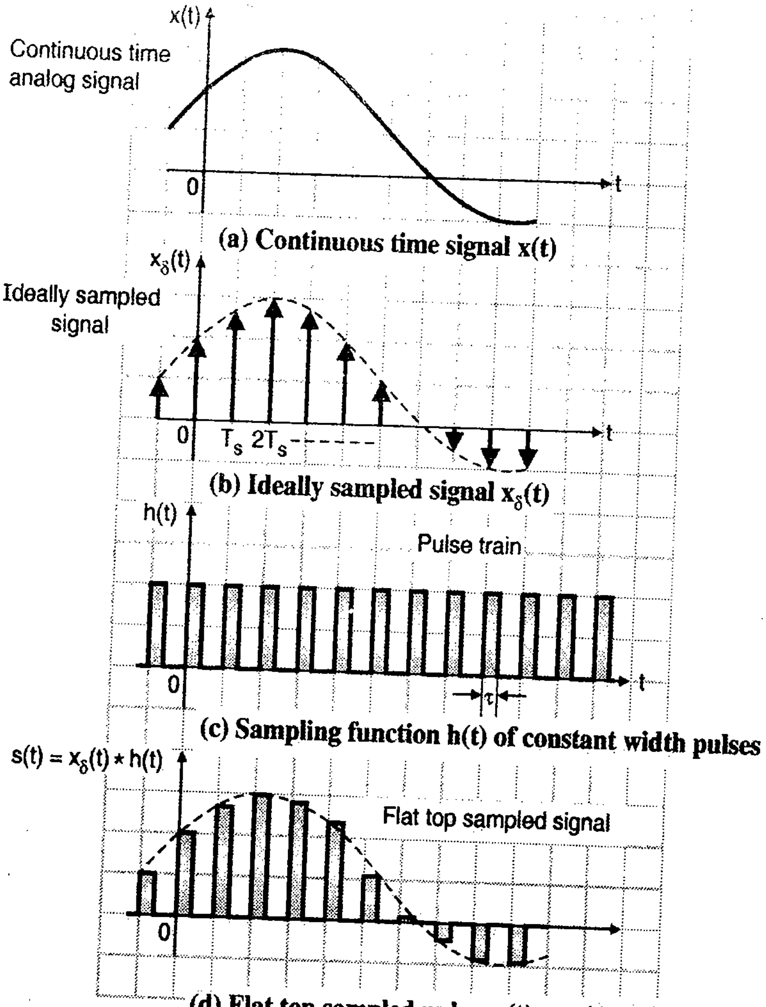
- From Fig. G.1.2 it is clear that only the rising edge of each pulse represents the instantaneous value of the analog signal x (t).
- Therefore the flat top sampled pulses can be obtained by "convolution" of the instantaneous sample and a pulse h (t) of duration τ.
- This is true because convolution of any function with the delta function results in the same function.

$$\text{i.e. } x(t) * \delta(t) = x(t) \qquad ...(G.1.1)$$

- This is the replication property of the delta function. This property is being used for the generation of flat top sampled pulses.



(D-435) **Fig. G.1.3 : Principle of generating the flat top sampled pulses**



(a) Continuous time signal x(t)

(b) Ideally sampled signal $x_\delta(t)$

(c) Sampling function h(t) of constant width pulses

(d) Flat top sampled pulses $s(t) = x_\delta(t) * h(t)$

(D-436) **Fig. G.1.4**

- This principle is graphically explained in Fig. G.1.3. From Fig. G.1.3 we can write that,

$$h(t) = \delta(t) * h(t) \quad ...(G.1.2)$$

Actual generation of flat top pulses :

- In Equation (G.1.2), on RHS if we replace δ (t) by x_δ (t) i.e. the ideally sampled signal then we get the flat top sampled signal s (t).

Flat top sampled signal, $s(t) = x_\delta(t) * h(t)$

- This equation can be graphically explained as shown in Fig. G.1.4.

Thus the flat top sampled pulses are obtained by convolution of the ideally sampled signal x_δ (t) and a pulse train of finite pulse width h (t). The width of each pulse is τ sec.

Spectrum of flat top sampled signal :

- The flat top sampled signal is given by,

$$s(t) = x_\delta(t) * h(t) \quad ...(G.1.3)$$

- Therefore the width of s (t) is decided by h (t) and the amplitude of s (t) depends on x_δ (t).

- The ideally sampled signal x_δ (t) is expressed mathematically as,

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(n T_s)\,\delta(t - n T_s) \quad \text{...(G.1.4)}$$

$$\text{Now } s(t) = x_\delta(t) * h(t)$$

- Using the definition of convolution,

$$x_\delta(t) * h(t) = \int_{-\infty}^{\infty} x_\delta(v)\, h(t - v)\, dv \quad \text{...(G.1.5)}$$

- Note that "τ" has not been used deliberately instead "v" is being used.

Substitute Equation (G.1.4) into Equation (G.1.5) to get,

$$s(t) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(v - nT_s)\, h(t - v)\, dv$$

- Interchanging the order of summation and integration and rearranging we get,

$$s(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \int_{-\infty}^{\infty} \delta(v - nT_s)\, h(t - v)\, dv \quad \text{...(G.1.6)}$$

- Now let us use the shifting property of delta function as,

$$\int_{-\infty}^{\infty} f(t)\,.\,\delta(t - t_d)\, dt = f(t_d) \quad \text{...(G.1.7)}$$

- Apply this property to Equation (G.1.6) to write,

$$s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, h(t - nT_s) \quad \text{...(G.1.8)}$$

- This expression represents the flat top sampled signal in time domain, in terms of the instantaneous sample values x (nT$_s$) and train of fixed duration pulses.
- To obtain the spectrum of s (t), let us use the convolution theorem which states that, convolution in time domain is transformed into multiplication of transforms in frequency domain.

$$s(t) = x_\delta(t) * h(t).$$

- Taking Fourier transform of both the sides we get,

$$S(f) = X_\delta(f)\, H(f) \quad \text{...(G.1.9)}$$

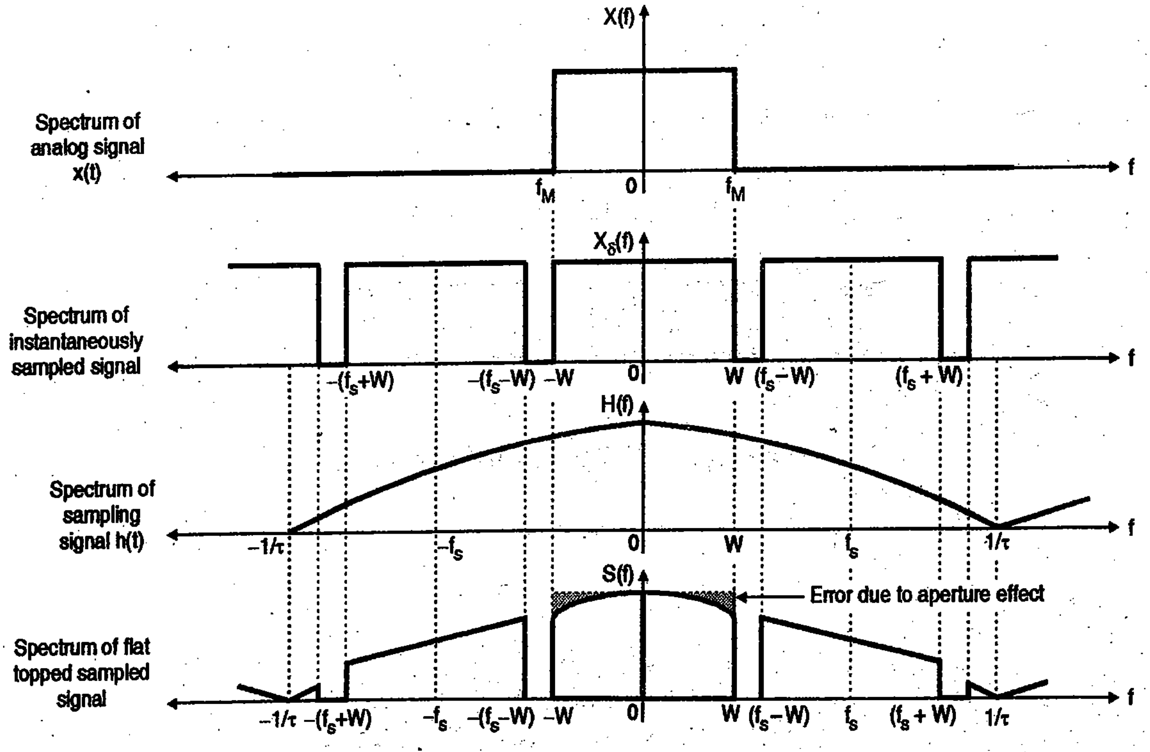
$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \quad \text{...(G.1.10)}$$

- Therefore Equation (G.1.9) becomes,

Spectrum of flat top sampled signal

$$S(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)\, H(f) \quad \text{...(G.1.11)}$$

- This is the expression for the spectrum of a flat top sampled signal. H (f) is the spectrum of h (t). As h (t) is a rectangular pulse, its spectrum is a sinc function.
- Therefore product of the spectrums X_δ (f) and H (f) is shown in Fig. G.1.5, because S (f) is equal to the product of X_δ (f) and H (f).



(D-437) **Fig. G.1.5 : Spectrum of a flat top sampled signal**

Observations from the Fig. G.1.5 :

1. The signal x (t) has a flat spectrum over its entire range from 0 to W. The transform of the instantaneous signal X_δ (f) has been drawn below it. The sampling frequency $f_s = 1/T_s$ is large enough to allow the guard band.
2. The spectrum of the sampling signal h (t) is a sinc function.
3. The spectrum of the flat topped signal is the product of these two spectrum. Due to the multiplication with the sinc function, this spectrum goes to zero at $f = 1/\tau$.

Aperture effect :

- Consider the spectrum of the flat topped signal. We are interested in the portion of the spectrum upto frequency W.
- The spectrum should have been flat in this portion of the spectrum but it is not as shown in Fig. G.1.5.
- The shaded portion shows an error due to an effect called "aperture effect".

The high frequency roll off characteristics of a typical H (f) acts like a low pass filter and attenuates the upper portion (high frequency) of the message signal spectrum. This loss of high frequency content is called as the "aperture effect". The aperture effect is due to the finite pulse width "τ" of the sampling signal. With increase in the width τ, the frequency 1/τ will reduce and the error will increase.

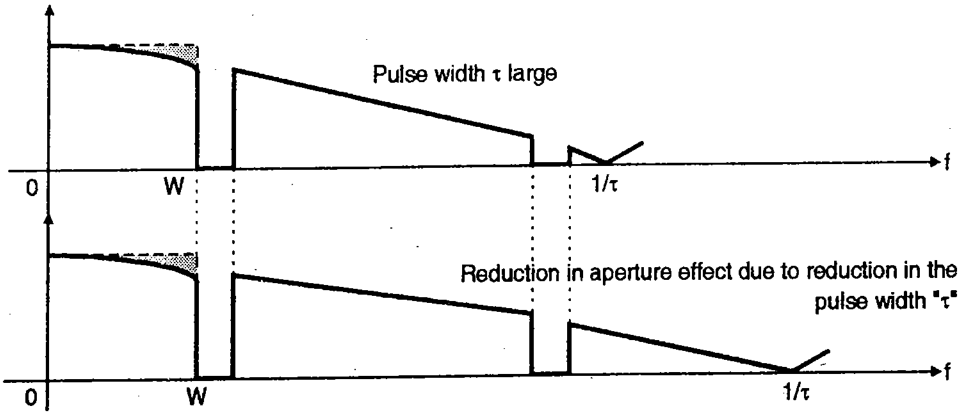
- The aperture effect can be reduced by reducing the pulse width τ as shown in Fig. G.1.6. The aperture effect can be corrected in reconstruction by including an equalizer.

Reconstruction of original signal x (t) :

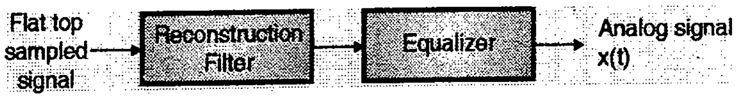
- Due to the aperture effect discussed earlier, an amplitude distortion as well as a delay is introduced in the flat top sampled signal.
- This distortion can be corrected by connecting an equalizer after the reconstruction filter (low pass filter) as shown in Fig. G.1.7.

Merits and demerits of flat top sampling :

1. Better SNR due to increased signal power. This is due to the finite width "τ" of the pulses.
2. Generation is easy.
3. Practical filters can be used for reconstruction.
4. Aperture effect introduces distortion.



(D-438) **Fig. G.1.6 : Effect of pulse width "τ" on the aperture effect**



(D-439) **Fig. G.1.7 : Reconstruction of x (t)**

Note